THE

LONDON, EDINBURGH AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE

CONDUCTED BY

SIR OLIVER JOSEPH LODGE, D.Sc., LL.D., F.R.S.
SIR JOSEPH JOHN THOMSON, O.M., M.A., Sc.D., LL.D., F.R.S.

AND

ALLAN FERGUSON, M.A., D.Sc.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex allenis libamus ut apes."

JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

VOL. XXX.—SEVENTH SERIES

JULY—DECEMBER 1940

LONDON:

TAYLOR & FRANCIS, LTD., RED LION COURT, FLEET STREET, E.C. 4.

Agents: WM. DAWSON & SONS, LTD., CANNON HOUSE, PILGRIM STREET, E.C. 4.

"Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

—— "Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

CONTENTS OF VOL. XXX. (Nos. 198–203)

(SEVENTH SERIES)

NUMBER 198—JULY

Page

I. The Use of the Wave Guide for Measurement of Microwave Dielectric Constants. By H. R. L. Lamont, M.A., B.Sc., Carnegie and Breadalbane Scholar in the University of Glasgow 1

II. On the Integral Equations of Continuous Dynamical Systems. By W. H. Ingram 16

III. A Mechanical Appliance for the Smoothing of Time Series. By T. E. W. Schumann, M.Sc., Ph.D. (Plates I. & II.) 39

IV. Modification of a Relativity Postulate. By Sir Shah Sulaiman, Federal Court, India 49

V. A Redetermination of the Crystal Structure of Lithium. By Harald Perlitz, Professor of Physics at the University of Tartu, and Endel Aruja, Assistant in the Physical Laboratory at the Technical University of Tallinn. (Plate III.) 55

VI. The Fabric Theory of Protein Structure. By Dorothy Wrinch, M.A., D.Sc., Lecturer in Chemistry, Johns Hopkins University, Research Fellow of Somerville College, Oxford 64

VII. Dark Bands in the Spectra of Double Diffraction Gratings. By R. A. Houstoun, D.Sc., University of Glasgow 68

VIII. Notices respecting New Books :—

American Mathematical Society Semicentennial Publications (in two volumes) 79

Internal Constitution of the Earth. Edited by Beno Gutenberg. 82

Mr. M. M. Macdermott's Vowel Sounds in Poetry 84

NUMBER 199—AUGUST

IX. Three-dimensional Vortex Motion in a Viscous Fluid. By Sydney Goldstein 85

X. Finite Complexes in Crystals : a Classification and Review. By A. F. Wells, M.A., Ph.D., Crystallographic Laboratory, Cambridge 103

Page

XI. Lorentz Transformation and Hamilton's Quaternions. By Otto F. Fischer 135

XII. The Sine Relation and Minimum Ellipticity in Foucault's Pendulums with Copper Suspensions. By B. Dasannacharya, M.A., Dr.Phil., F.Inst.P., Professor of Physics, Benares Hindu University, Benares, India, and Balram Singh Gautom, M.Sc. (Benares) 151

XIII. An Analogy between the Theories of Potential and Vibrations. By Harold Jeffreys 161

XIV. Some Integral Representations of the Associated Legendre Functions. By A. Erdélyi 168

XV. Notices respecting New Books :—

Mr. Gerhard Tintner's The Variate Difference Method 172

Messrs. H. Le Boiteux and R. Boussard's Elasticité et Photoélasticimétrie 172

Dr. E. G. Richardson's Sound 172

NUMBER 200—SEPTEMBER

XVI. Some Notes on End-corrections. By L. C. Tyte, B.Sc., Ph.D., F.Inst.P., Projectile Development Establishment 173

XVII. Solution of the Equation $f'(x)=f(1/x)$. By Ludwik Silberstein, Ph.D. 185

XVIII. The Fermi-Dirac Functions. By J. L. B. Cooper, D.Phil..... 187

XIX. Partial Derivatives of Derivatives. By W. S. Kimball, Ph.D., Robert College, Istanbul, Turkey 190

XX. Some Acoustical Source-Observer Problems. By S. W. Redfearn 223

XXI. Measurements of the Noise from Aerofoils and Streamline Wires. By W. F. Hilton, Ph.D., A.R.C.Sc., B.Sc., D.I.C., A.F.R.Ae.S. (of the National Physical Laboratory) 237

XXII. Some Measurements of High Frequency Permeability. By Willis Jackson, D.Sc., Electrotechnics Department, Manchester University 247

XXIII. Castigliano's Principle of Minimum Strain-Energy, and the Conditions of Compatibility for Strain. By R. V. Southwell 252

XXIV. Notices respecting New Books :—

Profs. J. E. Mack and M. J. Martin's The Photographic Process 259

Mr. H. T. Pledge's Science since 1500. A short History of Mathematics, Physics, Chemistry, Biology 259

Page
Prof. O. K. Rice's Electronic Structure and Chemical Binding.. 260
Messrs. H. S. Allen and R. S. Maxwell's A Text-book of Heat.—Part II. 260

NUMBER 201—OCTOBER

XXV. Quaternion Analogy of Wave-tensor Calculus. By W. H. McCrea 261
XXVI. The Propagation of Electromagnetic Waves in an Ionized Medium and the Calculation of the True Heights of the Ionized Layers of the Atmosphere. By Olof Rydbeck, Cruft Laboratory, Harvard University, and the Chalmers Institute of Technology, Gothenburg 282
XXVII. On the Nature of Negative Resistances and Negative Resistance Sections. By S. P. Chakravarti, M.Sc. (London), D.I.C., A.M.I.E.E., Mem.I.R.E., University Lecturer, University of Calcutta 294
XXVIII. Notes on Mathieu Functions.—I. A Class of Hyperbolic Mathieu Functions. By W. G. Bickley, D.Sc., Imperial College of Science and Technology 312
XXIX. The Thermal and Electrical Conductivities of Metals and Alloys. By W. F. Chubb, Ph.D., B.Sc. 323
XXX. The Idea of Minimum Proper Time, and some Consequences of it. By M. Fahmy, The Egyptian University, Cairo..... 331
XXXI. The Mutual Inductance of Two Circles : a Problem in Approximation. By E. H. Neville 340
XXXII. Obituary :—Sir Oliver Joseph Lodge, D.Sc., Sc.D., LL.D., F.R.S. 341
XXXIII. Notices respecting New Books :—
Mr. D. Laugharne Thornton's Mechanics Applied to Vibrations and Balancing..... 344
Messrs. J. V. Uspensky and M. A. Heaslet's Elementary Number Theory 345
Mr. W. W. Rouse Ball's Mathematical Recreations and Essays 345
Prof. L. E. Dickson's Modern Elementary Theory of Numbers..... 346
Prof. I. S. Sokolnikoff's Advanced Calculus 347
Dr. Harris Hancock's Development of the Minkowski Geometry of Numbers..... 347
Mr. Blamey Stevens's The Psychology of Physics..... 347
Reports on Progress in Physics.—Vol. VI..... 348

NUMBER 202—NOVEMBER

Page

XXXIV. Research on Combined Stress and Anisotropy. By James J. Guest 350

XXXV. The Matrix Theory of Four-Terminal Networks. By Louis A. Pipes, Ph.D., Graduate School of Engineering, Harvard University, Cambridge, Mass. 370

XXXVI. The Lateral Diffusion of a Stream of Ions in a Gas. By L. G. H. Huxley, M.A., D.Phil., F.Inst.P., F. W. Bennett Lecturer in Physics, University College, Leicester 396

XXXVII. A Simple Application of the Laplace Transformation. By H. S. Carslaw 414

XXXVIII. A Contribution to the Theory of Elasticity of Non-Isotropic Materials (with Applications to Problems of Bending and Torsion). By Eric Reissner, Ph.D., Massachusetts Institute of Technology 418

XXXIX. Note on the Barrier-layer Resistance of Selenium Photo-elements. By Arne Eld Sandström 428

XL. Obituary :—Sir J. J. Thomson, O.M., M.A., D.Sc., Sc.D., LL.D., F.R.S. 430

XLI. Notices respecting New Books :—

Mr. Bruce Chalmers's The Physical Examination of Metals.—Vol. I. Optical Methods 432

Messrs. I. H. B. and A. G. H. Spiers's The Physical Treatises of Pascal 433

Stephen Timoshenko : 60th Anniversary Volume 434

Otto Klemperer's Electron Optics (Cambridge Physical Tracts). 434

Drs. G. D. Parkes and J. W. Mellor's Mellor's Modern Inorganic Chemistry 435

Mr. N. Rashevsky's Advances and Applications of Mathematical Biology 435

Mr. J. W. L. Glaisher's British Association Mathematical Tables.—Vol. VIII. Number-Divisor Tables 435

Prof. Felix Klein's Elementary Mathematics from an Advanced Standpoint 436

The Raman Jubilee Volume 436

NUMBER 203—DECEMBER

XLII. The Ignition of Inflammable Gases by Hot Moving Particles.—II. By Stewart Paterson, M.A., B.Sc., formerly Metcalfe Research Fellow and Thomson Experimental Scholar, Glasgow University 437

Page

XLIII. Solutions of Mathieu's Equation.—I. By J. G. Brainerd, Sc.D., and C. N. Weygandt, M.S., Assistant Professor and Instructor respectively in the Moore School of Electrical Engineering of the University of Pennsylvania 458

XLIV. Einstein's Distant Parallelism and Dirac's Equation. By H. W. Haskey 478

XLV. On the Inelastic Scattering of a Beam of Particles by Hydrogen and Helium. By Prof. K. C. Kar, D.Sc. 487

XLVI. The Dielectric Constants of some Metallic Sulphates containing various amounts of Water of Crystallization. By C. K. Cheng, Ph.D., Assistant Demonstrator, Department of Physics, University of Toronto... 505

XLVII. A Table of Complete Elliptic Integrals. By Alan Fletcher, M.A., Ph.D., University of Liverpool..... 516

XLVIII. Notices respecting New Books :—

Mr. Gerhard Herzberg's Molecular Spectra and Molecular Structure.—I. Diatomic Molecules 520

Index 521

PLATES

I. & II. Illustrative of Dr. T. E. W. Schumann's Paper on a Mechanical Appliance for the Smoothing of Time Series.

III. Illustrative of Prof. Harald Perlitz and Mr. Endel Aruja's Paper on a Redetermination of the Crystal Structure of Lithium.

9774-108-29

CONTENTS OF No. 198.

Page

I. The Use of the Wave Guide for Measurement of Microwave Dielectric Constants. By H. R. L. Lamont, M.A., B.Sc., Carnegie and Breadalbane Scholar in the University of Glasgow 1

II. On the Integral Equations of Continuous Dynamical Systems. By W. H. Ingram 16

III. A Mechanical Appliance for the Smoothing of Time Series. By T. E. W. Schumann, M.Sc., Ph.D. (Plates I. & II.) 39

IV. Modification of a Relativity Postulate. By Sir Shah Sulaiman, Federal Court, India 49

V. A Redetermination of the Crystal Structure of Lithium. By Harald Perlitz, Professor of Physics at the University of Tartu, and Endel Aruja, Assistant in the Physical Laboratory at the Technical University of Tallinn. (Plate III.) 55

VI. The Fabric Theory of Protein Structure. By Dorothy Wrinch, M.A., D.Sc., Lecturer in Chemistry, Johns Hopkins University, Research Fellow of Somerville College, Oxford 64

VII. Dark Bands in the Spectra of Double Diffraction Gratings. By R. A. Houstoun, D.Sc., University of Glasgow 68

VIII. Notices respecting New Books :—

American Mathematical Society Semicentennial Publications (in two volumes) 79

Internal Constitution of the Earth. Edited by Beno Gutenberg. 82

Mr. M. M. Macdermott's Vowel Sounds in Poetry 84

PLATES.

I. & II. Illustrative of Dr. T. E. W. Schumann's Paper on a Mechanical Appliance for the Smoothing of Time Series.

III. Illustrative of Prof. Harald Perlitz and Mr. Endel Aruja's Paper on a Redetermination of the Crystal Structure of Lithium.

*** It is requested that all Communications for this Work may be addressed, post-paid, to the care of Messrs. Taylor and Francis, Ltd., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

THE

LONDON, EDINBURGH & DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE

[SEVENTH SERIES—VOL. 30]

I. *The Use of the Wave Guide for Measurement of Microwave Dielectric Constants.*

By H. R. L. Lamont, M.A., B.Sc.,
Carnegie and Breadalbane Scholar in the University of Glasgow *.

[Received August 22, 1939.]

Introduction.

Many measurements of dielectric constants at high frequencies have been made since the time of Drude, whose classic experiments are the basis of most of these. Drude [1] used the resonant properties of the Lecher wire system coupled to a high-frequency oscillator. In his "first method" the wave-length along the wires was measured, first in air, then with the wires immersed in liquid of dielectric constant ϵ. If λ_1 and λ_ϵ are these respective lengths, ϵ is given by the relation $\lambda_1 = \sqrt{\epsilon}\lambda_\epsilon$. In Drude's "second method" a simple form of condenser filled with the liquid to be measured was placed across the ends of the parallel wires and the resultant resonant length found. The condenser was calibrated in terms of known liquids.

A source of inaccuracy in all such experiments is the impossibility of including the complete electromagnetic field surrounding the wires.

---

* Communicated by John Thomson, M.A., D.Sc.

This error is removed when the Lecher wires are replaced by a pair of concentric tubes. The field then exists entirely between the tubes, and the outer serves as a screen. Drake, Pierce, and Dow [2] have used such a system for measurements on water by Drude's first method, and Kalinin [3] has used Drude's second method with similar apparatus.

A third conductor system in which stationary waves may be set up has recently gained prominence [4]. This consists of a single hollow metal tube filled with a dielectric material which forms a guide for electromagnetic waves of suitably high frequency. A bibliography of papers relating to this form of resonant line—the so-called wave guide—is given in a previous paper by the author [5]. The wave guide has the same advantages as those noted above for the concentric line, and its properties make it very suitable for the investigation of high-frequency dielectric constants, but, with the exception of some elegant measurements by Kašpar [6] with dielectric guides, the little experimental work so far published has been directed to the production and detection of the waves and to verification of the theory.

In this paper a description is given of dielectric constant measurements made with a wave guide at wave-lengths from 15 to 25 cm. (frequencies 2000–1200 Mc./sec.). Method *A* is simply Drude's first method. Methods *B*, *C*, and *D* are based on the second method, but give an absolute determination (Drude's method gave only relative values). After discussion of the results there is a section dealing with verifications of some deductions from the theory used.

Apparatus.

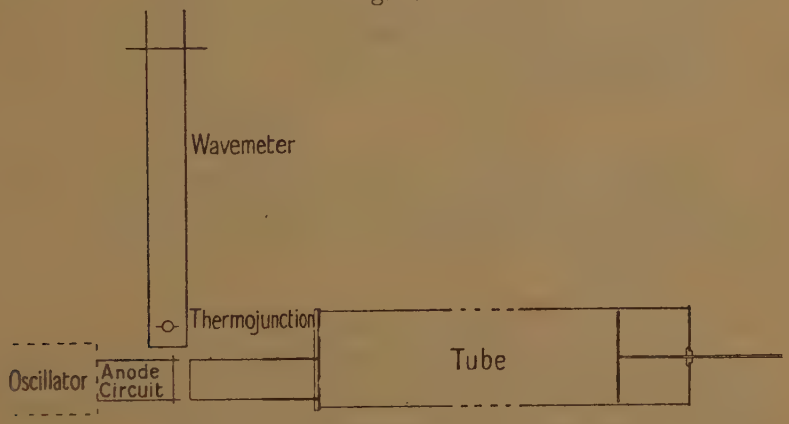
A magnetron oscillator was used capable of oscillating at wave-lengths down to 15 cm. The anode circuit is a pair of parallel wires attached to the anode leads and fitted with a sliding bridge for tuning purposes. To facilitate the coupling of this circuit to the wavemeter and tube system the bridge was kept fixed and the magnetron made movable upon rails. Coupled loosely to the fixed end of the anode circuit and at right angles to it is the wavemeter. This consists of two parallel copper wires, closed at the coupled end, carrying a brass reflecting plate through which the wires pass in close-fitting holes. The plate is moved by a screw and its position read by a travelling microscope. A vacuum thermo-junction connected to a microammeter is fixed near the coupled end. The distance between two successive positions of the reflecting plate which produce a maximum deflection on the microammeter represents a half wave-length. The length of the parallel wires allows three to four half wave-length measurements, and these are consistent to about 0·1 per cent. The corrections to be applied for wire spacing and skin effect have been dis-

cussed by Hund [7]; for the frequencies used the correction is of the order of 0·02 per cent., which is here negligible.

Some difficulty was experienced with frequency instability, which was found to be due mainly to fluctuation of the anode potential supplied from a motor generator through a potentiometer. A voltage stabilizer was built similar to one described by Nergaard [8], which is of the electronic type with degenerative triodes, voltage control being applied through two steep-slope pentode amplifying stages. This, in conjunction with large and well-charged accumulators for filament and field currents, gives satisfactory stability over the short period required for any set of measurements.

The tube used as wave guide is of copper, 7 inches internal diameter and some 5 feet long. It is shown in section in fig. 1, which illustrates

Fig. 1.



the disposition of the apparatus. To the end of the tube nearest the oscillator was soldered a brass ring, and a brass disk concentric with this is held in position by a trolitul insulating disk. Such an arrangement forms a suitable means of exciting the E_0 type of wave inside the tube. Coupling to the oscillator is by means of a trombone arrangement fixed as shown between the ring and the disk. The trombone length can be varied to tune the coupling line to one-quarter wave-length, and the tube can be moved on rollers to vary the degree of coupling. Later it was found quite satisfactory to omit the side of the trombone which connects to the tube, the remaining L-shaped part giving adequate coupling. Actually the anode circuit and tube coupling line are not coplanar as shown, their planes being parallel and about 1 cm. apart, to allow free movement of the anode circuit. At the other end of the tube is a movable piston sliding in a bush with liquid-tight gland fixed in a trolitul end-

plate. The piston head is a disk of diameter slightly less than the internal diameter of the tube, and contact between the piston head and the walls of the tube is maintained by 24 phosphor-bronze springs arranged round the periphery of the piston head. By means of a scale, displacements of the piston can be measured. A longitudinal slot was cut along part of the tube, and a 5 ma. vacuum thermojunction moving on a slider over the slot carries a short probe which projects into the tube. From readings on a microammeter connected to the thermojunction the radial field distribution along the length of the tube can be plotted. This slot should not materially affect the field, as for E-waves conduction currents in the tube walls are everywhere along the generators. Various methods were tried of indicating resonance in the tube when the piston was moved, and the most satisfactory was found to be by observing the reaction on the anode current of the oscillator. The no-load current was balanced out in the usual way and the increase at resonance observed on a 0–500 μa. meter. With sufficiently loose coupling the positions of resonance of the piston were reproducible to 0·1 per cent. The displacement between two resonance positions measures a half wave-length in the tube.

Preliminary Measurements.

A comparison test was first of all made between the oscillator wave-length as determined by the wavemeter and by tube measurements. When the tube is filled with air the wave-length inside, λ_t, is given by the formula

$$\frac{1}{\lambda_t} = \sqrt{\frac{1}{\lambda^2} - \left(\frac{k}{2\pi}\right)^2}, \quad \ldots\ldots\ldots \quad (1)$$

where λ is the oscillator wave-length and k a constant given for the E_0-wave by $2{\cdot}405/a$, a being the radius of the tube. Errors due to radiation are non-existent and those due to tube losses are very small. From measurements of λ_t the oscillator wave-length can be determined. The measurements made showed agreement with wavemeter readings to less than 0·25 per cent. at a number of points in the range 15 to 22 cm. The critical oscillator wave-length (above which transmission does not take place) is given by $\lambda = \frac{2\pi}{k} = 23{\cdot}2$ cm. for the tube used.

Determination of Dielectric Constant.

Method A.

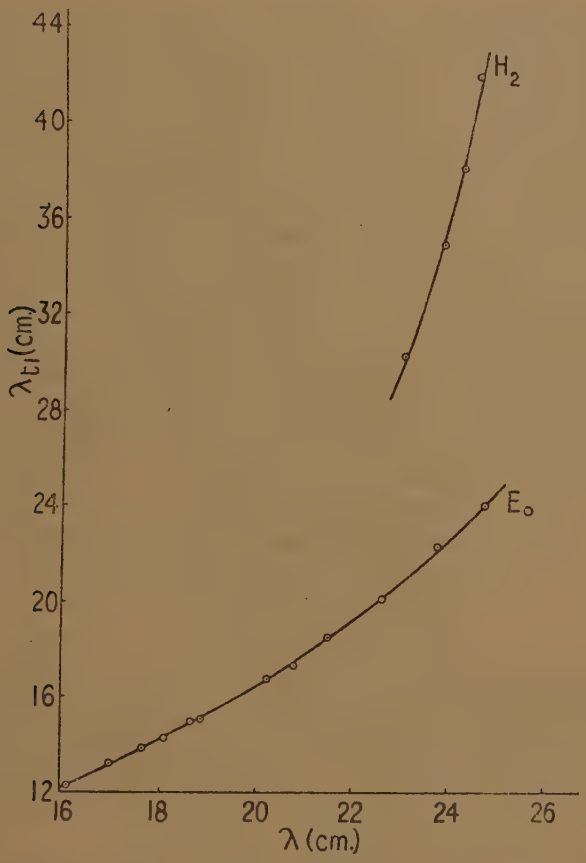
For a tube filled with an insulator of dielectric constant ϵ the wave-length in the tube, λ_{t1}, is given by

$$\frac{1}{\lambda_{t1}} = \sqrt{\frac{\epsilon}{\lambda^2} - \left(\frac{k}{2\pi}\right)^2}. \quad \ldots\ldots\ldots \quad (2)$$

If λ and λ_{t1} are measured ϵ can be determined from this equation.

In the experimental arrangement the tube is filled with a liquid dielectric. The spaces between the contact springs round the piston head allow transference of the liquid when the piston is moved, and the gland at the end prevents leakage. Some of the measurements of λ_{t1} were made from the distance which the piston had to be moved between successive resonance positions, others were found by plotting the field distribution along

Fig. 2.



the tube by means of the moving probe with the piston in a resonance position. As it is inconvenient to use a very long piston, the first method gives at most three maxima; with the second as many as eight half wave-lengths were plotted, but the process is tedious.

For the measurements pure liquid paraffin, a non-polar dielectric, was used. Corresponding values of λ, λ_{t1} obtained are plotted as coordinates in fig. 2, and from each pair ϵ was calculated. The lower curve shown is

the curve of equation (2) for ϵ equal to the mean of the values obtained. In the upper curve a few points are shown for the first wave of the H_2 series ($k=3{\cdot}04/a$; 3·04 being the first zero of $J_2'(x)$[4]), of which there was a component at some frequencies. The mean value of the dielectric constant of liquid paraffin obtained from twelve readings was $2{\cdot}191\pm 0{\cdot}007$.

Method B.

The obvious disadvantage of the method just described lies in the large quantity of liquid required—ten gallons were used in the above measurements ; also it is unsuitable for solid dielectrics since the piston cannot then be moved. The technique now to be described is suitable for solids and requires only a small volume of material. It is adapted from a similar method described by King [9] for use with a Lecher wire system.

A short cylinder of solid dielectric is made which can slide smoothly inside the tube which forms the wave guide. The introduction of the dielectric will produce a decrease in the resonant length of the tube and the piston will have to be moved inwards to restore resonance ; the distance to be moved is a function of the position of the dielectric slab in the tube. The theory has been investigated by the author in a previous paper [5], and on that the following work is based. Equation numbers given here in italics will be understood to refer to equations in that paper.

If δ is the piston shift then turning values of δ are given by the equations

$$\tan \tfrac{1}{2}\beta(s_1+\delta_1)=n \tan \tfrac{1}{2}\beta_1 s_1, \quad \ldots\ldots\ldots \quad (13)$$

$$\tan \tfrac{1}{2}\beta(s_1+\delta_2)=\frac{1}{n}\tan \tfrac{1}{2}\beta_1 s_1, \quad \ldots\ldots\ldots \quad (12)$$

where s_1 is the thickness of the slab, β, β_1 are the propagation constants in the tube equal to $\frac{2\pi}{\lambda_t}$, $\frac{2\pi}{\lambda_{t1}}$ respectively, and n is given for E-waves by

$$n=\sqrt{\frac{1-\left(\frac{k\lambda}{2\pi}\right)^2}{\epsilon-\left(\frac{k\lambda}{2\pi}\right)^2}}. \quad \ldots\ldots\ldots \quad (22)$$

n varies between values approaching 0 and values approaching $\sqrt{\epsilon}$ for values of λ below the critical wave-length. Which of (*12*) and (*13*) determines $\delta_{max.}$ and which $\delta_{min.}$ depends on the value of s_1 and on whether $n>1$ or <1 (ref. (5), p. 525). If $\delta_{max.}$ or $\delta_{min.}$ is determined experimentally, ϵ can be obtained from the appropriate equation, (*12*) or (*13*). Either gives an awkward transcendental equation for ϵ since both n and β_1 are

functions of ϵ, but by using both an explicit solution can be obtained as follows :—

Dividing the two equations gives

$$\frac{\tan \frac{1}{2}\beta(s_1+\delta_1)}{\tan \frac{1}{2}\beta(s_1+\delta_2)} = n^2 = \left(\frac{\epsilon\beta}{\beta_1}\right)^2.$$

On multiplying the two equations we get

$$\beta_1 = \frac{2}{s_1}\tan^{-1}\{\tan \tfrac{1}{2}\beta(s_1+\delta_1).\tan \tfrac{1}{2}\beta\,(s_1+\delta_2)\}^{\frac{1}{2}},$$

and hence

$$\epsilon = \frac{2}{\beta s_1}\tan^{-1}\{\tan \tfrac{1}{2}\beta(s_1+\delta_1).\tan \tfrac{1}{2}\beta(s_1+\delta_2)\}^{\frac{1}{2}}.\left[\frac{\tan \frac{1}{2}\beta(s_1+\delta_1)}{\tan \frac{1}{2}\beta(s_1+\delta_2)}\right]^{\frac{1}{2}}, \quad (3)$$

an explicit value for ϵ in terms of measurable quantities λ, s_1, δ_1, δ_2.

For the experiments a cylinder of paraffin wax was moulded, 4 cm. long and a sliding fit for the tube. It was inserted by removing the piston, which was afterwards replaced. The slot along the top of the tube allows the wax to be moved and its position to be read by a scale. The wax was moved in steps of 1 cm., the resonance position of the piston determined for each step, and a "shift curve" obtained. When the approximate positions for maximum and minimum piston movement were located, smaller steps were taken and the turning values of piston position determined accurately. Before and after these measurements the resonance position of the piston was found for the empty tube, and the wave-length of the oscillator determined.

Using formula (3) above, a number of determinations of ϵ were made for readings between 15 and 21 cm. wave-length. The mean value obtained for paraffin wax was $2{\cdot}22 \pm 0{\cdot}01$.

Method C.

It is some inconvenience to have to remove the dielectric after each measurement, and it would be simpler if the difference only between δ_1 and δ_2 could be used, this being obtained from the shift curve. This is made possible by eliminating δ_1 and δ_2 from (*12*), (*13*), and $\Delta = \delta_1 - \delta_2$, giving

$$\Delta = \frac{2}{\beta}\tan^{-1}\left\{\frac{1}{2}\left(n-\frac{1}{n}\right)\sin \beta_1 s_1\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

This equation is again not suitable for calculation, but in the particular case for which $n=1$, Δ becomes zero, *i. e.*, the shift curve degenerates to a straight line. The wave-length for which $n=1$ is given by

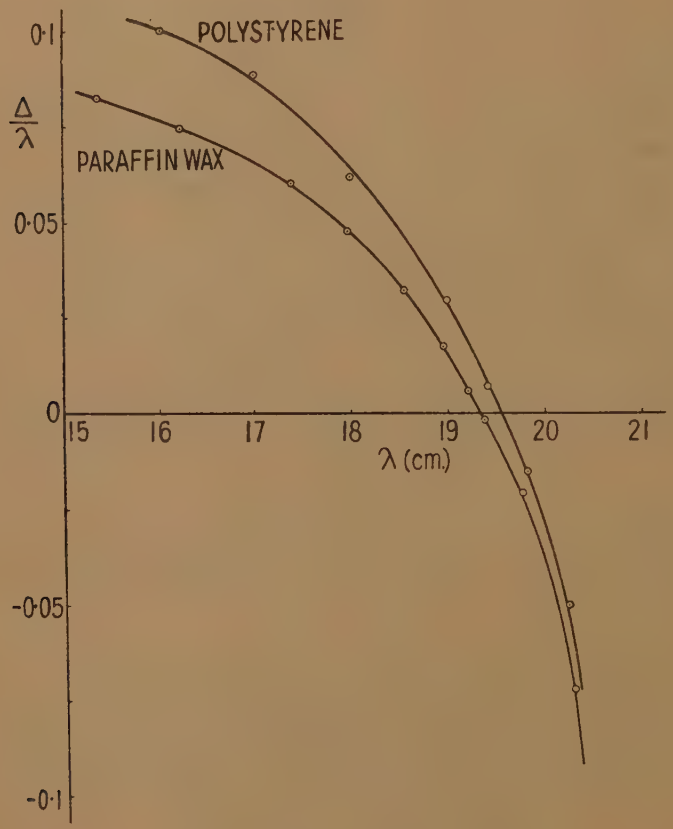
$$\lambda = \frac{2\pi}{k}\sqrt{\frac{\epsilon}{\epsilon+1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and if this is found an easy means of determining ϵ is available.

A cylinder of dielectric is used and shift curves found as in *B*. In this case, however, Δ can be found directly from the shift curve, and measurements of piston position in the empty tube are not required.

The points on the curve marked " paraffin wax " in fig. 3 are experimental values of Δ/λ plotted against λ, and are obtained from the previous measurements described under method *B*. The value of λ for which

Fig. 3.



$\Delta=0$ can be determined by interpolation. In this case $\Delta=0$ at $\lambda=$ 19·30 cm., which, using equation (*24*), gives $\epsilon=2{\cdot}21$. This agrees well with the previous value. For an actual determination only a few points close to $\Delta=0$ are required, and a linear interpolation is sufficient.

Of the recently developed plastics, one, polystyrene [10], is of great value for high-frequency insulation on account of its low power factor *.

* Most of the high-frequency insulation in the apparatus described here is of polystyrene.

It was thought of interest to measure its dielectric constant at these frequencies. From a 3-cm. slab of the material supplied under the trade name of " Trolitul " a cylinder was cut and machined to the 7-in. diameter of the tube and was used in the same way as the paraffin wax cylinder.

The values of Δ obtained are shown also in fig. 3. In this case the wave-length for which Δ is zero is 19·62 cm., which gives for ϵ the value 2·47.

It should be noted that by this method ϵ can be found at only one wave-length, this being determined by ϵ itself and by the tube radius. If a range of wave-length is required, as in the investigation of an absorption band, method *B* will be found suitable.

The H-waves are not suitable for use with method *C*, since for H-waves n is given by

$$n=\sqrt{\frac{\epsilon-\left(\frac{\lambda}{2\pi}\right)^2}{1-\left(\frac{k'\lambda}{2\pi}\right)^2}}, \quad \ldots\ldots\ldots \quad (21)$$

and as, from this expression, n is always $>\sqrt{\epsilon}$, there is no point at which $\Delta=0$ *.

Method D.

This is similar to method *C*, but is adapted for measurements on liquid dielectrics. A short cylindrical container is made to fit inside the tube and this is filled with the dielectric to be measured. Equations are now derived for corrections to be made for the container.

King [9] has shown in his analysis of the corresponding problem in the Lecher wire system that, if the ends of the dielectric sample have thin retaining walls, each of these can be regarded as a small lumped susceptance, $-ib$, across the wires at that point. This is realizable as a small condenser across the wires, and, although the same is not possible for the wave guide, the concept of a lumped susceptance across a section of the tube is mathematically identical. King's equation corresponding to (*13*) is

$$\tan \tfrac{1}{2}\beta(s_1+\delta_1)=-\mathrm{N}b+n \tan \tfrac{1}{2}\beta_1 s_1, \quad \ldots\ldots \quad (4)$$

where N is the characteristic impedance of the line. The concept of characteristic impedance is also possible for the wave guide (see note, p. 535, of ref. (5)). Equation (4) will therefore be taken as giving a turning

---

* If method *B* is to be used for H-waves the explicit equation for ϵ corresponding to (3) can be readily shown to be

$$\epsilon=\left(\frac{k'\lambda}{2\pi}\right)^2+\left(\frac{\beta\lambda}{2\pi}\right)^2 \frac{\tan \tfrac{1}{2}\beta(s_1+\delta_1)}{\tan \tfrac{1}{2}\beta(s_1+\delta_2)}.$$

No use is made here of these waves for a reason mentioned later.

value of the piston shift for a sample of dielectric with thin retaining walls. It can be shown also that the equation corresponding similarly to (*12*) is

$$\tan \tfrac{1}{2}\beta(s_1+\delta_2)=\frac{1}{\mathrm{N}b+n\cot \tfrac{1}{2}\beta_1 s_1}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Nb can be found by measuring the maximum or minimum shift for the empty cell and putting $\epsilon=1$ in (4) or (5). As before, equations (4) and (5) are not amenable to calculation, but if we obtain shift curves for the cell filled with dielectric and from them obtain, as in *C*, the wave-length for which $\Delta=0$, then at this point, although n will not be equal to unity because of the effect of the cell walls, it will not be far different from unity. To find this difference put $\delta_1=\delta_2$ in (4) and (5), and we get

$$(\mathrm{N}b+n\cot \tfrac{1}{2}\beta_1 s_1)(-\mathrm{N}b+n\tan \tfrac{1}{2}\beta_1 s_1)=1,$$

whence, for Nb small,

$$n^2-2n\mathrm{N}b\cot \beta_1 s_1=1,$$

i. e.,

$$n \doteqdot 1+\mathrm{N}b\cot \beta_1 s_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To find Nb put $\epsilon=1$ in (4) and (5); then for δ_1' and δ_2' small

$$\mathrm{N}b=-\tfrac{1}{2}\beta\delta_1'\sec^2 \tfrac{1}{2}\beta s_1$$

and

$$\mathrm{N}b=-\tfrac{1}{2}\beta\delta_2'\operatorname{cosec}^2 \tfrac{1}{2}\beta s_1,$$

whence

$$\mathrm{N}b\cos \beta s_1=-\tfrac{1}{2}\beta(\delta_1'-\delta_2')=-\tfrac{1}{2}\beta\Delta'. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The quantity Δ' is the difference between maximum and minimum shifts produced by the empty cell, and, if it is measured at the appropriate wave-length, Nb can be determined from (7). A sufficiently accurate value of β_1 for use in (6) can be got by finding ϵ from equation (*24*), which is now only approximately true, viz.,

$$(\lambda)_{\Delta=0} \doteqdot \frac{2\pi}{k}\sqrt{\frac{\epsilon}{\epsilon+1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

n can now be found from (6), and hence ϵ from equation (*22*).

The cell used is made from celluloid sheet in the form of a cylinder 4 cm. long and a sliding fit in the tube, and is fitted with a small filling plug. It was filled with the liquid paraffin used in *A*, and from readings taken with it a curve like those of fig. 3 was constructed.

From this curve for $\Delta=0$ the value $\lambda=19{\cdot}29$ cm. was obtained. At this wave-length a shift curve was drawn for the empty cell which gave for Δ' a value of 0·21 cm. The first approximation to ϵ from (8) is 2·21. Hence, using equations (6) and (7), we get $n=0{\cdot}997$. Using this value of n and putting $\epsilon=2{\cdot}21-\eta$ in (*22*) we get $\eta=0{\cdot}02$. Hence $\epsilon=2{\cdot}19$, which agrees with the value obtained in *A*.

It is to be noted that if the dielectric constant of the liquid is already known approximately (*e. g.* its low-frequency value or its optical value) the correction to be made can be reduced considerably by choosing the

cell length such that $\cot \beta_1 s_1$ in (6) is small. If the value of this term is large the approximations are of little value. A rough calculation was made when designing the cell ; for the above the value of $\cot \beta_1 s_1$ is 0·13.

Accuracy of Results.

A. Individual readings obtained in *A* showed a maximum variation from the mean of 1·4 per cent. It is considered that a considerable part of this variation is due not to errors in measurement, but to distortion of the wave by slight deformations in the tube. The tube used in the experiments had evidently suffered previous vicissitudes, but it is unlikely that the mean values are appreciably affected.

B. Again the dispersion of the results is largely due to tube irregularities, but the probable error shown is again reasonably small.

C. On account of the form of (*24*) the error in method *C* caused by an error in $(\lambda)_{\Delta=0}$ is rather large. For $\epsilon=2$ a change of 1 per cent. in λ produces a 6 per cent. change in ϵ. From a large-scale curve it was found possible to interpolate for $(\lambda)_{\Delta=0}$ to 0·1 per cent., and if, on the basis of the preliminary measurements, we assume that wavemeter readings are correct to 0·25 per cent., the possible error in ϵ is of the order of 2 per cent. The consistency of results suggests that the actual error is much less than this.

D. The same remarks apply as to method *C*. Also if the length of the cell is suitably chosen the errors introduced by the approximations made are small, and comparison of the results obtained with that of *A* shows a satisfactory agreement.

The method is of little value for liquids of high dielectric constant such as water and polar mixtures. The variation of ϵ with $\frac{\epsilon}{\epsilon+1}$ then becomes large, and large errors in ϵ are caused by small errors in $(\lambda)_{\Delta=0}$.

Consideration of Results.

The results for liquid paraffin and paraffin wax are within the usual range for hydrocarbons of 2·1 to 2·3. The composition of paraffin wax being indefinite, the result obtained is of value merely in indicating the order of the dielectric constant at these frequencies. The values of $2{\cdot}191 \pm 0{\cdot}007$ and 2·19 for liquid paraffin compare with Müller's [11] value $2{\cdot}18 \pm 0{\cdot}04$ obtained by Drude's second method at 60 cm. wavelength. The optical refractive index for the Na D lines was measured using a spectrometer and a hollow prism filled with the liquid. The dielectric constant is the square of the refractive index and was here found to be 2·20, exactly Müller's value for the Na lines. Liquid paraffin being non-polar, no dispersion should occur, but the difference between the values for the two frequencies is slight and, being within the estimated error, cannot be said to be significant.

Published values for the dielectric constant of polystyrene do not show entire agreement, the variations being due, perhaps, to differences in purity or in the state of polymerization, for the substance is not a pure chemical compound, being a mixture of chain molecules of different lengths. The value given in a descriptive pamphlet on "Distrene," the British product, is 2·3 (at $800–75.10^4$ c./sec.), that by Race and Leonard [12] 2·4 (at 10^6 c./sec.) for the German Polystyrol, and by Matheson and Goggin [10] 2·55–2·60 (at $60–2.10^7$ c./sec.). A value of 2·5 would seem to be a good average. The result of 2·47 obtained here indicates that no large dispersion occurs at frequencies corresponding to microwaves. To compare this value with that for optical frequencies a prism of polystyrene ("Trolitul") was made. The square of the index of refraction for the Na lines obtained with a spectrometer was 2·51, the same value as obtained by Matheson and Goggin. The difference between the two values is again too small to be significant.

Experiments on Wave Distribution inside the Tube.

The consistency of the foregoing measurements is a sufficient proof of the correctness of the general theory used in methods *B–D*. The following experiments are verifications of deductions made in the author's previous theoretical paper [5], and are intended to be read in conjunction with it. The experiments were done concurrently with the development of the theory, but their description was postponed to this paper to avoid repetition in describing the apparatus and technique. No attempt is made here to recapitulate the theory involved.

It was shown that for maximum and minimum bridge displacement the wave distribution is symmetrical with respect to the dielectric slab (ref. (5), p. 530, and fig. 6). By means of the probe moving along the tube through the slot the radial electric field on both sides of the dielectric slab can be measured. Figs. 4 and 5 illustrate the results of such measurements, fig. 4 showing the slab in the position for minimum piston displacement, and fig. 5 the slab in the position for maximum piston displacement. The abscissæ are distances along the tube from an arbitrary origin, the ordinates are microammeter readings proportional to the square of the field. The symmetrical nature of the distribution is well evidenced. As in the theory, for minimum displacement the first node is at a distance between $\frac{1}{4}\lambda_t$ and $\frac{1}{2}\lambda_t$ from the face of the slab, and for maximum displacement between 0 and $\frac{1}{4}\lambda_t$ (ref. (5), p. 529).

The shift curves are of the form previously calculated, and show the increase of skewness with increase of n (ref. (5), p. 528). The values of n in the series of readings for paraffin wax decrease from 1·26 to 0·9 with increase of λ, passing through the value 1 at the point where $\Delta=0$. Similar readings were taken for the H_1-wave. As there are circulatory currents

in the tube walls for H-waves the longitudinal slot will produce some disturbance of the wave. While this was so, and these readings were

Fig. 4.

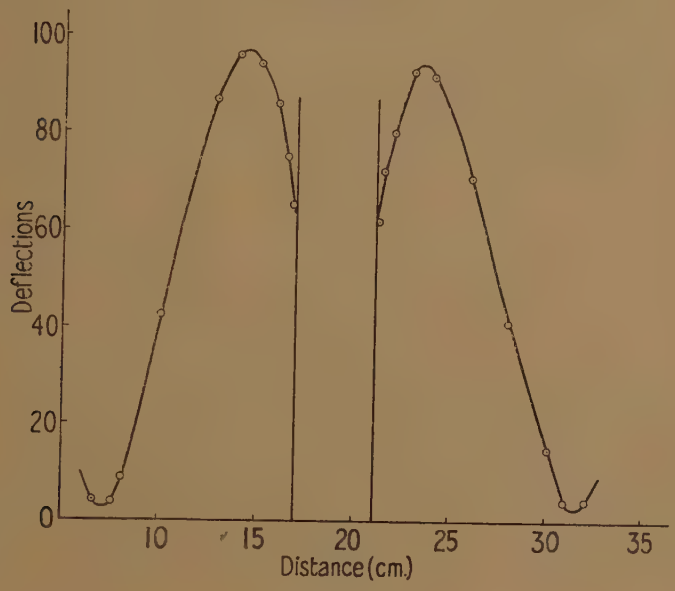
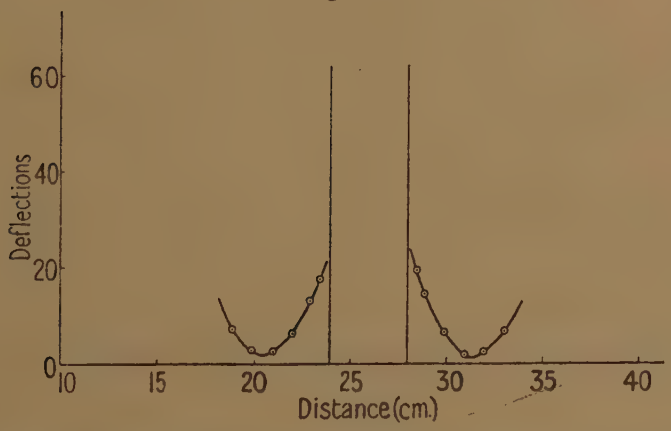


Fig. 5.

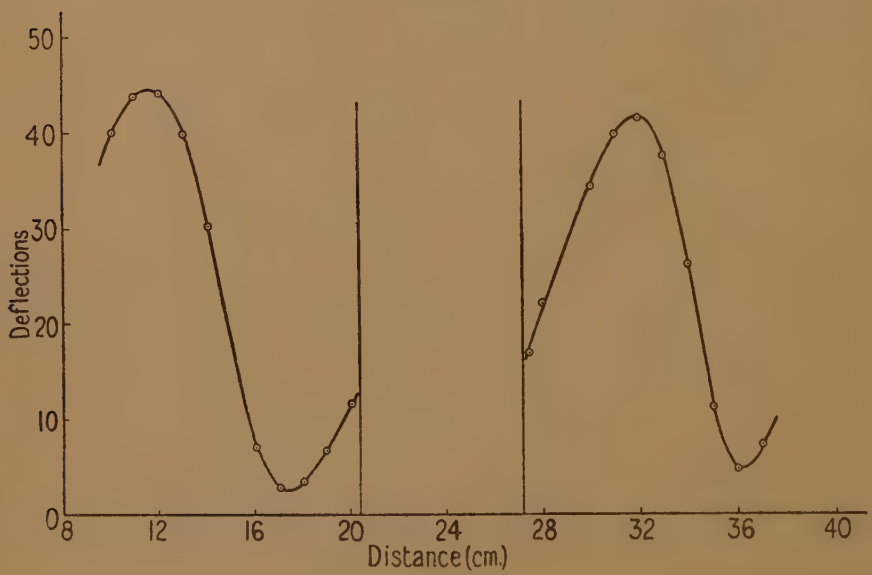


slightly erratic, the steady increase of n with λ as required by theory was clearly indicated.

To demonstrate the particular case for which the thickness of the dielectric slab is a multiple of $\frac{1}{2}\lambda_{11}$ (ref. (5), p. 530, and fig. 7), another

cylinder of paraffin wax was made, of length 6·75 cm. Taking for the dielectric constant the value 2·22 found in *B*, an oscillation of free space wave-length 17·39 cm. will give an E_0-wave in the tube such that $\frac{1}{2}\lambda_{t1}$ is equal to the length of the cylinder. The oscillator was set as nearly as possible to this wave-length, and it was verified that the shift curve then obtained was substantially straight. By means of the moving probe the field on both sides of the wax was plotted, and this is shown in fig. 6. The figure demonstrates that the parts of the wave on the two sides of the dielectric are complementary, the external effect of the

Fig. 6.



dielectric being to divide the original wave into two parts and separate them. The amplitudes are not exactly equal, as they should be, but are roughly so.

Summary.

The properties of the wave guide are used to provide methods of measurement of dielectric constant for microwaves, and experimental results are obtained at wave-lengths from 15 to 25 cm. (2000–1200 Mc./sec.). Four methods are described, of which the first and last are suitable for liquids and the other two for solids. As wave guide a hollow copper tube is used which in the first method is filled with the dielectric ; in the other methods a short cylinder only of the dielectric is required. The

theory has been considered in an earlier paper. Experimental results obtained are as follows :—

Substance.	Dielectric constant by method :			
	A.	*B.*	*C.*	*D.*
Liquid paraffin	2·191±0·007	..	..	2·19
Paraffin wax	..	2·22±0·01	2·21	..
Polystyrene	..	..	2·47	..

The paper concludes with some experiments to verify various points of theory.

Acknowledgment.

The author wishes to thank Professor Taylor Jones for helpful advice and for the provision of facilities in the Department of Natural Philosophy of Glasgow University, where the work was done. To Dr. John Thomson thanks are again due for much valuable guidance, and also to Dr. R. W. Sloane, who provided the germ of the idea from which this work has developed.

References.

(1) P. Drude, *Z. Phys. Chem.* xxiii. p. 267 (1897).
(2) F. H. Drake, G. W. Pierce, and M. T. Dow, Phys. Rev. xxxv. p. 613 (1930).
(3) W. I. Kalinin, *Phys. Z. Sowjet.* x. p. 257 (1936).
(4) See, for example, G. C. Southworth, Bell Syst. Tech. J. xv. p. 284 (1936); W. L. Barrow, Proc. Inst. Radio Engrs. xxiv. p. 1298 (1936).
(5) H. R. L. Lamont, Phil. Mag. xxix. p. 521 (1940).
(6) E. Kašpar, *Ann. Phys., Lpz.* xxxii. p. 353 (1938).
(7) A. Hund, Sci. Pap. U.S. Bur. Stand. xix. no. 491 (1924).
(8) L. S. Nergaard, Proc. Inst. Radio Engrs. xxiv. p. 1207 (1936).
(9) R. King, Rev. Sci. Instrum. viii. p. 201 (1937).
(10) See, for example, L. A. Matheson and W. C. Goggin, Industr. Engng. Chem. xxxi. p. 334 (1939) ; G. F. Bloomfield, T. & R. Bull. xiv. p. 635 (1939).
(11) W. Müller, *Ann. Phys., Lpz.* xxiv. p. 99 (1935).
(12) H. H. Race and S. C. Leonard, Elect. Engng., N.Y. lv. p. 1347 (1936).

II. *On the Integral Equations of Continuous Dynamical Systems.*

By W. H. Ingram †.

[Received July 7, 1939.]

Introduction.

The system of differential equations and boundary conditions

$$\left.\begin{aligned} \frac{\partial y_i}{\partial x} &= \Sigma a_{ik}(x)\frac{\partial y_k}{\partial t} + \Sigma b_{ik}(x)y_k, \\ \Sigma\{\alpha_{ik}y_k(a)+\beta_{ik}y_k(b)\} &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$i, j=1, 2, \ldots n$, $a \leqslant x \leqslant b$, includes as special cases the differential equations and boundary conditions for the non-uniform dissipative vibrating string in the form

$$\left.\begin{aligned} \frac{\partial y}{\partial x} &= \mathrm{T}^{-1}(x)\frac{\partial u}{\partial t}, \\ \frac{\partial u}{\partial x} &= m(x)\frac{\partial y}{\partial t} + r(x)y, \\ y(a) &= y(b) = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the differential equations ‡ and boundary conditions for the electrical transmission line

$$\left.\begin{aligned} -\frac{\partial v}{\partial x} &= l(x)\frac{\partial i}{\partial t} + r(x)i, \\ -\frac{\partial i}{\partial x} &= c(x)\frac{\partial v}{\partial t} + s(x)v, \\ v(a) - r_a i(a) &= 0, \\ v(b) - r_b i(b) &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the cases of the rotating rope, vibrating bar, and certain one-dimensional diffusion problems.

The substitution $y_i = e^{\lambda t}u_i(x)$ into (1) gives

$$\left.\begin{aligned} \mathrm{Y}'(x) &= [\mathrm{A}(x)\lambda + \mathrm{B}(x)]\mathrm{Y}(x), \\ \mathrm{W}_a\mathrm{Y}(a) + \mathrm{W}_b\mathrm{Y}(b) &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

---

† Communicated by the Author.

‡ Heaviside, ' Electrical Papers,' vol. 1, p. 142.

where $Y(x)$ is a columnar matrix having elements $u_1(x)$, $u_2(x)$, . . . $u_n(x)$, and where $A(x)$, $B(x)$, W_a, and W_b are square nth order matrices of the variable and constant coefficients in (1).

The system (4) with the restriction that it have a Green's function has been studied by Birkhoff and Langer † and with the restriction that it be self-adjoint by Bliss ‡ and Temple §. In either case it is equivalent to the integral equation

$$Y(x)=\lambda\int_a^b K(x,\xi)Y(\xi)\,d\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $K(x,\xi)=G(x,\xi)A(\xi)$, $G(x,\xi)$ being the $n\times n$ matrix of Green's functions.

Some special cases of the system (1) have no Green's matrix. The condition that one exist is that the steady-state system

$$\left.\begin{aligned} Y'(x)&=B(x)Y(x),\\ W_aY(a)+W_bY(b)&=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

be incompatible, *i. e.*, have no solution other than the solution $Y\equiv 0$. Th s (2) has no Green's matrix, but the differential equations and boundary conditions in their more usual form

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)&=m(x)\frac{\partial^2 y}{\partial t^2}+r(x)\frac{\partial y}{\partial t},\\ y(a)=y(b)&=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

has a Green's function, since no not-identically-zero solution of $T\,dy/dx=$ const., $T(x)$ being always positive, can satisfy both boundary conditions at the same time. Thus (1), if it does not imply (5), may imply a similar integral equation indirectly.

Any λ satisfying (5) is real when the Green's matrix is symmetrical. In the cases of the electrical transmission line (where there is an asymmetry in the Green's matrix) and of the damped vibrating string, it would seem reasonable to conclude, from physical experience, that any λ of (5) would then be complex. However, $G(x,\xi)$ and $B(\xi)$ being real, λ cannot be complex without Y being complex. If the elements of $Y(x)$, namely u_1, u_2, . . ., u_n, are complex, say v_1+iw_1, v_2+iw_2, . . ., v_n+iw_n, and $\lambda=\alpha+i\omega$, then (5) becomes

$$\left.\begin{aligned} V(x)&=\alpha\textstyle\int KV\,d\xi-\omega\int KW\,d\xi,\\ W(x)&=\omega\textstyle\int KV\,d\xi+\alpha\int KW\,d\xi.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

† Birkhoff & Langer, Proc. Am. Acad. Arts & Sciences, 58–2 (1923).
‡ G. A. Bliss, Trans. Am. Math. Soc. xxviii. p. 561 (1926).
§ G. Temple, Proc. Lond. Math. Soc. xxix. p. 257 (1928).

But if $e^{\lambda t}(v_k(x)+iw_k(x))$ satisfies (1), then $v_k(x)$ and $w_k(x)$ do separately at time $t=0$, and hence one is a multiple of the other; *i. e.*, $W(x)=cV(x)$, where c is a scalar multiplier, which substituted into (8) gives

$$\alpha-c\omega=(\omega+c\alpha)/c.$$

From this $c=\sqrt{-1}$ in contradiction to Y being complex. Hence the solutions of (5) have only limited application to the system (1).

In the case of damped vibrating systems satisfying (1), the substitution

$$y_i=e^{\alpha t}\cos(\omega t-\beta)u_i(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

results in a system of linear homogeneous integral equations of the second kind in two parameters, α and ω, with the same Green's functions as in (5). The derivation and solution of this system, and a determination of the conditions of admissibility of the substitution (9), are the ultimate concerns of the present research. Our immediate concern in the present paper is with the special cases of the vibrating string and the electrical transmission line, cases which have interest in themselves and where the treatment suggests fairly plainly the more general theory.

The integral equations are solved through the systems of linear equations implied by them, it being assumed that, conversely, the finite systems imply the integral equations. This return is strictly justifiable in those cases where (1) is derivable as the limiting form for a discrete physical system for which a Green's function exists and where the system of linear equations implied by the integral equations has application to the discrete physical system.

In § 1 a connexion is established between a method of solution recently † evolved for discrete dynamical systems and a method for the numerical solution of integral equations. In § 2 a method for the numerical solution of the system of integral equations in the two parameters mentioned is given in the cases of the dissipative vibrating string and the electrical transmission line. The new method for the numerical solution of integral equations here solves for the first time by means of integralequations characteristic value problems in cases where the characteristic values are complex.

Notation.

A columnar matrix of elements $u_{1i}, u_{2i}, \ldots, u_{ni}$ will be designated U_i.

A line-matrix of elements $v_{j1}, v_{j2}, \ldots, v_{jn}$ will be designated V_j.

An $n\times n$ square matrix will be designated $[w]$ or (w_{ij}).

The element of a matrix (a_{ij}) belonging to the ith row and jth column is written a_{ij}.

† Duncan & Collar, Phil. Mag. xix. p. 203 (1935).

On occasion, but not consistently, a columnar matrix is designated with two vertical dots on the right and a line-matrix with two horizontal dots on the left as a memory aid to the reader; thus, $\cdot\cdot VU$: is a scalar, $U : \cdot\cdot V$ is an $n \times n$ matrix.

The $n \times n$ matrix with 1's on the principal diagonal and 0's elsewhere will be designated I_n.

A columnar matrix of an enumerably infinite many elements $\chi_{1i}, \chi_{2i}, \ldots$ is written $\chi_i(x)$; similarly a line-matrix of an enumerably infinite many elements $\eta_{j1}, \eta_{j2} \ldots$ is written $\eta_j(x)$.

The complex number $\alpha + \sqrt{-1}\omega$ will be called a *characteristic number* of the system (1) to which (9) applies.

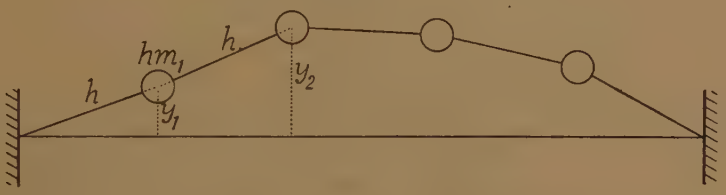
§ 1, *real* λ.

A method for the numerical solution of (5) when λ is real may be arrived at by a generalization of algebraic methods for the numerical solution of the finite system

$$u_i = \lambda \Sigma h G_{ij} A_j u_j \quad \ldots\ldots\ldots \quad (10)$$

implied by (5).

Fig. 1.



Alternatively, any *à priori* discrete system whose limiting form is governed by the system (1) may be taken as a heuristic starting point. For example, the vibrating string may be taken as the limiting form of the discrete system of fig. 1. The differential equations are

$$\left.\begin{aligned} hm_1\ddot{y}_1 &= T\left(\frac{y_2 - y_1}{h} - \frac{y_1}{h}\right) + hf_1, \\ hm_2\ddot{y}_2 &= T\left(\frac{y_3 - y_2}{h} - \frac{y_2 - y_1}{h}\right) + hf_2, \\ &\cdots\cdots\cdots \\ hm_n\ddot{y}_n &= T\left(\frac{-y_n}{h} - \frac{y_n - y_{n-1}}{h}\right) + hf_n, \end{aligned}\right\}$$

where f_i is the external force per unit length of string at the ith particle, m the mass per unit length of the ith particle, and T the uniform tension of the string. It is easily verified † that for a system of variable external forces the behaviour of the system of fig. 1 is governed by the equation

$$y_i = \Sigma h G_{ij}(f_j - m_j \ddot{y}_j), \quad \ldots\ldots \quad (10\,a)$$

where G_{ij} gives the displacement at the ith particle for a unit constant external transverse force acting on the jth particle against the transverse component there of the tension T.

The system of n total differential equations for the discrete system of fig. 1 implies, when $h \to 0$, $n \to \infty$, the system (7) with $r=0$, and the latter in turn implies the integral equation

$$y(x) = \int_a^b G(x, \xi)[f(\xi) - m(\xi)\ddot{y}(\xi)]\,d\xi, \quad \ldots\ldots \quad (10\,b)$$

which is thus implied by (10 a).

Equation (10) may be written

$$U = \lambda[hK]U, \quad \ldots\ldots\ldots \quad (11)$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}, \quad [hK] = \begin{bmatrix} hK_{11} hK_{12} \ldots hK_{1n} \\ hK_{21} hK_{22} \ldots \\ \cdot \\ \cdot \\ \cdot \\ hK_{n1} \qquad\qquad hK_{nn} \end{bmatrix}, \quad \ldots \quad (12)$$

where $K_{ij} = G_{ij} m_j$. From a fundamental theorem ‡

$$[hK] = (u_{ij}) \Lambda (u_{kl})^{-1}, \quad \ldots\ldots\ldots \quad (13)$$

$$[hK]^s = (u_{ij}) \Lambda^s (v_{kl}), \quad \ldots\ldots\ldots \quad (14)$$

where

$$\Lambda = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 & \ldots & \\ 0 & \lambda_2^{-1} & 0 & \ldots & \\ 0 & 0 & \lambda_3^{-1} & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ & & & & \lambda_n^{-1} \end{bmatrix}$$

and where $(v_{ij}) = (u_{kl})^{-1}$ is the matrix reciprocal to the modal matrix (u_{kl}).

† *Cf.* Frank & von Mises, ' Differential und Integralgleichungen,' 2nd edition, p. 497.

‡ Frazer, Duncan & Collar, ' Elementary Matrices,' p. 66.

Since $(v_{ij})(u_{kl})=\mathrm{I}_n$, it follows that

$$\Sigma v_{ij}u_{jk}=\left\{\begin{matrix}0, & k\neq i,\\ 1, & k=i,\end{matrix}\right\}, \quad \ldots\ldots\ldots \quad (15)$$

i. e., the columnar matrix U_k of elements $u_{1k}, u_{2k}, \ldots, u_{nk}$ corresponding to $\lambda=\lambda_k$ and the line-matrix V_l of elements $v_{l1}, v_{l2}, \ldots, v_{ln}$ corresponding to $\lambda=\lambda_l$ are biorthogonal.

Equation (14) can be expanded to

$$[h\mathrm{K}]^s=\sum_{r=1}^{n}\left(\frac{1}{\lambda_r}\right)^s\begin{bmatrix}u_{1r}v_{r1} & u_{1r}v_{r2}\ldots & u_{1r}v_{rn}\\ u_{2r}v_{r1} & u_{2r}v_{r2} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ u_{nr}v_{r1} & & u_{nr}v_{rn}\end{bmatrix}, \quad \ldots \quad (16)$$

a series whose successive terms drop off rapidly in importance when $|\lambda_1|<|\lambda_2|<\ldots<|\lambda_n|$ and when s is large.

Pre-multiplication by the arbitrary line-matrix E_i of elements $(\eta_{i1}, \eta_{i2}, \ldots \eta_{in})$ and post-multiplication by the columnar matrix Ξ_j of elements $(\chi_{1j}, \chi_{2j}, \ldots, \chi_{nj})$ gives, respectively,

$$\cdot\cdot\ \mathrm{E}_i[h\mathrm{K}]^\sigma=\Sigma_r\lambda_r^{-\sigma}a_{ir}\mathrm{V}_r, \quad \ldots\ldots\ldots \quad (17)$$

$$[h\mathrm{K}]^s\Xi_j:=\Sigma_r\lambda_r^{-s}b_{rj}\mathrm{U}_r, \quad \ldots\ldots\ldots \quad (18)$$

where

$$a_{ir}=\Sigma\eta_{ij}u_{jr}, \quad \ldots\ldots\ldots \quad (19)$$

$$b_{rj}=\Sigma v_{ri}\chi_{ij}. \quad \ldots\ldots\ldots \quad (20)$$

Computation of the series $[h\mathrm{K}]\Xi$, $[h\mathrm{K}]^2\Xi, \ldots$ and letting n become infinite yields the induction

$$\mathfrak{P}_j^s(x)=\int_a^b\ldots\int_a^b\mathrm{K}(x, t_s)\mathrm{K}(t_s, t_{s-1})\ldots\mathrm{K}(t_2, t_1)\chi_j(t_1)\,dt_1\,dt_2\ldots dt_s, \quad (21)$$

defining $\mathfrak{P}_j^s(x)$ as our limiting form for the left-hand side of (18). A similar induction for the left-hand side of (17) indicates the limiting form

$$\mathfrak{Q}_i^\sigma(x)=\int_a^b\ldots\int_a^b\eta_i(t_1)\mathrm{K}(t_1, t_2)\mathrm{K}(t_2, t_3)\ldots\mathrm{K}(t_\sigma, x)\,dt_1\,dt_2\ldots dt_\sigma, \quad (22)$$

so that (17) and (18) become, respectively,

$$\mathfrak{Q}_i^\sigma(x)=\Sigma_r\lambda_r^{-\sigma}\mathrm{V}_r(x)\int_a^b\eta_i(\xi)\mathrm{U}_r(\xi)\,d\xi, \quad \ldots\ldots \quad (23)$$

$$\mathfrak{P}_j^s(x)=\Sigma_r\lambda_r^{-s}\mathrm{U}_r(x)\int_a^b\mathrm{V}_r(\xi)\chi_j(\xi)\,d\xi. \quad \ldots\ldots \quad (24)$$

When the absolute value of λ_1 is smaller than the absolute value of any other λ, the right-hand side of (24) becomes

$$\mathfrak{P}_1^s(x)=\mathrm{U}_1(x)\times\text{const.} \quad \ldots\ldots\ldots \quad (25)$$

approximately, for s large enough, provided

$$\int_a^b V_1(\xi)\chi_1(\xi)\,d\xi \neq 0,$$

where, by (14), V_1 is the solution of

$$V(x)=\lambda\int_a^b V(y)K(y,x)\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

for $\lambda=\lambda_1$. Under these conditions the first characteristic number is obtained as the ratio

$$\lambda_1=\mathfrak{P}_1^s(x)/\mathfrak{P}_1^{s+1}(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

an appropriate generating function being $\chi_1=$const. and x being any fixed value on the range $a<x<b$.

In the case of the uniform dissipationless vibrating string, formula (27), with $\chi_1=$const., gives results correct to 0·1 of 1 per cent. with $s=3$. Possibly better results may be obtained with numerators and denominators averaged in various ways †.

The approximate formulas (25) and (27) have been obtained before ‡ on the assumption that $\chi_1(x)$ can be expressed in the form

$$\chi_1(x)=\alpha_1 U_1(x)+\alpha_2 U_2(x)+\ldots,$$

where at least the numerical coefficient $\alpha_1\neq 0$ and where the U's form an orthogonal system satisfying a homogeneous integral equation of the form of (5) for different λ's. The formulas were obtained for the case $\chi(x)=1$, $K(x,t)=K(t,x)$, by E. Schmidt §.

The second characteristic function $U_2(x)$ is given approximately by $\mathfrak{P}_2^s(x)$, s suitably large, when $\chi_2(x)$ is orthogonal to $V_1(x)$, and then λ_2 is given approximately by $\lambda_2=\mathfrak{P}_2^s(x)/\mathfrak{P}_2^{s+1}(x)$. In the case of the uniform dissipationless vibrating string the generating function $\chi_2(x)=2x-(a+b)$ gives the second characteristic function and number accurate to about 0·1 of 1 per cent. with $s=4$. This generating function happens to be exactly orthogonal to $V_1(x)$, and in consequence the computation of $U_2(x)$ is no more difficult than the computation of $U_1(x)$. In general, nevertheless, a generating function orthogonal to $\mathfrak{D}_1^s(x)$, s fixed, will not be orthogonal to V_1, so that in the computation of any characteristic function above the first the Fourier components of all lower characteristic functions have, as a rule, to be subtracted out of $\mathfrak{P}_i^\sigma(x)$, $\lambda\leqslant s$, at one or more stages including the final one s in the computation of $\mathfrak{P}_i^s(x)$.

† *Cf.* Grammel, 'Ergebnisse der Exakten Naturwissenschaften,' Bd. 1, p. 93.
‡ Hohenemser, 'Ergebnisse der Mathematik,' Band 1, Heft 4, p. 16.
§ E. Schmidt, *Math. Annalen*, lxiii. p. 433 (1907).

Parenthetic on the Suppression of Lower Modes.

It may be remarked that when $y_1, y_2 \ldots y_n$ of equation (1) can be written in the form

$$Y := \Sigma a_i \cos(\omega_i t - \beta_i) U_i(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

premultiplication by the line-matrix V_1 gives

$$\cdot\cdot V_1 Y := a_1 \cos(\omega_1 t - \beta_1) U_1(x)$$

in view of (15). Hence writing $\cdot\cdot V_1 Y := 0$ imposes the value 0 on a_1, *i. e.*, suppresses the first mode. Solving the system (17) for V_1 gives

$$\cdot\cdot V_1 = \lambda_1^{\sigma} \Sigma E_i [hK]^{\sigma} D_1 / D, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where

$$D = \begin{vmatrix} \Sigma\eta_{1j}u_{j1} & \Sigma\eta_{1j}u_{j2} \ldots & \Sigma\eta_{1j}u_{jn} \\ \Sigma\eta_{2j}u_{j1} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \Sigma\eta_{nj}u_{j1} \ldots & & \Sigma\eta_{nj}u_{jn} \end{vmatrix}$$

and where D_{i1} is the minor of the element in the ith row of the first column of D. When the arbitrary vectors $(\eta_{1j}, \eta_{2j}, \ldots, \eta_{nj})$, $j = 1, 2, \ldots n$, are taken normal and orthogonal to the vectors U_j and U_k respectively, then $D = I_n$ and the condition $V_1 Y = 0$ takes the form $\cdot\cdot E_1 [hK]^{\sigma} Y := 0$.

In the finite case the second modal function can be obtained from the system

$$\left.\begin{aligned} \Sigma Q_{iv}^{s} u_v &= ,0 \\ u_i &= \lambda \Sigma h K_{ij} u_j, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where

$$\Sigma \cdots \Sigma h^{s} \eta_{1i} K_{ij} K_{jk} \cdots K_{sv} = Q_{1v}^{s}, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

by first eliminating one of the variables to get the reduced system $U = \lambda[hK^*]U$ having the same general form as (11) but with U a columnar matrix of $n-1$ elements and $K_{ij}^{*} = K_{ij} - K_{il} Q_{ij}^{s} / Q_{1l}^{s}$ being an $(n-1) \times (n-1)$ matrix, and l fixed. Accordingly one might expect that the new kernel $K^*(x, y) = K(x, y) - K(x, t) \Omega_1^{s}(y) / \Omega_1^{s}(t)$, t anywhere on the range (a, b), substituted into the right-hand side of (21), would give $\mathfrak{P}_2(x)$; its failure to do so emphasizes the generally tentative nature of our limiting process.

Alternatively, the second modal function may be obtained as the solution of the simultaneous system

$$\left.\begin{aligned} \int_a^b \Omega_1^{s}(t) u(t)\, dt &= 0, \\ u &= \lambda \int_a^b K(x, t) u(t)\, dt, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32\,a, b)$$

which is our limiting form of (30).

Any solution $u(x)$ of (32 b) if orthogonal to $\mathfrak{D}_1^s(x)$ is also orthogonal to $\mathfrak{D}_1^{s+1}(x)$, and conversely, as is proved by multiplying both sides of (32 b) by $\mathfrak{D}_1^s(x)$ and integrating. In particular any solution orthogonal to $\eta_1(x)$ is orthogonal to $\mathfrak{D}_1^s(x)$, any s. But a choice of χ_2 orthogonal to V_1 ensures that (32 a) will be satisfied at least approximately, s large, for clearly

$$\int \mathfrak{D}_1^s(t)\mathfrak{P}_2^\sigma(t)\,dt = \int \mathfrak{D}_1^{s+\sigma}(t)\chi_2(t)\,dt = \int V_1(t)\chi_2(t)\,dt, \text{ approx.}$$

Since $V_1(x)$ is known only approximately in most cases, the best that can be done is to take $\chi_2(x)$ orthogonal to $\mathfrak{D}_1^\sigma(x)$ for σ as large as practicable. When this is done $\chi_2(x)$ will not be also orthogonal to $\mathfrak{D}_1^{\sigma+1}(x)$, or to $V_1(x)$, necessarily, for, on multiplying both sides of (23) by $\chi_2(x)$ and integrating, there results

$$\int \mathfrak{D}_1^\sigma \chi_2\,dx = \lambda_1^{-\sigma}A_{11}B_{12} + \lambda_2^{-\sigma}A_{12}B_{22} + \lambda_3^{-\sigma}A_{13}B_{32} + \ldots,$$

$$\int \mathfrak{D}_1^{\sigma+1} \chi_2\,dx = \lambda_1^{-\sigma-1}A_{11}B_{12} + \lambda_2^{-\sigma-1}A_{12}B_{22} + \lambda_3^{-\sigma-1}A_{13}B_{32} + \ldots,$$

expressions which cannot both be zero except when the coefficients $A_{1r}B_{r2}$ are all zero, the A's and B's being the integral forms of (19) and (20). Hence even if the first term on the right of (24) is small relative to the second for $s=\sigma$ it will be λ_2/λ_1 times greater in relative importance to the second term for $s=\sigma+1$, in general. The exception occurs in the case of the uniform vibrating string and uniform transmission line, where A_{1r} is zero for r even and B_{r2} is zero for r odd.

§ 2, *complex* λ.

When the ith particle of fig. 1 encounters the frictional resistance $hr_i\dot{y}_i$, the governing equations may be expressed in the form

$$y_i = \Sigma h G_{ij}(f_j - m_j\ddot{y}_j - r_j\dot{y}_j). \quad \ldots\ldots\ldots \quad (33)$$

The tentative assumption that the y's will have the form

$$y_j = e^{\alpha t}\cos(\omega t - \beta)u_j(x) \quad \ldots\ldots\ldots \quad (34)$$

subsequent to the removal of the external forces gives, on substitution, the simultaneous equations

$$\left.\begin{aligned} u_i &= \Sigma h G_{ij}\{-m_j(\alpha^2+\omega^2) - r_j\alpha\}u_j, \\ 0 &= \Sigma h G_{ij}\{2m_j\alpha\omega + r_j\omega\}u_j, \end{aligned}\right\} \quad \ldots\ldots \quad (35)$$

obtained on equating sine and cosine terms in the time. The second equation of (35) gives

$$\alpha = -\frac{1}{2}\frac{\Sigma h G_{ij} r_j u_j}{\Sigma h G_{ij} m_j u_j}, \quad \ldots\ldots\ldots \quad (36)$$

a quantity independent of i when r_j is proportional to m_j, and then

$$u_i = (\alpha^2 - \omega^2)\Sigma h G_{ij} m_j u_j. \quad \ldots\ldots\ldots \quad (37)$$

Hence the differential equation and boundary condition

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\mathrm{T}\frac{\partial y}{\partial x}\right)&=m(x)\frac{\partial^2 y}{\partial x^2}+r(x)\frac{\partial y}{\partial x} \\ y(a)&=y(b)=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

has a solution of the form (34) when $r(x)/m(x)=$const., and then α, ω, and u are given by

$$u(x)=(\alpha^2-\omega^2)\int_a^b \mathrm{G}(x,\xi)m(\xi)u(\xi)\,d\xi, \quad . \quad . \quad . \quad . \quad (38)$$

$$\alpha=-\frac{1}{2}\frac{\int \mathrm{G}(x,\xi)r(\xi)u(\xi)\,d\xi}{\int \mathrm{G}(x,\xi)m(\xi)u(\xi)\,d\xi}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where (38) is solvable by the method of § 1. The restriction on the physical constants r and m is a condition on the permissibility of the substitution (34). When the condition is not satisfied there are nc characteristic functions nor characteristic numbers.

Writing $\lambda_i=\alpha_i+\sqrt{-1}\omega_i$, the general solution of (7) is

$$y=\Sigma\Re e^{\lambda_i t}\mathrm{A}_i u_i(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where the A's are arbitrary constants and where $\Re$ indicates that the real part is to be taken.

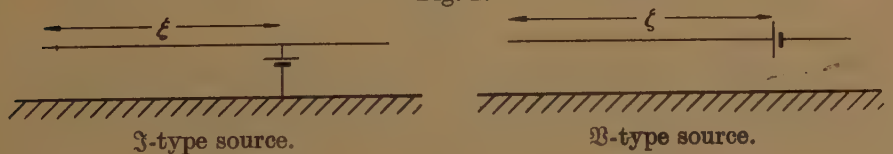
Transmission Line.

The electrical transmission line illustrates a more general case. Any solution of the system

$$\left.\begin{aligned} v&=\int_a^b \{\mathrm{G}_{v\mathfrak{J}}(x,\xi)c(\xi)\partial v(\xi,t)/\partial t+\mathrm{G}_{v\mathfrak{B}}(x,\xi)l(\xi)\partial i(\xi,t)/\partial t\}\,d\xi \\ i&=\int_a^b \{\mathrm{G}_{i\mathfrak{J}}(x,\xi)c(\xi)\partial v(\xi,t)/\partial t+\mathrm{G}_{i\mathfrak{B}}(x,\xi)l(\xi)\partial i(\xi,t)/\partial t\}\,d\xi \end{aligned}\right\} \quad . \quad (41)$$

satisfies the differential equations and boundary conditions (3). The Green's functions $\mathrm{G}_{v\mathfrak{B}}(x,\xi)$, $\mathrm{G}_{i\mathfrak{B}}(x,\xi)$ are steady-state solutions of (3) for v and i, respectively, with a unit $\mathfrak{B}$-type source at ξ, and $\mathrm{G}_{v\mathfrak{J}}(x,\zeta)$, $\mathrm{G}_{i\mathfrak{J}}(x,\zeta)$ are steady-state solutions for a unit $\mathfrak{J}$-type source at ζ.

Fig. 2.



Since the sources are unit sources

$$\mathrm{G}_{i\mathfrak{J}}(x,\xi)\Big|_{\zeta-0}^{\zeta+0}=\mathrm{G}_{v\mathfrak{B}}(x,\zeta)\Big|_{\zeta-0}^{\zeta+0}=1, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

then, for a continuous distribution of sources of both kinds on the same line, current and potential are given by

$$\left.\begin{aligned} v(x)&=\int_a^b \{G_{v\mathfrak{V}}(x,\xi)\mathfrak{V}(\xi)+G_{v\mathfrak{J}}(x,\xi)\mathfrak{J}(\xi)\}d\xi,\\ i(x)&=\int_a^b \{G_{i\mathfrak{V}}(x,\xi)\mathfrak{J}(\xi)+G_{i\mathfrak{J}}(x,\xi)\mathfrak{J}(\xi)\}d\xi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (43)$$

where $\mathfrak{V}(x)$, $\mathfrak{J}(x)$ are source intensity distributions. From (43), $v(x)$ and $i(x)$ satisfy

$$\left.\begin{aligned} -dv/dx&=r(x)i+\mathfrak{V}(x),\\ -di/dx&=s(x)v+\mathfrak{J}(x), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and the boundary conditions †.

The existence of continuous solutions of the steady-state equations

$$\left.\begin{aligned} -dv/dx&=r(x)i,\\ -di/dx&=s(x)v,\\ v(a)-r_a i(a)&=0,\\ v(b)-r_b i(b)&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

follows from the fact that the Picard process gives an absolutely and uniformly convergent series solution which is a continuous function of the initial values of the variables even when r and s are not continuous. Linearly independent functions $v_a(x)$, $v_b(x)$ therefore exist which satisfy the boundary conditions

$$\left.\begin{aligned} v_a(a)-r_a i_a(a)&=0,\\ v_b(b)-r_b i_b(b)&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where $i_a(x)$, $i_b(x)$ are connected with $v_a(x)$, $v_b(x)$ respectively by the differential equations, and are also linearly independent.

The Green's functions can be expressed in terms of these solutions as follows :—

$$\left.\begin{aligned} G_{v\mathfrak{V}}(x,t)&=\begin{cases} -v_a(x)i_b(t), & a\leqslant x<t,\\ -v_b(x)i_a(t), & t<x\leqslant b, \end{cases}\\ G_{v\mathfrak{J}}(x,t)&=\begin{cases} -v_a(x)v_b(t), & a\leqslant x<t,\\ -v_b(x)v_a(t), & t<x\leqslant b, \end{cases}\\ G_{i\mathfrak{V}}(x,t)&=\begin{cases} i_a(x)i_b(t), & a\leqslant x<t,\\ i_b(x)i_a(t), & t<x\leqslant b, \end{cases}\\ G_{i\mathfrak{J}}(x,t)&=\begin{cases} i_a(x)v_b(t), & a\leqslant x<t,\\ i_b(x)v_a(t), & t<x\leqslant b, \end{cases} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47)$$

† On source-representation, see Kneser, ' Die Integralgleichungen,' § 2, p. 5.

and from this we have the three reciprocal theorems :

$$\left.\begin{aligned} G_{\mathfrak{v}\mathfrak{z}}(x, y) &= G_{\mathfrak{v}\mathfrak{z}}(y, x), \\ G_{\mathfrak{i}\mathfrak{w}}(x, y) &= G_{\mathfrak{i}\mathfrak{w}}(y, x), \\ G_{\mathfrak{v}\mathfrak{w}}(x, y) &= -G_{\mathfrak{i}\mathfrak{z}}(y, x). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (48)$$

The tentative substitution

$$\left.\begin{aligned} v &= e^{\alpha t}\cos(\omega t-\beta)\mathrm{V}(x), \\ &= e^{\alpha t}\cos i(\omega t-\gamma)\mathrm{I}(x), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (49)$$

into (41) gives the simultaneous system

$$\left.\begin{aligned} \mathrm{V}\cos\beta &= \alpha\int\{G_{\mathfrak{v}\mathfrak{z}}c\mathrm{V}\cos\beta+G_{\mathfrak{v}\mathfrak{w}}l\mathrm{I}\cos\gamma\}d\xi+\omega\int\{G_{\mathfrak{v}\mathfrak{z}}c\mathrm{V}\sin\beta+G_{\mathfrak{v}\mathfrak{w}}l\mathrm{I}\sin\gamma\}d\xi, \\ \mathrm{I}\cos\gamma &= \alpha\int\{G_{\mathfrak{i}\mathfrak{z}}c\mathrm{V}\cos\beta+G_{\mathfrak{i}\mathfrak{w}}l\mathrm{I}\cos\gamma\}d\xi+\omega\int\{G_{\mathfrak{i}\mathfrak{z}}c\mathrm{V}\sin\beta+G_{\mathfrak{i}\mathfrak{w}}l\mathrm{I}\sin\gamma\}d\xi, \\ \mathrm{V}\sin\beta &= -\omega\int\{G_{\mathfrak{v}\mathfrak{z}}c\mathrm{V}\cos\beta+G_{\mathfrak{v}\mathfrak{w}}l\mathrm{I}\cos\gamma\}d\xi+\alpha\int\{G_{\mathfrak{v}\mathfrak{z}}c\mathrm{V}\sin\beta+G_{\mathfrak{v}\mathfrak{w}}l\mathrm{I}\sin\gamma\}d\xi, \\ \mathrm{I}\sin\gamma &= -\omega\int\{G_{\mathfrak{i}\mathfrak{z}}c\mathrm{V}\cos\beta+G_{\mathfrak{i}\mathfrak{w}}l\mathrm{I}\cos\gamma\}d\xi+\alpha\int\{G_{\mathfrak{i}\mathfrak{z}}c\mathrm{V}\sin\beta+Gl_{\mathfrak{i}\mathfrak{w}}\mathrm{I}\sin\gamma\}d\xi, \end{aligned}\right\}$$

. . . . . (50)

in the unknowns $\mathrm{V}(x)\cos\beta$, $\mathrm{I}(x)\cos\gamma$, $\mathrm{V}(x)\sin\beta$, $\mathrm{I}(x)\sin\gamma$, and implying† the algebraic system

$$\left.\begin{aligned} \mathrm{U} &= \alpha[h\mathrm{K}]\mathrm{U}+\alpha[h\mathrm{L}]\mathrm{W}+\omega[h\mathrm{K}]\mathrm{X}+\omega[h\mathrm{L}]\mathrm{Y}, \\ \mathrm{W} &= \alpha[h\mathrm{M}]\mathrm{U}+\alpha[h\mathrm{N}]\mathrm{W}+\omega[h\mathrm{M}]\mathrm{X}+\omega[h\mathrm{N}]\mathrm{Y}, \\ \mathrm{X} &= -\omega[h\mathrm{K}]\mathrm{U}-\omega[h\mathrm{L}]\mathrm{W}+\alpha[h\mathrm{K}]\mathrm{X}+\alpha[h\mathrm{L}]\mathrm{Y}, \\ \mathrm{Y} &= -\omega[h\mathrm{M}]\mathrm{U}-\omega[h\mathrm{N}]\mathrm{W}+\alpha[h\mathrm{M}]\mathrm{X}+\alpha[h\mathrm{N}]\mathrm{Y}, \end{aligned}\right\} \quad . \; . \quad (51)$$

where U, W, X, Y are columnar matrices of n elements each, $[h\mathrm{K}]$, $[h\mathrm{L}]$, $[h\mathrm{M}]$, $[h\mathrm{N}]$ are square matrices and α and ω scalars, and where

$$\begin{aligned} \mathrm{K}(x, \xi) &= G_{\mathfrak{v}\mathfrak{z}}(x, \xi)c(\xi), \\ \mathrm{L}(x, \xi) &= G_{\mathfrak{v}\mathfrak{w}}(x, \xi)l(\xi), \\ \mathrm{M}(x, \xi) &= G_{\mathfrak{i}\mathfrak{z}}(x, \xi)c(\xi), \\ \mathrm{N}(x, \xi) &= G_{\mathfrak{i}\mathfrak{w}}(x, \xi)l(\xi). \end{aligned}$$

Among other ways, (51) may be written

$$\begin{bmatrix} \mathrm{U} & \mathrm{X} \\ \mathrm{W} & \mathrm{Y} \end{bmatrix} = \begin{bmatrix} [h\mathrm{K}] & [h\mathrm{L}] \\ [h\mathrm{M}] & [h\mathrm{N}] \end{bmatrix} \begin{bmatrix} \mathrm{U} & \mathrm{X} \\ \mathrm{W} & \mathrm{Y} \end{bmatrix} \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}, \quad . \; . \quad (52)$$

† I am indebted to Prof. F. J. Murray, who has been so kind as to read the MS. of this paper, for the observation that the first two terms on the right-hand side of the first equation of (51) must be linearly dependent and the last two terms of the second equation must be linearly dependent if $\tan\beta$ and $\tan\gamma$ are not to be imaginary and ω is not to be zero. I add that these conditions on the columnar matrices of the right-hand side of (51) essentially are the conditions that A_1 of (69) shall be proportional to A_2 and that B_1 shall be proportional to B_2.

where the left-hand side is a matrix of $2n$ rows and 2 columns. The n solutions of (52) for the n characteristic numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ may be combined into the single equation

$$\begin{pmatrix} U_{ik} & X_{il} \\ W_{jk} & Y_{jl} \end{pmatrix} = \begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix} \begin{pmatrix} U_{ik} & X_{il} \\ W_{jk} & Y_{jl} \end{pmatrix} \Lambda, \quad . \quad . \quad . \quad (53)$$

where

$$\begin{pmatrix} U_{ik} & X_{il} \\ W_{jk} & Y_{jl} \end{pmatrix} = \begin{pmatrix} U_{11} \ldots U_{1n} & X_{11} \ldots X_{1n} \\ \\ U_{n1} \quad U_{nn} & X_{n1} \quad X_{nn} \\ W_{11} \quad W_{1n} & Y_{11} \quad Y_{1n} \\ \\ W_{n1} \quad W_{nn} & Y_{n1} \quad Y_{nn} \end{pmatrix}, \quad . \quad . \quad . \quad . \quad (54)$$

$$\Lambda = \begin{pmatrix} \alpha_1 & 0 & 0 & \ldots & 0 & -\omega_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & 0 & & 0 & 0 & -\omega_2 & & 0 \\ 0 & 0 & \alpha_3 & & 0 & 0 & 0 & & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & & & \alpha_n & 0 & 0 & & -\omega_n \\ \omega_1 & 0 & & & 0 & \alpha_1 & 0 & & 0 \\ 0 & \omega_2 & & & 0 & 0 & \alpha_2 & & 0 \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ 0 & 0 & & \ldots & \omega_n & & 0 & \ldots & \alpha_n \end{pmatrix}, \quad . \quad (55)$$

$$\begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix} = \begin{pmatrix} hK_{11}hK_{12} \ldots hK_{1n}hL_{11} \ldots hL_{1n} \\ hK_{21}hK_{22} \quad hK_{2n}hL_{21} \quad hL_{2n} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ hK_{n1}hK_{n2} \ldots hK_{nn}hL_{n1} \ldots hL_{nn} \\ hM_{11}hM_{12} \ldots hM_{1n}hN_{11} \ldots hN_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ hM_{n1}hM_{n2} \ldots hM_{nn}hN_{n1} \ldots hN_{nn} \end{pmatrix} \quad . \quad (56)$$

Repeated premultiplication of both sides of (53) by (56) gives

$$\begin{pmatrix} U_{ik}X_{il} \\ W_{jk}Y_{jl} \end{pmatrix} = \begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix}^s \begin{pmatrix} U_{ik}X_{jl} \\ W_{jk}Y_{jl} \end{pmatrix} \Lambda^s, \quad . \quad . \quad . \quad (57)$$

whence

$$\begin{pmatrix} U_{ik}X_{il} \\ W_{jk}Y_{jl} \end{pmatrix} \Lambda^{-s} \begin{pmatrix} U^*_{mn}W^*_{mq} \\ X^*_{pn}Y^*_{pq} \end{pmatrix} = \begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix}^s, \quad . \quad . \quad . \quad (58)$$

where

$$\begin{pmatrix} U^*_{mn}W^*_{mq} \\ X^*_{pn}Y^*_{pq} \end{pmatrix} = \begin{pmatrix} U_{ik}X_{il} \\ W_{jk}Y_{jl} \end{pmatrix}^s \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

A simple induction shows that Λ^{-s} has the form

$$\Lambda^{-s} = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & \nu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & & 0 & 0 & \nu_2 & & 0 \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ 0 & 0 & \dots & \mu_n & 0 & 0 & \dots & \nu_n \\ -\nu_1 & 0 & \dots & 0 & \mu_1 & 0 & \dots & 0 \\ 0 & -\nu_2 & & 0 & 0 & \mu_2 & & 0 \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ 0 & 0 & \dots & -\nu_n & 0 & & \dots & \mu_n \end{pmatrix}, \quad . \quad . \quad (60)$$

so that (58) may be written

$$\begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix}^s = \Sigma\mu_i \begin{pmatrix} U_{1i}U^*_{i1} & \dots & U_{1i}W^*_{in} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ W_{ni}U^*_{i1} & \dots & W_{ni}W^*_{in} \end{pmatrix} - \Sigma\nu_i \begin{pmatrix} X_{1i}U^*_{i1} & & X_{1i}W^*_{in} \\ & & \\ & & \\ Y_{ni}U^*_{i1} & & Y_{ni}W^*_{in} \end{pmatrix}$$

$$+ \Sigma\mu_i \begin{pmatrix} X_{1i}X^*_{i1} & & X_{1i}Y^*_{in} \\ & & \\ & & \\ Y_{ni}X^*_{i1} & & Y_{ni}Y^*_{in} \end{pmatrix} + \Sigma\nu_i \begin{pmatrix} U_{1i}X^*_{i1} & & U_{1i}Y^*_{in} \\ & & \\ & & \\ W_{ni}X^*_{i1} & & W_{ni}Y^*_{in} \end{pmatrix}$$

$$. \quad . \quad . \quad . \quad (61)$$

Also, equations (57) and (59) lead to

$$\begin{pmatrix} U^*W^* \\ X^*Y^* \end{pmatrix} = \Lambda^s \begin{pmatrix} U^*W^* \\ X^*Y^* \end{pmatrix} \begin{pmatrix} [hK][hL] \\ [hM][hN] \end{pmatrix}, \quad . \quad . \quad . \quad (62)$$

an equation equivalent to a system of integral equations in the variables $U^*(x)$, $W^*(x)$, $X^*(x)$, $Y^*(x)$, and (59) implies the relations of biorthogonality :

$$\int_a^b \{U_m^*(x)U_k(x)+W_m^*(x)W_k(x)\}dx=\begin{cases}1, & m=k,\\ 0, & m\neq k,\end{cases}$$

$$\int_a^b \{U_m^*(x)X_l(x)+W_m^*(x)Y_l(x)\}dx = \quad 0, \text{ any } m,\ l,$$

$$\int_a^b \{X_p^*(x)U_k(x)+Y_p^*(x)W_k(x)\}dx= \quad 0, \text{ any } p,\ k,$$

$$\int_a^b \{X_p^*(x)X_l(x)+Y_p^*(x)Y_l(x)\}dx=\begin{cases}1, & p=l,\\ 0, & p\neq l,\end{cases} \quad . \ . \ . \quad (63)$$

having application later in the computation of the modal components of a composite function.

Post-multiplication of both sides of (61) by either of the columnar matrices of $2n$ elements :

$$\begin{pmatrix}\Xi\\ 0\end{pmatrix} = \begin{bmatrix}\chi_1\\ \chi_2\\ \cdot\\ \cdot\\ \cdot\\ \chi_n\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix}, \qquad \begin{pmatrix}0\\ \Xi\end{pmatrix} = \begin{bmatrix}0\\ \cdot\\ \cdot\\ 0\\ \chi_1\\ \chi_2\\ \cdot\\ \cdot\\ \cdot\\ \chi_n\end{bmatrix} \quad . \ . \ . \ . \quad (64)$$

gives a columnar matrix whose first n elements are each a linear combination of U's and X's, and whose second n elements are each a linear combination of W's and Y's. When the first of (64) is not orthogonal to U_1^*, and when $(\alpha_1^2+\omega_1^2)^s$ is much smaller than $(\alpha_2^2+\omega_2^2)^s$, this multiplication gives, approximately,

$$\begin{pmatrix}h\mathrm{K} & h\mathrm{L}\\ h\mathrm{M} & h\mathrm{N}\end{pmatrix}^s \begin{pmatrix}\Xi\\ 0\end{pmatrix} = \begin{bmatrix}(d_1\mu_1+c_1\nu_1)U_{11}+(c_1\mu_1-d_1\nu_1)X_{11}\\ (d_1\mu_1+c_1\nu_1)U_{21}+(c_1\mu_1-d_1\nu_1)X_{21}\\ \cdot \qquad\qquad \cdot\\ \cdot \qquad\qquad \cdot\\ \cdot \qquad\qquad \cdot\\ (d_1\mu_1+c_1\nu_1)U_{n1}+(c_1\mu_1-d_1\nu_1)X_{n1}\\ (d_1\mu_1+c_1\nu_1)W_{11}+(c_1\mu_1-d_1\nu_1)Y_{11}\\ \cdot \qquad\qquad \cdot\\ \cdot \qquad\qquad \cdot\\ \cdot \qquad\qquad \cdot\\ (d_1\mu_1+c_1\nu_1)W_{n1}+(c_1\mu_1-d_1\nu_1)Y_{n_1},\end{bmatrix} \quad . \ . \quad (65)$$

where $d_1=\Sigma U_{1j}^*\chi_j$, $c_1=\Sigma X_{1j}^*\chi_j$, and a similar expression is given by the second of (64). In the limit, therefore, equation (65) gives $V_1(x)$ times a constant for the first range and $I_1(x)$ times a constant for the second range.

Multiplying out the left-hand side of (65) for the cases $s=1$, $s=2$, ... gives the induction and indicates the limiting form

$$\left.\begin{aligned} \mathfrak{P}^1(x)&=\int_a^b K(x,\xi)\chi(\xi)\,d\xi,\\ \mathfrak{S}^1(x)&=\int_a^b M(x,\xi)\chi(\xi)\,d\xi,\\ \mathfrak{P}^s(x)&=\int_a^b K(x,\xi)\mathfrak{P}^{s-1}(\xi)\,d\xi+\int_a^b L(x,\xi)\mathfrak{S}^{s-1}(\xi)\,d\xi\\ \mathfrak{S}^s(x)&=\int_a^b M(x,\xi)\mathfrak{P}^{s-1}(\xi)\,d\xi+\int_a^b N(x,\xi)\mathfrak{S}^{s-1}(\xi)d\xi, \end{aligned}\right\} \quad . \quad . \quad . \quad (66)$$

$s>1$, where $\mathfrak{P}^s(x)$ is an approximation to $V_1(x)$ and $\mathfrak{S}^s(x)$ is an approximation to $I_1(x)$ for $\chi=\chi_1=$const. or for $\chi=\chi_1$ any function not orthogonal to the functions $U_1^*(x)$, $X_1^*(x)$.

Writing $X=U\tan\beta$, $Y=W\tan\gamma$ in (65) it is found that $\mathfrak{P}^s(x)=c_1^{-1}U(x)$, $\mathfrak{S}^s(x)=c_2^{-1}W(x)$, where c_1 and c_2 are constants depending upon β and γ respectively. Substituting

$$\left.\begin{aligned} V\cos\beta&=c_1\mathfrak{P}^s,\\ I\cos\gamma&=c_2\mathfrak{S}^s,\\ V\sin\beta&=c_1\tan\beta\mathfrak{P}^s,\\ I\sin\gamma&=c_2\tan\gamma\mathfrak{S}^s, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

into (50) gives

$$\left.\begin{aligned} c_1A=&\ \alpha(c_1A_1+c_2A_2)+\omega(c_1A_1\tan\beta+c_2A_2\tan\gamma),\\ c_2B=&\ \alpha(c_1B_1+c_2B_2)+\omega(c_1B_1\tan\beta+c_2B_2\tan\gamma),\\ c_1A\tan\beta=&-\omega(c_1A_1+c_2A_2)+\alpha(c_1A_1\tan\beta+c_2A_2\tan\gamma),\\ c_2B\tan\gamma=&-\omega(c_1B_1+c_2B_2)+\alpha(c_1B_1\tan\beta+c_2B_2\tan\gamma), \end{aligned}\right\} \quad . \quad . \quad (68)$$

equations to be solved for α and ω in terms of the A's and B's, where

$$\left.\begin{aligned} A&=\mathfrak{P}^s(x), & B&=\mathfrak{S}^s(x),\\ A_1&=\int_a^b K(x,\xi)\mathfrak{P}^s(\xi)d\xi, & B_1&=\int_a^b M(x,\xi)\mathfrak{P}^s(\xi)d\xi,\\ A_2&=\int_a^b L(x,\xi)\mathfrak{S}^s(\xi)d\xi, & B_2&=\int_a^b N(x,\xi)\mathfrak{S}^s(\xi)d\xi. \end{aligned}\right\} \quad . \quad . \quad (69)$$

Their relative magnitudes only being of interest, one of the angles β and γ may be assigned any numerical value arbitrarily. Writing $\beta=0$,

solving the first two equations of (68) for α and ω, and substituting into the last two, gives

$$\frac{c_2}{c_1} = \frac{AB_2 - BA_1}{2BA_2}, \quad \ldots\ldots\ldots\ldots \quad (70)$$

$$\tan^2\gamma = \frac{4AB(A_1B_2 - A_2B_1) - (AB_2 + BA_1)^2}{(AB_2 - BA_1)^2}, \quad \ldots \quad (71)$$

$$\alpha = \frac{AB_2 + BA_1}{2(A_1B_2 - A_2B_1)}, \quad \ldots\ldots\ldots\ldots \quad (72)$$

$$\omega = \frac{AB_2 - BA_1}{2(A_1B_2 - A_2B_1)} \tan\gamma. \quad \ldots\ldots\ldots \quad (73)$$

A necessary condition for the permissibility of the substitution (49) is that $\tan\gamma$, α, and ω be independent of x or become independent of x as s becomes infinite. This ultimate independence is assured when the ratios $A : A_1 : A_2$, $B : B_1 : B_2$ become independent of x as s becomes infinite; for finite s, a value of x on the range $a<x<b$ is chosen arbitrarily for approximate results.

A further necessary condition for the permissibility of the substitution (49) is, of course, that $\tan\gamma$ as given by (71) shall be real.

When the first matrix of (64) is orthogonal to U_1^*, U_2^*, ... U_{i-1}^* and is not orthogonal to U_i^*, and when $(\alpha_1^2+\omega_1^2)$, $(\alpha_2^2+\omega_2^2)$, ... form an increasing sequence, equations (72) and (73) then give α_i and ω_i provided that the ratios $A : A_1 : A_2$, $B : B_1 : B_2$ become independent of x as s becomes infinite and provided $\tan\gamma_i$ comes out real.

Uniform Transmission Line.

The foregoing theory may be illustrated by application to the case of the uniform leakageless electrical transmission line grounded at both terminals, say at $x=0$ and $x=b$. The characteristic functions in this case are known † to be sine and cosine functions of x, and the Green's functions are

$$\left.\begin{aligned} G_{v\mathfrak{I}}(x,t) &= \frac{r_0}{b}x(b-t), \quad x<t, & G_{v\mathfrak{V}}(x,t) &= \frac{-1}{b}x, & x<t,\\ &= \frac{r_0}{b}t(b-x), \quad x>t, & &= \frac{1}{b}(b-x), & x>t,\\ G_{i\mathfrak{I}}(x,t) &= \frac{-1}{b}(b-t), \quad x<t, & G_{i\mathfrak{V}}(x,t) &= \frac{1}{br_0} & \\ &= \frac{1}{b}t, \quad x>t. & & & \end{aligned}\right\} \quad (74)$$

† K. W. Wagner, ' Elektromagnetische Ausgleichsvorgaenge in Freileitungen und Kabeln,' Teubner, 1908.

The generating function $\chi=\chi_1=1$ and these Green's functions substituted into (66) give

$$\mathfrak{P}_1^1(x)=\frac{c_0r_0}{2!}\mathrm{S}_1^1(x),$$

$$\mathfrak{P}_1^2(x)=\frac{c_0^2r_0^2}{4!}\mathrm{S}_1^2(x)-\frac{l_0c_0}{2!}\mathrm{S}_1^1(x),$$

$$\mathfrak{P}_1^3(x)=\frac{c_0^3r_0^3}{6!}\mathrm{S}_1^3(x)-2\frac{r_0l_0c_0^2}{4!}\mathrm{S}_1^2(x),$$

$$\mathfrak{P}_1^4(x)=\frac{c_0^4r^4}{8!}\mathrm{S}_1^4(x)-3\frac{r_0^2l_0c_0^3}{6!}\mathrm{S}_1^3(x)+\frac{c_0^2l_0^2}{4!}\mathrm{S}_1^2(x),$$

$$\mathfrak{P}_1^5(x)=\frac{c_0^5r_0^5}{10!}\mathrm{S}_1^5(x)-4\frac{r_0^3l_0c_0^4}{8!}\mathrm{S}_1^4(x)+3\frac{c_0^3r_0l_0^2}{6!}\mathrm{S}_1^3(x),$$

$$\mathfrak{P}_1^6(x)=\frac{c_0^6r_0^6}{12!}\mathrm{S}_1^6(x)-5\frac{r_0^4l_0c_0^5}{10!}\mathrm{S}_1^5(x)+6\frac{c_0^4r_0^2l_0^2}{8!}\mathrm{S}_1^4(x)-\frac{c_0^3l_0^3}{6!}\mathrm{S}_1^3(x),$$

$$\mathfrak{P}_1^7(x)=\frac{c_0^7r_0^7}{14!}\mathrm{S}_1^7(x)-6\frac{r^5l_0c_0^6}{12!}\mathrm{S}_1^6(x)+10\frac{c_0^5r_0^3l_0^2}{10!}\mathrm{S}_1^5(x)-4\frac{c_0^4l_0^3r_0}{8!}\mathrm{S}_1^4(x), \quad . \quad . \quad (75)$$

$$\mathfrak{S}_1^s(x)=\frac{-1}{r_0}\frac{d\mathfrak{P}_1^s(x)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where

$$\mathrm{S}_1^1(x)=-x^2+bx,$$
$$\mathrm{S}_1^2(x)=\ x^4-2bx^3+b^3x,$$
$$\mathrm{S}_1^3(x)=-x^6+3bx^5-5b^3x^3+3b^5x,$$
$$\mathrm{S}_1^4(x)=\ x^8-4bx^7+14b^3x^5-28b^5x^3+17b^7x,$$
$$\mathrm{S}_1^5(x)=-x^{10}+5bx^9-30b^3x^7+126b^5x^5-255b^7x^3+155b^9x,$$
$$\mathrm{S}_1^6(x)=\ x^{12}-6bx^{11}+55b^3x^9-396b^5x^7+1683b^7x^5-3410b^9x^3+2073b^{11}x,$$
$$\mathrm{S}_1^7(x)=-x^{14}+7bx^{13}-91b^3x^{11}+1001b^5x^9-7293b^7x^7+31031b^9x^5$$
$$-62881b^{11}x^3+38227b^{13}x. \quad . \quad . \quad . \quad (77)$$

The S's are polynomial representations of const. $\times \sin \pi x/b$ of increasing accuracy. The numerical coefficients in (75), when detached, form an array

1					
1	−1				
1	−2				
1	−3	1			
1	−4	3			
1	−5	6	−1		
1	−6	10	−4		
:	:	:	:	:	. . . . . (75)

which can be obtained from Pascal's triangle by a simple rule ; hence the $\mathfrak{P}$'s are also polynomial representations of const. $\times \sin x/b$ on the range 0 to b.

Just as $U(x)$, $W(x)$, $X(x)$, $Y(x)$ are linear combinations of $\mathfrak{P}^s(x)$ and $\mathfrak{S}^s(x)$, s large, so are $U^*(x)$, $W^*(x)$, $X^*(x)$, $Y^*(x)$ linear combinations of $\mathfrak{Q}^s(x)$ and $\mathfrak{T}^s(x)$, defined by

$$\left.\begin{aligned}
\mathfrak{Q}^1(x) &= \int_a^b \eta(y)\mathrm{K}(y, x)dy, \\
\mathfrak{T}^1(x) &= \int_a^b \eta(y)\mathrm{L}(y, x)dy, \\
\mathfrak{Q}^s(x) &= \int_a^b \{\mathfrak{Q}^{s-1}(y)\mathrm{K}(y, x)+\mathfrak{T}^{s-1}(y)\mathrm{M}(y, x)\}dy, \\
\mathfrak{T}^s(x) &= \int_a^b \{\mathfrak{Q}^{s-1}(y)\mathrm{L}(y, x)+\mathfrak{T}^{s-1}(y)\mathrm{N}(y, x)\}dy,
\end{aligned}\right\} \quad . \quad . \quad . \quad (79)$$

expressions derived from (61) by premultiplication by either of the line-matrices $(\eta_1, \eta_2, \ldots \eta_n, 0, 0, \ldots 0)$ or $(0, 0, \ldots 0, \eta_1, \eta_2, \ldots \eta_n)$ of $2n$ elements each. This premultiplication shows that simultaneously η must not be orthogonal to both $U_1(x)$, $X_1(x)$ and χ must not be orthogonal to both $U_1^*(x)$, $X_1^*(x)$ if (66) is to give an approximation to the first modal function. When this orthogonality exists, the second modal function will be given approximately by (66) provided the orthogonality does not simultaneously exist with respect to $U_2(x)$, $X_2(x)$ for $\eta(x)$ (*i. e.*, for $\eta_2(x)$) and with respect to $U_2^*(x)$, $X_2^*(x)$ for $\chi(x)$ (*i. e.*, for $\chi_2(x)$).

In the case of the uniform line, $\mathrm{M}(y, x) = -\mathrm{L}(x, y)$ and $\mathfrak{Q}_1^s(x)$ is found to be a linear combination of the sine-polynomials (77), for $\eta_1 = 1$, all of which are orthogonal to the function $(-2x+b)$. Hence the function $\chi_2 = -2x+b$ will generate the second modal function provided this χ_2 is not orthogonal to $U_2^*(x)$, $X_2^*(x)$, *i. e.*, not orthogonal to $\mathfrak{Q}_2^s$, any s, generated by a similarly appropriate η_2.

The generating function $\chi_2 = -2\xi+b$ in (66) gives

$$\begin{aligned}
\mathfrak{P}_2^1(x) &= \frac{c_0 r_0}{3\,!}\mathrm{S}_2^1(x), \\
\mathfrak{P}_2^2(x) &= \frac{c_0^2 r_0^2}{5\,!}\mathrm{S}_2^2(x) - \frac{c_0 l_0}{3\,!}\mathrm{S}_2^1(x), \\
\mathfrak{P}_2^3(x) &= \frac{c_0^3 r_0^3}{7\,!}\mathrm{S}_2^3(x) - 2\frac{c_0^2 l_0 r_0}{5\,!}\mathrm{S}_2^2(x), \\
\mathfrak{P}_2^4(x) &= \frac{c_0^4 r_0^4}{9\,!}\mathrm{S}_2^4(x) - 3\frac{c_0^3 l_0 r_0^2}{7\,!}\mathrm{S}_2^3(x) + \frac{c_0^2 l_0^2}{5\,!}\mathrm{S}_2^2(x), \quad . \quad . \quad . \quad (80)
\end{aligned}$$

$$\mathfrak{S}_2^s(x) = \frac{-1}{r_0}\frac{d\,\mathfrak{P}_2^s(x)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

where

$$\mathrm{S}_2^1(x) = (2x^3 - 3bx^2 + b^2x),$$
$$\mathrm{S}_2^2(x) = \tfrac{1}{3}(-6x^5 + 15bx^4 - 10b^2x^3 + b^4x),$$
$$\mathrm{S}_2^3(x) = \tfrac{1}{3}(6x^7 - 21bx^6 + 21b^2x^5 - 7b^4x^3 + b^6x),$$
$$\mathrm{S}_2^4(x) = \tfrac{1}{15}(-30x^9 + 135bx^8 - 180b^2x^7 + 126b^4x^5 - 60b^6x^3 + 9b^8x). \quad . \quad (82)$$

The curves of (82) are symmetrical about, and pass through, the point $x=b/2$ on the x-axis; plotting shows that they are representations of const. $\sin\frac{2\pi x}{b}$ of increasing accuracy, $\mathrm{S}_2^4(x)$ representing such a function accurately to within 0·5 of 1 per cent. for 18 out of 20 divisions of 360°, and the accuracy is almost † as good for the remaining two divisions.

In the case of the uniform non-leaking line, the $\mathfrak{Q}$'s come out the same as the $\mathfrak{P}$'s, but

$$\mathfrak{T}_i^\sigma = \frac{l_0}{c_0 r_0}\frac{d\mathfrak{Q}_i^\sigma}{dx},$$

and thus is not equal to $\mathfrak{S}_i^\sigma$ in general.

The computation of the third modal function illustrates greater but more usual difficulties. We seek a function χ_3 orthogonal both to $\mathfrak{Q}_1^s(x)$ and $\mathfrak{Q}_2^\sigma(x)$ for infinitely large values of s and σ. Even moderately large values being excluded by the labour involved in the computation of the $\mathfrak{Q}$'s, small values of s and σ have to be used: at worst $s=\sigma=1$. Polynomials proportional to $\mathfrak{Q}_1^1(x)$ and $\mathfrak{Q}_2^1(x)$ are $(-x^2+bx)$ and $(2x^3-3bx^2+b^2x)$, respectively. The simplest polynomial orthogonal to both of these is easily found to be $\chi_3(x)=5x^2-5bx+b^2$. With this generating function the first approximation to the third modal function comes out

$$\mathfrak{P}_3^1(x) = \frac{r_0\mathcal{C}_0}{12}(b^3x - 6b^2x^2 + 10bx^3 - 5x^4), \quad . \quad . \quad . \quad . \quad (83)$$

the curve of fig. 3. This curve has a component of the first modal function which may be subtracted out before proceeding to the computation of $\mathfrak{P}_3^2(x)$; from the symmetry of the curve no component of the second modal function is present.

The evaluation of components of lower modal functions at any stage s in the computation of a modal function above the first in the general case depends upon the relations of biorthogonality (63). These can be expressed in simpler form, for, just as the U's and X's are proportional to the $\mathfrak{P}$'s and the W's and Y's are proportional to the $\mathfrak{S}$'s, so the starred letters are correspondingly proportional to the $\mathfrak{Q}$'s and $\mathfrak{T}$'s as follows

† Our accuracy is thus superior to that obtained by Bateman, "On the Numerical Solution of Linear Integral Equations," Proc. Roy. Soc. A, vol. c. p. 445 (1921).

from the definition of the latter. Self-consistency of the system (63) imposes the relations

$$U^*=\sin\gamma V^*,\quad W^*=-\sin\beta I^*,\quad X^*=-\cos\gamma V^*,\quad Y^*=\cos\beta I^*, \quad . \quad (84)$$

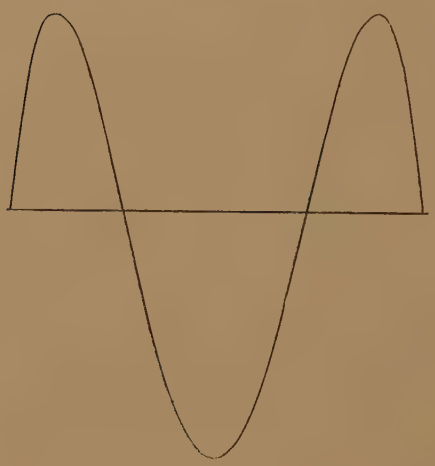
corresponding to

$$U=\cos\beta V,\quad W=\cos\gamma I,\quad X=\sin\beta V,\quad Y=\sin\gamma I, \quad . \quad . \quad (85)$$

implied by (50) and (51). In (84), V^* and I^* are given by $\mathfrak{D}^s$ and $\mathfrak{I}^s$ when the generating functions χ_i, η_i are biorthogonal to $\mathfrak{D}_j$, $\mathfrak{P}_j$ respectively. In terms of V, V^*, I, I^*, (63) implies and is implied by

$$\left.\begin{aligned}\sin(\gamma_m-\beta_n)\int_a^b V_m^* V_n\,dx&=\begin{cases}1, & m=n,\\ 0, & m\neq n,\end{cases}\\ \sin(\gamma_m-\beta_n)\int_a^b I_m^* I_n\,dx&=\begin{cases}1, & m=n,\\ 0, & m\neq n,\end{cases}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Fig. 3.



biorthogonality relations which can also be expressed in the form

$$\left.\begin{aligned}\int_a^b \mathfrak{D}_m^s\,\mathfrak{P}_n^\sigma\,dx&=\begin{cases}\text{const.}, & m=n,\\ 0 & m\neq n,\end{cases}\\ \int_a^b \mathfrak{I}_m^s\,\mathfrak{Q}_n^\sigma\,dx&=\begin{cases}\text{const.}, & m=n,\\ 0 & m\neq n,\end{cases}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (87)$$

valid for any σ, s when the generating functions are as before. When, simultaneously, $\mathfrak{P}_n^\sigma(x)$ can be expressed as the sum of terms each the product of a function of the line constants by a function of x, as in (75), and

$\mathfrak{Q}_m^s(x)$ can be expressed similarly, then relations of biorthogonality of the same kind as (87) exist among the functions of x. For example, the S's of (75) are orthogonal to the S's of (82). The use of these simpler relations of orthogonality may be illustrated in the computation of the component of the first modal function to be found in $\mathfrak{P}_3^1(x)$ of (83). Writing

$$-5x^4+10bx^3-6b^2x^2+b^3x=a_1S_1^1+a_2S_2^1+a_3S_3,$$

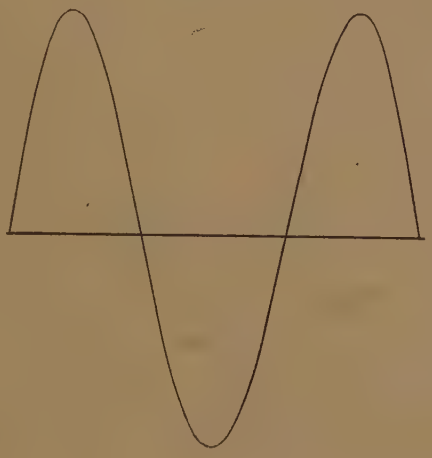
where S_3 is unknown, a_1 is computed by multiplying both sides by S_1^σ and integrating from 0 to b. Using S_1^1 for this purpose, there results

$$a_1=-b^2/14,$$
$$a_2=0,$$

so that, assuming $\int_a^b S_1^1 S_3\,dx=0$,

$$S_3^1=\text{const.}\times(-14x^4+28bx^3-17b^2x^2+3bx).$$

Fig. 4.



Taking S_3^1 as just found as a generating function χ_3 in (66) gives

$$\mathfrak{P}_3^2=\text{const.}\times(28x^6-84bx^5+85b^2x^4-30b^3x^3+b^5x)+\ldots$$

This approximation to const. $\times\sin\dfrac{3\pi x}{b}$ has, in its turn, an undesirable component of $\sin \pi x/b$. Proceeding as before to subtract this out, there results

$$S_3^2(x)=\text{const.}\times(1176x^6-3528bx^5+3570b^2x^4-1260b^3x^3-5b^4x^2+47b^5x),$$

the curve of fig. 4, a better approximation to $\sin 3\pi x/b$, and which, in its turn, can be used as a generating function χ_3 in (66), and so on.

Our procedure in this case, where $\mathfrak{P}_i^s = \mathfrak{D}_i^s$, thus is essentially the same as in the less general method of numerical solution of integral equations to Koch †.

Since the series of generating functions starting with $\chi_1 = 1$, $\chi_2 = b - 2x$, can be replaced by the generating functions $\chi_1 = \sin \pi x/b$, $\chi_2 = \sin 2\pi x/b$, $\chi_3 = \sin 3\pi x/b$ . . . with the greatest possible advantage in the cases of the uniform vibrating string and of the uniform transmission line, obviously, it is suggested that in general the first generating function $\chi_1(x)$ should be taken as close as possible, or as computationally convenient, to the assumed form of the first characteristic function when the computer's experience gives him sufficient faith in his assumption as to said form.

Having found $\mathfrak{P}_i^s(x)$ and $\mathfrak{S}_i^s(x)$ corresponding to $\chi_i(x)$, the A's and B's of (68) are computed for substitution into (71) to give γ_i, and into (72) and (73) to give α_i and ω_i.

The limiting form of $\mathfrak{P}^s(x)$ gives, of course, exact results. Thus, taking $\sin \kappa\pi x/b$, $\kappa = 1, 2, \ldots$ for $\mathfrak{P}_\kappa^s$ gives

$$A = \sin \kappa\pi x/b, \qquad B = -\frac{\kappa\pi}{br_0} \cos \kappa\pi x/b,$$

$$A_1 = \frac{c_0 r_0 b^2}{\kappa^2\pi^2} \sin \kappa\pi x/b, \qquad B_1 = -\frac{c_0 b}{\kappa\pi} \cos \kappa\pi x/b,$$

$$A_2 = -\frac{l_0}{r_0} \sin \kappa\pi x/b, \qquad B_2 = 0,$$

whence

$$\tan^2 \gamma = \frac{4\kappa^2\pi^2 l_0 - r_0^2 c_0 b^2}{b^2 r_0^2 c_0},$$

$$\alpha = \frac{r_0}{2l_0},$$

$$\omega = \sqrt{\frac{\kappa^2\pi^2}{c_0 l_0 b^2} - \left(\frac{r_0}{2l_0}\right)^2},$$

in agreement with well-known results.

New York,
June 30, 1939.

† Hohenemser, p. 19.

III. *A Mechanical Appliance for the Smoothing of Time Series.*

By T. E. W. Schumann, M.Sc., Ph.D.*

[Received May 15, 1939.]

[Plates I. & II.]

The smoothing or graduation of irregularly fluctuating data spaced at equal intervals is a statistical problem of considerable importance. It is a process which is daily applied by the economist who wishes to determine the trend of interest rates and of commodity or security prices, by the industrialist who keeps a check on the trend of production costs, and by many scientists and business men who have to deal with figures showing more or less erratic fluctuations.

In general it may be said that the object of any graduation process is to free the actual data from fluctuations which are considered to be of minor importance and to adjust the data in such a way that the picture, in the form of a curve, accentuates those variations or trends which it is desired to reveal.

In order to attain this object, the statistician has at his disposal a number of methods of varying merit. In the first place the data may be plotted on paper and freehand (or French curve) smoothing may be applied. This is undoubtedly the simplest and most direct method, but probably also the most unsatisfactory, and consequently it is seldom employed by the serious statistician. Of the many mathematical formulæ used for obtaining smooth curves the method of moving averages is probably the one which is the most widely in use, but for many purposes this method is also far too crude, and hence many other methods have been proposed, some of which are widely used. However, all mathematical methods more elaborate than that of moving averages involve a great amount of labour, and there cannot be much doubt that if a rapid mechanical smoothing process could be devised, it would be of immense practical value to all statisticians who are required to deal with problems of graduation.

The object of this paper is to describe a machine which not only reduces the labour of graduation to a minimum, but which automatically

---

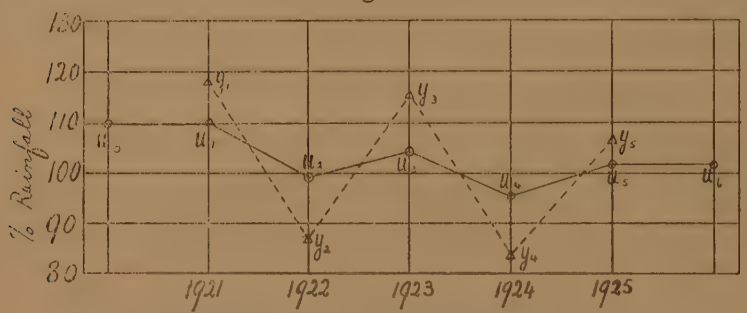
* Communicated by the Author.

applies to the crude data a very effective smoothing formula, which, moreover, rests upon a strictly logical basis.

The theory of this smoothing process has already been published *, but in order to explain the principle on which the machine is based we may quote from the said article :—

" The underlying idea of this new method, which for the sake of brevity we shall refer to as the *equilibrium method*, may most conveniently be explained with the help of the accompanying diagram, fig. 1. In this figure y_1 to y_5 are the crude data, and represent the annual rainfalls for the Cape Town district during the years 1921 to 1925 inclusive. For the sake of clarity only a few points are taken, but it is to be understood that there is no limit to the number of datum points which can be included

Fig. 1.



Illustrative of the smoothing principle.

in the graduation process. Our object now is to draw a smooth curve to represent the data.

" In order to accomplish this we imagine an inextensible string tied to the two points u_0 and u_6, whose values are left arbitrary for the time being. Initially the string is not stretched tightly, but is supposed to exhibit a certain degree of slackness. We now imagine springs to be attached at the points y_1 to y_5 and to the string, each spring drawing the string in a positive or negative y direction with a force proportional to its own length. Unless the string is too slack at the beginning it will take up an intermediate position such as the curve u_0, u_1, u_2, u_3, u_4, u_5, u_6 in fig. 1. Evidently this is a smoothed curve of the original data.

" If the original amount of slackness in the string be decreased a smoother curve is obtained, but the fit to the crude data would be inferior. By lengthening or shortening the string any degree of smoothness may be obtained, but in general the closeness of fit to the original data is an

* Schumann, Phil. Mag. xxvi. p. 970 (1938).

inverse function of the smoothness of the fitted curve. This is a principle of general application, and the investigator is always obliged to decide up to what point he is prepared to sacrifice closeness of fit in order to obtain sufficient smoothness."

In order to put into practice what is here presented in theory, a very simple mechanism is evidently required ; but in devising a machine the principle on which it is to operate should be clearly understood, and this is concerned mainly with the action of the springs. The springs should be constrained to act only in the positive or negative y direction, and hence frictional forces at the points of contact between the string and the different springs should be entirely absent. Also it should be clearly borne in mind that theory demands that the force exerted on the string by the spring attached to the point y_1 must be proportional to $y_1 - u_1$, the same condition applying to the other springs. If an ordinary helical spring be attached to the point y_1 and the string be slipped through a hole at the free end of the spring, and tension applied, then the force exerted by the spring would be proportional not to its total extended length but merely to its *extension*, which of necessity is less than $y_1 - u_1$, however short the unextended spring may be. Hence helical springs operating in this way would not answer the purpose. The main mechanical problem to be solved is therefore to find a type of spring which could be constrained to act in the y direction only, and which would exert a force proportional to the deviation $y_1 - u_1$.

Various ways of accomplishing this, *e. g.*, by the use of a system of pendulums or by the use of initially extended helical springs, suggest themselves ; but only the method actually adopted will be described here.

The apparatus, as built may best be explained with the aid of the accompanying photographs. Fig. 2 (Pl. I.) shows the complete apparatus capable of dealing with 18 datum points. It consists of a wooden board approximately 6 feet by 4 feet, mounted at a height convenient for observation. A scale covers the upper part of the board. Glass cylinders, numbered 1 to 18, each about 50 cm. long, are spaced at equal intervals, and these can be moved vertically up or down along suitable grooves and fixed in any required position by means of supporting clamps. Each cylinder is about half-filled with mercury in which there floats a cylindrical iron rod also of length 50 cm. To the top of each iron rod a short stem of lesser diameter is rigidly attached, and this stem has a small hole near its upper end. The function of the floating rods is to act as springs. Fig. 5 (Pl. II.) gives a closer view of the cylinders and rods.

A steel wire, which should be as fine as possible and suitably tempered, is attached to the hook H (fig. 2, Pl. II.), which can also be shifted

vertically up or down as required. After threading the eighteen holes in succession the wire passes over the pulley P on the right. Any desired tension may now be applied to the wire by placing suitable weights in the pan attached to its free end. The pulley, like the hook H, is movable along a vertical iron rod and can be fixed in position at any height.

With this brief description of the machine, and bearing in mind the principles of the smoothing process, the method of its operation must be practically self-evident. Given the data y_1 to y_n, which are to be graduated, cylinder No. 1 is so adjusted that the hole in the stem corresponds to the value y_1 on the scale, the rod being left to float quite freely. Cylinders nos. 2, 3, etc., are similarly adjusted to fit the values y_2, y_3, etc. If a thin steel wire be now passed through all the holes in succession, and if this wire be lightly adjusted so that it exerts no force on the rods it will, in effect, be a graph of the original unsmoothed data, as illustrated in fig. 2 (Pl. I.). It may be mentioned here that for purposes of clearer photography, a fairly thick string was employed in place of the fine steel wire.

In order to obtain a graduated curve of the data it is now only necessary to apply tension to the wire by placing a suitable weight in the pan attached to the wire. Furthermore, since the complete theory demands that $u_0=u_1$ and $u_n+1=u_n$, the hook and the pulley should be so adjusted that the portions of the wire between H and cylinder 1, and between P and cylinder 18, are horizontal. When this has been done, care being taken that any frictional forces are eliminated by rapid vibration of the whole machine, the wire will form a smoothed curve as shown in fig. 3 (Pl. I.). If greater smoothness is desired, it is only necessary to place the appropriate greater weight in the pan and again adjust the hook and the pulley to their proper positions. Thus a smoother curve will be obtained such as illustrated in fig. 4 (Pl. II.), which, like fig. 3 (Pl. I.), was taken at an angle in order to reduce the width of the picture.

The model of the machine here described is capable of dealing only with 18 datum points at a time, but it must be evident that, theoretically at least, a machine could be built for an almost indefinitely long series of data, although practical considerations of space may curtail the actual size.

The Theory of the Graduation Process.

The following is a brief *résumé* of the theory of the graduation process, given in greater detail in the publication already referred to.

In fig. 1 consider the vertical forces acting at the point u_r. The force exerted by the floating rod at this point is proportional to the distance y_r-u_r through which the rod is displaced from its position of free-floating

equilibrium. This force is equal to $P(y_r-u_r)$, where it may readily be proved that

$$P=\frac{ab\rho g}{b-a}.$$

a=cross-sectional area of the rod.

b=internal cross-sectional area of the glass tube.

ρ=density of mercury.

g=acceleration due to gravity.

Two other forces act at u_r, and these are due to the tension T in the wire. If we postulate that the distance L between neighbouring cylinders is so large that the square of any first difference $(u_{r+1}-u_r)^2$ may be neglected in comparison with L^2, then it will readily be seen that the vertical forces exerted by the wire are $\frac{T}{L}\left(u_{r-1}-u_r\right)$ and $\frac{T}{L}\left(u_{r+1}-u_r\right)$ respectively.

Equating these three forces to zero, since every point on the wire is in equilibrium, we obtain n equations of the type

$$T(u_{r+1}+u_{r-1}-2u_r)+PL(y_r-u_r)=0,$$

or

$$u_r=\frac{T}{2T+PL}\left(u_{r+1}+u_{r-1}\right)+\frac{PL}{2T+PL}y_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By solving these n equations we are able to determine the n unknown quantities u_1 to u_n.

Now it may be demonstrated that this mechanical method corresponds to a graduation formula which is very effective and which can be developed on sound mathematical principles.

In any graduation process there must of necessity be a compromise between two opposing factors, namely, smoothness of the graduated curve and closeness of fit to the original data. It is, therefore, in every way logical that a graduation formula should be such that maximum smoothness is obtained with minimum departure from the observed facts. This principle may also be formulated thus: the graduation formula should be such that the roughness or lack of smoothness of the graduated curve plus a constant k times the departure from the observed facts is a minimum.

If we now define the roughness of the curve by the sum of the squares of the first differences $\sum_1^{n-1}(u_{r+1}-u_r)^2$, and if the departure from the observed facts be defined as the sum of the squares of the deviations of the data

from the graduated curve $\sum_{1}^{n}(y_r-u_r)^2$, then $\sum_{1}^{n-1}(u_{r+1}-u_r)^2+k\sum_{1}^{n}(y_r-u_r)^2$ should be a minimum.

If this expression is to be made a minimum, it must be differentiated with respect to all the u's in turn, and the results equated to zero. In this way we derive n equations of the type

$$u_r=\frac{1}{2+k}(u_{r+1}+u_{r-1})+\frac{k}{2+k}y_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is identical with (1), provided

$$\left.\begin{aligned} k&=\frac{\mathrm{PL}}{\mathrm{T}},\\ u_0&=u_1,\\ u_{n+1}&=u_n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This means that the mechanical process automatically applies to the crude data a graduation formula which is based on the mathematical minimum principle explained above.

The solution of the n equations (1) or (2) leads to the result

$$u_r=\frac{2p-1}{p^n-(1-p)^n}\left\{\phi_{n-r+1}\sum_{1}^{r}\alpha^{r-s}\phi_s y_s+\phi_r\sum_{r+1}^{n}\alpha^{s-r}\phi_{n-s+1}y_s,\right\} \quad . \quad (4)$$

where

$$\left.\begin{aligned} \alpha&=\frac{\mathrm{T}}{2\mathrm{T}+\mathrm{PL}}=\frac{1}{2+k},\\ \alpha^2&=p(1-p),\\ \phi_r&=\frac{1}{\alpha+p}\left\{p^r+\frac{p}{\alpha}(1-p)^r\right\}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

According to eq. (4) the graduated values of u_1 to u_n are expressed in terms of the crude data y_1 to y_n and of certain numerical coefficients. By adopting different values of k or α we may, with the aid of (4) and (5), construct tables containing these numerical coefficients.

As an illustrative example let us assume

$$\frac{\mathrm{T}}{\mathrm{PL}}=\frac{1}{k}=6,$$

in which case

$$\alpha=6/13,$$
$$p=9/13.$$

Also let

$$n=18.$$

TABLE I.

Numerical Coefficients for the Case $\frac{T}{PL}=6$.

	y_1·	y_2·	y_3·	y_4·	y_5·	y_6·	y_7·	y_8·	y_9·	y_{10}·	y_{11}·	y_{12}·	y_{13}·	y_{14}·	y_{15}·	y_{16}·	y_{17}·	y_{18}·
u_1 ...	·334	·223	·148	·099	·066	·044	·029	·020	·013	·009	·006	·004	·002	·002	·001	—	—	—
u_2 ...	·223	·260	·173	·116	·077	·051	·034	·023	·015	·010	·007	·004	·003	·002	·001	—	—	—
u_3 ...	·148	·173	·227	·151	·101	·067	·045	·030	·020	·013	·009	·006	·004	·002	·002	·001	—	—
u_4 ...	·099	·116	·151	·212	·141	·094	·063	·042	·028	·018	·012	·008	·006	·004	·002	·002	·001	—
u_5 ...	·066	·077	·101	·141	·205	·137	·091	·061	·041	·027	·018	·012	·008	·005	·004	·002	·002	·001
u_6 ...	·044	·051	·067	·094	·137	·204	·135	·090	·060	·040	·026	·018	·012	·008	·006	·004	·003	·002
u_7 ...	·029	·034	·045	·063	·091	·135	·202	·134	·089	·059	·040	·026	·018	·012	·008	·006	·004	·004
u_8 ...	·020	·023	·030	·042	·061	·090	·134	·200	·134	·089	·059	·040	·026	·018	·012	·009	·007	·006
u_9 ...	·013	·015	·020	·028	·041	·060	·089	·134	·200	·134	·089	·060	·040	·027	·018	·013	·010	·009
u_{10} ...	·009	·010	·013	·018	·027	·040	·060	·089	·134	·200	·134	·089	·060	·041	·028	·020	·015	·013
u_{11} ...	·006	·007	·009	·012	·018	·026	·040	·059	·089	·134	·200	·134	·090	·061	·042	·030	·023	·020
u_{12} ...	·004	·004	·006	·008	·012	·018	·026	·040	·059	·089	·134	·202	·135	·091	·063	·045	·034	·029
u_{13} ...	·002	·003	·004	·006	·008	·012	·018	·026	·040	·060	·090	·135	·204	·137	·094	·067	·051	·044
u_{14} ...	·001	·002	·002	·004	·005	·008	·012	·018	·027	·041	·061	·091	·137	·205	·141	·101	·077	·066
u_{15} ...	—	·001	·002	·002	·004	·006	·008	·012	·018	·028	·042	·063	·094	·141	·212	·151	·116	·099
u_{16} ...	—	—	·001	·002	·002	·004	·006	·009	·013	·020	·030	·045	·067	·101	·151	·227	·173	·148
u_{17} ...	—	—	—	·001	·002	·003	·004	·007	·010	·015	·023	·034	·051	·077	·116	·173	·260	·223
u_{18} ...	—	—	—	·001	·002	·002	·004	·006	·009	·013	·020	·029	·044	·066	·099	·148	·223	·334

The numerical coefficients by which the y's have to be multiplied are contained in Table I., the method of using the table being quite obvious, thus :—

$$u_1 = \cdot 334y_1 + \cdot 223y_2 + \cdot 148y_3 + \cdot 099y_4 + \text{etc.}$$

TABLE II.

Annual Rainfall Deviations together with the Values graduated Mathematically and Mechanically.

Year.	y, percentage deviations of annual rainfall.	u, theoretical graduated values.	Experimental graduated values.
1881.....	36·5	— 4·9	— 5·0
1882.....	—24·5	—11·9	—11·9
1883.....	—55·0	—16·8	—16·7
1884.....	—29·3	—15·2	—15·5
1885.....	— 0·6	—11·4	—11·7
1886.....	—38·4	— 9·3	— 9·4
1887.....	6·4	— 2·3	— 2·5
1888.....	—12·9	3·2	3·1
1889.....	21·5	11·5	11·2
1890.....	43·3	18·1	18·0
1891.....	50·2	20·4	20·4
1892.....	19·0	17·8	17·6
1893.....	8·8	15·1	14·8
1894.....	67·2	13·3	13·2
1895.....	— 0·9	2·5	2·2
1896.....	—40·8	— 7·6	— 7·8
1897.....	—43·8	—12·3	—12·5
1898.....	— 8·2	—11·7	—12·0

These coefficients were applied to the data given in the second column of Table II., which represent the percentage deviations from normal annual rainfall in a Central Cape district for the years 1881 to 1898 inclusive. The results of this mathematical graduation are given in the third column.

Experimental Results.

In order to obtain experimental values corresponding to the theoretical graduated values given in Table II., it is necessary to determine the tension T, or the mass m which should be suspended from the wire to ensure the correct degree of smoothness.

The values of the constants of the machine are :—

Diameter of iron rods=0·798 cm.

Internal diameter of glass cylinders=1·43 cm.

a= 0·50 sq. cm.

b= 1·606 sq. cm.

L= 9·48 cm.

ρ=13·55 grams per c.c.

$$P=\frac{abpg}{b-a}=9{\cdot}83\,g.$$

In the particular case where $\frac{T}{PL}=6$

$$m=T/g=\frac{6PL}{g}=559 \text{ grams.}$$

A mass of 559 grams was accordingly suspended from the wire, and the experimental values as read off on the scale are given in the last column of Table II., and these show very satisfactory agreement with the computed values u_1 to u_{18}.

Mechanical Imperfections and Possible Improvements.

The machine here described was the first model to be built, and the fact that it gives results in such close agreement with the theoretical values must be regarded as most gratifying.

However, it exhibits certain mechanical imperfections, which will probably be overcome in the second model now being built.

In practice it appears that the surface tension of the mercury plays a part which cannot be entirely disregarded. This defect might be minimized by using cylinders and rods of larger diameter, but in the second model the mercury is being replaced by water and the thin iron rods by hollow metal cylinders each about 5 cm. in diameter closed and suitably loaded at the bottom. These cylinders do not float in separate containers, but in a common trough filled with water. A long metal stem is attached to each cylinder, and along this there slides a steel pointer, provided with a hole through which the wire passes. In the first model the iron rods and containing cylinders had to be moved bodily to fit the crude

data, whereas in the new machine it is only necessary to adjust the pointers to their proper heights so as to correspond with the data.

The advantage of this method is that a common trough can be used, that water can be conveniently substituted for mercury, that the effects of surface tension can be practically eliminated, and that the pointers allow more accurate readings on the scale.

The most important shortcoming of the machine, as described, is the effect of friction where the wire passes through the holes in the metal stems or pointers. To a great extent this difficulty may be overcome by polishing the surface after the holes have been drilled, or preferably by using a loop of polished wire, suitably bent, in lieu of the holes drilled in the stems. If every great sensitivity is required, it will probably be advisable to provide each stem with two tiny pulleys between which the wire must be made to pass, but experiments on the elimination of friction are still proceeding, and in the end it may not prove necessary to introduce such an elaboration.

My thanks are due to Mr. R. C. Brown of this office who was responsible for the construction of the apparatus.

Meteorological Office,
Pretoria, S. Africa.

IV. *Modification of a Relativity Postulate.*

By Sir Shah Sulaiman, Federal Court, India *.

[Received August 21, 1939.]

1. The fundamental postulate of relativity is that the velocity of light (though finite) relative to any moving object, no matter how fast and in what direction it may be moving, and whether light be overtaking it or receding from it, is always the same and equal to the velocity of light itself. Such an extraordinary postulate, attributing in effect the properties of infinity to a finite velocity, would not have been easily accepted but for the fact that its results in the special theory (for certain special reasons) work well, and its results in the general theory are believed to have been confirmed by observation. But if the postulate is only approximately and not exactly true it is bound to break down in the extreme case when the moving object is light itself and is affected by a strong gravitational field.

In the case of the deflexion of light from distant stars coming past the edge of the Sun at the time of a total solar eclipse, Einstein's value is 1″·74.

(1) But when second order terms of refraction and aberration are added to the Sobral value (1919) the deflexion comes to 2″·16 †.

(2) When correction is made for the deflexion of light from the outer ring of stars as well, which had been wrongly assumed by the Lick Expedition (1922) to be unaffected, the value comes out to be 2″·20.

(3) The observations made by Freundlich and his colleagues in North Sumatra (1929) gave a mean value of $2''{\cdot}24 \pm 0''{\cdot}10$ ‡.

(4) And what is more, when similar corrections are applied to the 1919 and 1922 values, the mean comes out to be much the same §.

(5) The Astronomer Royal has admitted that

(*a*) 2″·16 is certainly the correct result from the 1919 plates; and (*b*) the corrections given by the check plates should not have been neglected by the Lick Expedition ||.

---

* Communicated by the Author.

† Hopmann, *Phys. Zeitschrift*, xxiv. p. 476 (1923).

‡ *Zeitschrift für Astrophysik*, xxxi. pp. 171–198 (1931).

§ *Ibid.*

|| 'Observatory,' lv. no. 692 (January 1932).

The mere fact that Freundlich had an unsymmetrical field cannot be a fatal objection.

(6) The results of Michailov's observations of the 1936 eclipse have not yet been announced and are eagerly awaited *.

(7) Additional interest in the true value of the deflexion is created by the coming total solar eclipse of 1st October, 1940, in South Africa.

It is accordingly necessary to re-examine the soundness of the postulate as applicable to the general theory with a view to discovering whether the deficiency in the theoretical value can be otherwise accounted for. For economy of space the relativity relations are borrowed directly from Sir Arthur Eddington's 'Mathematical Theory of Relativity' (second edition) and references to the pages of this standard work are given.

2. The interval as between two neighbouring events is at p. 10 taken as

$$\begin{aligned} ds^2 = {} & g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 \\ & + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3 + 2g_{14}dx_1dx_4 \\ & + 2g_{23}dx_2dx_3 + 2g_{24}dx_2dx_4 + 2g_{34}dx_3dx_4, \end{aligned}$$

where the g's are some functions of the coordinates.

It is assumed at p. 13 that we can have "a small region of the world throughout which the g's can be treated as constants." It is submitted that as, although the first derivatives of the g's can by transformation of coordinates be made to vanish at a selected point (see footnote on p. 13) the other derivatives may not; the assumption that even in a small region the g's are constants is only approximately true to the first order terms, and not exactly true. In a non-gravitational field the g's are absolute constants, and the right-hand side can certainly be broken up into a sum of four squares, provided we do not mind passing from the real into an unreal world by admitting imaginary coefficients. But in a gravitational field the supposed reduction into a sum of four squares is only approximately and not exactly possible. The g's, which involve the distance r, can be regarded as nearly constant at any particular point, but there will be different values for those constants at different points varying with r. The whole track cannot properly be represented by an equation having only one set of such constants.

The equation given at p. 14,

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2,$$

is in a gravitational field only approximately and not exactly true; and the same is, of course, true for the relation given at p. 22,

$$\left(\frac{ds}{dt}\right)^2 = c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 = c^2 - v^2.$$

* Since this was written, he has, by his letter from Moscow dated 11th December, 1939, informed me that the deflexion is $2''{\cdot}72 \pm {\cdot}21$.

It follows that the relation $ds=0$, given for light when $v=c$, is in a gravitational field only approximately and not exactly true. As small differences can over a large region easily accumulate and produce a large cumulative effect, there is really no justification for the assumption that for light $ds=0$ all along its track throughout a large region. If, on account of the postulates of the special theory, that assumption has to be made in the general theory, it will have to be a wholly new and independent postulate. In this paper the postulate that $ds=0$ for light even in a gravitational field is not made, and ds is taken to be small but not zero.

3. The general condition for a geodesic, as at p. 36,

$$\int ds$$

is stationary is accepted, and therefore

$$\delta\int ds=0.$$

The whole of the mathematical analysis in § 37 (pp. 81–85) can be fully adopted, as it does not anywhere involve the assumption that $ds=0$ for light.

The result is the equation at p. 85,

$$ds^2=-1/\gamma\, dr^2-r^2\, d\theta^2-r^2\sin^2\theta\, d\phi^2+\gamma c^2\, dt^2,$$

where

$$\gamma=1-\frac{2\mu}{c^2 r}.$$

It must, however, be pointed out that when at p. 83 the equation

$$ds^2=-\mathrm{U}(r)\, dr^2-\mathrm{V}(r)(r^2\, d\theta^2+r^2\sin^2\theta\, d\phi^2)+\mathrm{W}(r)c^2\, dt^2$$

is transformed into

$$ds^2=-\mathrm{U}_1(r_1)\, dr_1^2-r_1^2\, d\theta^2-r_1^2\sin^2\theta\, d\phi^2+\mathrm{W}_1(r_1)c^2\, dt^2,$$

and then the suffixes are dropped, the significance of r is changed completely ; for instance, if r had previously represented the distance from the source, $r_1=r\sqrt{\mathrm{V}(r)}$ would cease to do so, though the difference may possibly be of only second order terms.

Following the method in § 39 (pp. 85–6) we obtain the two equations

$$\left.\begin{aligned} r^2\frac{d\phi}{ds}&=\frac{h}{c},\\ \frac{dt}{ds}&=\frac{\mathrm{C}}{c\,.\,\gamma}, \end{aligned}\right\}$$

where h and C are constants of integration.

Using the previous equation of p. 85, eliminating ds and dt, and substituting for γ and r, we get

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{(C^2-1)c^2}{h^2}+\frac{2\mu}{h^2}+\frac{2\mu}{c^2}\,.\,u^2.$$

Differentiating with respect to $d\phi$ and simplifying, we get, as at p. 86,

$$\frac{d^2u}{d\phi^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{c^2}\,.\,u^2.$$

4. Owing to the postulate of relativity that $ds=0$ for light there was apparently no option but to take

$$h=c\,.\,r^2\frac{d\phi}{ds}=\infty\,,$$

so that one term disappears, and the equation for light reduces to

$$\frac{d^2u}{d\phi^2}+u=\frac{3\mu}{c^2}\,.\,u^2.$$

Accordingly the value of the deflexion of light from stars past the Sun, as at p. 91, becomes $\frac{4\mu}{c^2R}$, where R is the perpendicular on the asymptote. This equals 1″·74 only.

5. We reject the postulate that $ds=0$ for light on two grounds: (a) that, as already indicated, it is not exactly so, unless a fresh postulate is made; and (b) that we are really not concerned with ds by itself, but with the relations

$$\frac{ds}{dt},\ \frac{ds}{dr},\ \frac{ds}{d\theta}\text{ or }\frac{ds}{d\phi};$$

even if the small differentials vanish, their ratios may not.

Hence we can eliminate ds from the relations

$$r^2\frac{d\phi}{ds}=\frac{h}{c}\quad\text{and}\quad\frac{dt}{ds}=\frac{C}{c\,.\,\gamma},$$

and get

$$r^2\frac{d\phi}{dt}=\frac{h}{C}\left(1-\frac{2\mu}{c^2r}\right).$$

For light coming from stars past the Sun, the hyperbolic path is very nearly straight, and the velocity of light changes only very slightly, hence $\frac{h}{C}$, which is the value of double the areal rate at $r=\infty$, is nearly equal to $c\,.\,R$, where R is the perpendicular to the asymptote, and can be taken as nearly equal to h.

Accordingly the two equations of motion for light are

$$\left.\begin{aligned} \frac{d^2u}{d\phi^2}+u&=\frac{\mu}{h^2}+\frac{3\mu}{c^2}\,u^2, \\ r^2\frac{d\phi}{dt}&=h\left(1-\frac{2\mu}{c^2r}\right), \end{aligned}\right\}\ \text{nearly,}$$

where h is nearly equal to c . R.

6. To evaluate the deflexion of light we consider the first equation, which gives the geometrical picture.

Its first approximate solution is $u=\dfrac{\cos\phi}{\mathrm{R}}$.

Therefore

$$\frac{d^2u}{d\phi^2}+u=\frac{\mu}{h^2}+\frac{3\mu}{c^2\mathrm{R}^2}\cos^2\phi.$$

Now

$$u_1=\frac{\mu}{h^2}+\frac{\mu}{c^2\mathrm{R}^2}(\cos^2\phi+2\sin^2\phi)$$

is found to be a particular integral, so that

$$u=\frac{\mu}{h}+\frac{\cos\phi}{\mathrm{R}}+\frac{\mu}{c^2\mathrm{R}^2}(\cos^2\phi+2\sin^2\phi).$$

Substituting $r\cos\phi=x$, $r\sin\phi=y$ and $u=\dfrac{1}{r}$, we get

$$x=\mathrm{R}-\frac{\mu}{h^2}\,.\,\mathrm{R}\sqrt{x^2+y^2}-\frac{\mu}{c^2\mathrm{R}}\,\frac{x^2+2y^2}{\sqrt{x^2+y^2}}.$$

The asymptotes are found by taking y very large as compared to x, then

$$x=\mathrm{R}-\frac{\mu}{h^2}\,.\,\mathrm{R}\,(x\pm y)-\frac{\mu}{c^2\mathrm{R}}(\pm 2y).$$

As $h=c$. R, the maximum deflexion given by the angle between the two asymptotes is

$$\epsilon=\frac{6\mu}{c^2\mathrm{R}}.$$

7. It cannot be denied that there is one serious fallacy underlying the relativity value. As a multiple body problem is insoluble in Relativity, it has been conveniently assumed that for purposes of its gravitational effect the whole mass of the Sun is concentrated at its centre. This was, of course, true for Newton's inverse square law, but may not hold for an altogether different law. When the necessary correction is made the relativity value may fall much below its supposed value.

8. It has been shown by me* that for the approximate law of attraction $-\frac{\mu}{r^2}-\frac{3\mu h^2}{c^2}\cdot\frac{1}{r^4}$, to which the equations for light would approximate to the first order, the attraction of a sphere of radius a and uniform density will be

$$-\frac{\mu}{r^2}-\frac{3\mu h^2}{c^2}\cdot\frac{1}{r^4}\left(1-\frac{1}{5}\cdot\frac{a^2}{r^2}\right).$$

The deflexion of light will be

$$\frac{\mu}{c^2\mathrm{R}}\left(2+4-\frac{16}{25}\cdot\frac{a^2}{\mathrm{R}^2}\right)=2''\cdot33,\text{ close to the edge.}$$

If the density increases towards the centre, the value will approach $2''\cdot61$.

---

* Proc. Nat. Acad. Sci. India. ix. pt. ii. pp. 48 and 50.

V. *A Redetermination of the Crystal Structure of Lithium.*

By Harald Perlitz, Professor of Physics at the University of Tartu,
and
Endel Aruja, Assistant in the Physical Laboratory at the Technical University of Tallinn *.

[Received May 11, 1939.]

[Plate III.]

Contents.

		Page
1.	Introduction	55
2.	Preparation of Specimens	56
3.	Diffraction Patterns	57
4.	The Lattice Constant	58
5.	Space-lattice and Space-groups	61
6.	Summary of Results	62
7.	References	63

1. *Introduction.*

Using the X-ray powder method, the lattice constant of lithium was determined by A. W. Hull [1], J. M. Bijvoet and A. Karssen [2], F. Simon and E. Vohsen [3], and E. Posnjak [4]. These constants, the numbers of α-reflexions that could be observed, and some other significant data are collected into Table I. As may be seen from the table, the lattice constant has been determined to only three significant figures, and generally only four moderately diffracted α-reflexions have been recorded. The only exception is Hull's first photograph, on which nine α-reflexions were recorded. From these Hull supposed two strong ones to be due either to an impurity or to the admixture of a second form of lithium, the remaining seven lines being consistent with, although not including all the lines of, a body-centred cubic lattice. It is the purpose of the present paper to give the results of a redetermination of the crystal structure of lithium based on ten α-reflexions ranging from 18° to 79°, a redetermination leading to four significant figures for the lattice constant and to suitable space-groups for the crystal symmetry.

---

* Communicated by the Authors.

2. *Preparation of Specimens.*

The lithium used was commercial *lithium metallicum*, and was said to be about 98 to 99 per cent. metallic, the rest being oxide, some silicon dioxide, and traces of iron. On analysis, however, it was found to contain also 0·9 per cent. sodium, but no potassium, calcium, barium, and aluminium *. Conveniently sized, freshly cut lumps of lithium were immersed in molten paraffin and heated beyond 186° C., the melting temperature of lithium. Then the liquid lithium was sucked into capillaries of a lithium-beryllium-borate glass [5], which previously to that had been freed from moisture by boiling in petroleum. The sucking was performed either at about 190° C. or 210° C. The capillaries used were

TABLE I.

Data of Powder Photographs of Lithium.

Authority.	Hull.	Bijvoet & Karssen.	Simon & Vohsen.	Posnjak.
a	3·50 A.	3·50 A.	3·50 A.†	3·51 A.
n	7+2 3	4	4	4
θ	28° 14½°	67½°	39·3°	—
hkl	420 211	022	022	—
λ	Mo–K	Cr–K	Cu–K	Mo–K

a=lattice constant.
θ=glancing angle of highest reflexions.
n=number of α-reflexions.
hkl=indices of highest reflexions.
λ=radiation used.

of 0·05 mm. wall thickness, and all but one of circular section from 0·5 to 0·9 mm. inner diameter; the remaining one was of oval section, its longest and shortest diameter being 1·6 mm. and 1·2 mm. respectively. The sucking having been done, all specimens in circular capillaries were quenched in paraffin oil kept in a salt and ice mixture at −20° C. Finally,

* For these statements we are indebted to Mr. H. Tamm, of the Department of Chemistry, who kindly performed the analysis for us.

† To facilitate comparison with the values at room temperature this value has been computed by the present authors from a=3·46 A., given by Simon and Vohsen as lattice constant at 100° K.

both ends of every circular capillary were carefully sealed with bees-wax. Until they were fixed to the camera specimen-holders all specimens were stored in paraffin oil at room temperature. After being fitted to holders every specimen was dipped once more into paraffin oil, to have its capillary coated with a thin film of oil impermeable to moisture. The specimen in the oval capillary was treated differently. After the sucking it was re-immersed in molten paraffin, quickly withdrawn, and rapidly cooled down to room temperature in a jet of air, thus preventing undue recrystallization. This capillary became coated with a thin protective layer of solidified paraffin impermeable to moisture. The specimens treated both ways kept their bright metallic lustre during exposures, but the general scattering and the additional lines due to paraffin oil were much fainter than the general scattering and additional lines due to solidified paraffin.

3. *Diffraction Patterns.*

Three cylindrical cameras, 57·4, 64·0, and 114·1 mm. in diameter respectively, were used, and ten diffraction patterns were obtained with Cu–K radiation. On the diffraction patterns, of which three are reproduced in Pl. III., two sets of reflexions are discernible :

(1) a first set consisting of more or less spotty reflexions, and

(2) a second one consisting of continuously blackened (not spotty) reflexions.

All the reflexions of the second set, but none of the first, tallied with reflexions due to either paraffin or paraffin oil or sodium. The presence of the rather well-developed diffraction pattern of sodium indicates that the amount of sodium, if any [6], which might be dissolved in lithium is considerably less than the whole amount of sodium present as contamination in our specimen of commercial lithium. Since, therefore, the amount of sodium, if any, which can be dissolved in lithium is considerably less than 0·9 per cent. by weight, or $0{\cdot}2_7$ atomic per cent., sodium is to be regarded as practically insoluble in lithium. If during exposure the specimens were only rotated in the usual way about their axes the reflexions of the first set were usually exceedingly spotty, as those in the upper photographs of Pl. III. (rotation photograph). But eventually less spotty reflexions, as those in the lowest photograph of Pl. III., could be obtained by shifting the specimens from time to time along their axes of rotation (rotation-translation photograph [7]), whilst uniformly blackened reflexions were obtained only with the specimen in the wide oval capillary. All the reflexions of the spotty sets were assigned to lithium, these sets containing up to 8 α-reflexions and 2 α_1-reflexions, ranging from glancing angle $\theta=18°$ to $\theta=79°$, and also the appropriate β- and α_2-reflexions.

4. *The Lattice Constant.*

The positions of the reflexions were read off under the microscope of a dividing machine, readings of the middles of reflexions being recorded. The positions of strong and not broad reflexions could be located within 0·05 mm. The effective radii of both the smaller cameras were computed by the method proposed by M. Straumanis and A. Jevinš [8], whilst the

TABLE II.

Rotation-translation Photographs of Polycrystalline Commercial Lithium. Copper radiation of a Hadding tube operated from 13 to 16 hours at 34 kv. effective voltage and 30 ma.

No.		400		398	399	401	402	403
D		114·11		$63{\cdot}31_8$	$57{\cdot}13_7$	$63{\cdot}24_8$	$57{\cdot}13_7$	114·11
d		0·7		0·6	0·5	0·8	0·9	0·9
Q^2.	λ.	θ.	a.	a.	a.	a.	a.	a.
6	α	32·60°	3·4974	3·5061	3·5078	3·5044	3·5001	3·5068
8	α	38·44°	3·5005	3·5030	3·4957	3·5001	3·4992	3·5011
10	α	44·01°	3·5014	3·5034	3·5023	3·5003	3·5012	—
12	α	49·58°	3·5008	3·5019	3·4997	3·5013	3·5025	—
14	α	55·22°	3·5016*	3·5037	3·5050	3·4979	3·5007	3·5022
16	α	61·47°	3·4997*	—	—	—	—	—
18	α_1	68·67°	3·5011	3·5028	3·5008	3·4982	3·5013	—
20	α_1	78·88°	3·5035	—	3·5033	3·5025	3·5038	3·5032
20	α_2	(79·72°)	—	—	3·5024	3·5022	3·5025	—
$a_{extr.}$			$3{\cdot}502_1$	$3{\cdot}502_8$	$3{\cdot}502_8$	$3{\cdot}502_7$	$3{\cdot}503_0$	$3{\cdot}503_4$
$t°$			23	23	23	26	26	26
a_{20}			$3{\cdot}501_5$	$3{\cdot}502_2$	$3{\cdot}502_2$	$3{\cdot}501_5$	$3{\cdot}501_8$	$3{\cdot}502_2$

Arithmetical mean at 20° C. $a=3{\cdot}501_9\pm0{\cdot}000_2$ A.
No.=number of photograph.
D=effective diameter of cylindrical cameras in millimetres.
d=diameter of capillaries in millimetres.
Q^2=sum of squares of indices of reflexions.
λ=spectral line of Cu–K radiation.
θ=glancing angle in degrees.
a=lattice constant in Ångström units.
$a_{extr.}$=extrapolated lattice constant in Ångström units.
$t°$=temperature in degrees Centigrade.
a_{20}=extrapolated lattice constant reduced to 20° C.

radius of the largest camera was deduced from a series of equidistant fiducial marks on the circumference of its cylinder. The values of the

* In calculating this value, λ_{α_1} has been substituted for λ_α.

lattice constant derived from all the α-reflexions but the first two are listed in Table II. The first two α-reflexions, 011 and 002, were omitted, since they were so broad that it was impossible to determine the exact position of their median lines and since they are of little importance for calculating the lattice constant. In computing the lattice constant the following values for wave-lengths of Cu–K radiation were used :—

$$\lambda_{\alpha_1}=1{\cdot}537395 \text{ A. and } \lambda_{\alpha_2}=1{\cdot}541232 \text{ A } [9].$$

$$\lambda_{\alpha}=\tfrac{1}{3}\,(2\lambda_{\alpha_2}+\lambda_{\alpha_1})=1{\cdot}538674 \ [10].$$

The value of λ_α was used with all unresolved α-reflexions except reflexions $Q^2=14$ and $Q^2=16$ in photograph no. 400, the widths of which already exceeded appreciably the radius of the specimen investigated. In these last cases the value $\lambda_{\alpha 1}$ was used instead. To exclude systematic errors due to eccentricity of and absorption in the specimens the values of the lattice constants of Table II. were plotted as ordinates against $\sin^2\theta$ as abscissa and smooth lines were drawn and extrapolated to $\sin^2\theta=1$, as shown in fig. 2 [11]. Lattice constants read off at $\sin^2\theta=1$ were reduced to 20° C., assuming $56{\cdot}0\times10^{-6}$ as the thermal coefficient of the linear expansion of lithium [12], and the arithmetical mean of these reduced values, the value

$$a=3{\cdot}501_9\pm0{\cdot}000_2 \text{ A.}$$

is taken as the lattice constant of commercial lithium at 20° C.

With two atoms to the unit cube the density of lithium is found to be

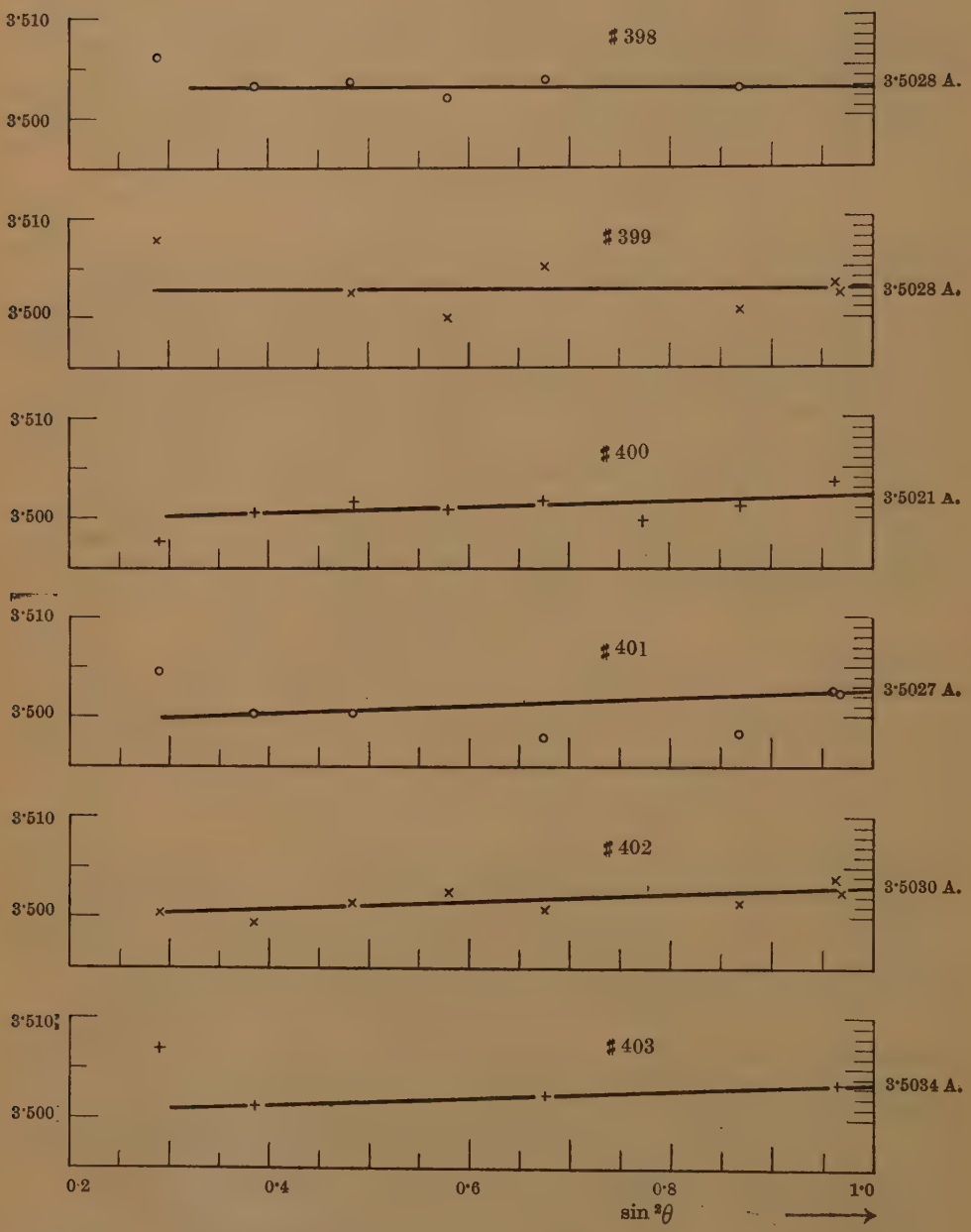
$$\rho_x=2\times6{\cdot}940\times1{\cdot}649\times10^{-24}/(3{\cdot}501_9\times10^{-8})^3$$
$$=0{\cdot}5330 \text{ g. cm.}^{-3},$$

in good agreement with $\rho=0{\cdot}534$ g. cm.$^{-3}$, the density of lithium in bulk [13]. That corroborates the inference that the amount of sodium, if any, which might be dissolved in lithium is considerably less than the amount of sodium present as contamination. It is therefore not to be expected that the lattice constant of pure lithium would be appreciably less than the value found for commercial lithium. In our case Vegard's law would be a sufficiently close approximation for calculating the lattice constant of pure lithium from those of commercial lithium and of the sodium admixed. Exaggerating the case by assuming only two-thirds of the sodium present to be in the elemental form and one-third wholly dissolved in the lattice of lithium, we have accordingly the relation

$$a=3{\cdot}501_9 \text{ A.}=0{\cdot}9991a_{\text{Li}}+0{\cdot}0009\ a_{\text{Na}}.$$

With $a_{\text{Na}}=4{\cdot}282_0$ A. [14], this yields $a_{\text{Li}}=3{\cdot}501_5$ A. for an underestimated value of the lower limit of the lattice constant of pure lithium. Therefore

Fig. 2.



Lattice constants of commercial lithium containing about $\frac{1}{4}$ atomic per cent. of sodium, plotted against $\sin^2\theta$, and extrapolated to $\sin^2\theta=1$.

it is to be concluded that the lattice constant of pure lithium lies in the interval

$$3{\cdot}501_5 \text{ A.} < a \leqslant 3{\cdot}501_9 \text{ A.},$$

being nearer to the upper limit than to the lower one.

5. *Space-lattice and Space-groups.*

From the sum of the squares of the indices of the reflecting planes in Table II. it is obvious that the diffraction pattern of lithium is that of a body-centred space-lattice. With a body-centred lattice containing two identical atoms in the unit cube, only the atomic positions 000 and $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ are compatible; but in this case the structure amplitude is either zero or twice the atomic scattering factor f. Therefore the proportion for calculating intensities I becomes

$$\mathrm{I} \propto \frac{1+\cos^2\theta}{\sin^2\theta \,.\, \cos\theta} \,.p.f^2 \,.\, \exp\left(-2\mathrm{B}\frac{\sin^2\theta}{\lambda^2}\right),$$

where p is the number of co-operating planes and the exponential factor the square of the temperature factor. In this formula no allowance for absorption has been made, since the significant product μr is for every specimen less than

$$r \,.\, \rho \,.\, \left(\frac{\mu}{\rho}\right) = 0{\cdot}05 \times 0{\cdot}534 \times 1{\cdot}10 \text{ [15]}$$
$$= 0{\cdot}029,$$

which is negligible. The atomic scattering factors f were taken from R. W. James and G. W. Brindley [16], and B was calculated from the formula

$$\mathrm{B} = \frac{6h^2}{mk\Theta} \cdot \left\{\frac{\Phi(x)}{x} + \frac{1}{4}\right\} \text{ [17]}.$$

With $\Theta = 363°$ K. as the characteristic temperature of lithium [18] the exponent B at 20° C. was found to be $\mathrm{B} = 3{\cdot}80 \times 10^{-16}$. The calculated intensities are compared with the visually estimated intensities of photographs no. 398, no. 399, and no. 384 in Table III. The estimates of both the former photographs coincide, but differ from the estimates of the last photograph, the difference being due to differences in peculiarities of reflexions on the photographs. The reflexions on both the former photographs are narrow but less uniformly blackened than those on the latter, which, on the contrary, owing to the thickness of the corresponding specimen, are very broad, so that the α-reflexion 022 and the β-reflexion 013 already overlap. But the correspondence of the calculated intensities with the visually estimated intensities is nevertheless such as to corroborate the assumption that lithium has a cubic body-centred lattice with two atoms located at 000 and $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ respectively.

Although occasionally, as may be seen from Table II., some of the weaker reflexions may be missing, there are no missing sets of reflexions, but, generally speaking, all the reflexions due to a body-centred cubic lattice are present. Therefore we may conclude that only one of the following space-groups, T^3—I23, T_d^3—I$\bar{4}$3m, T_h^5—Im3, 0^5—I43, and 0_h^9—Im3m is consistent with the crystal structure of lithium.

TABLE III.

Calculated and Estimated Intensities of Lithium.

λ.	Q^2.	hkl.	θ.	$I_{calc.}$	$I_{est.}$		
α	2	011	18·1°	252·7	10	10	10
α	4	002	26·1°	29·4	8	8	8
α	6	112	32·6°	46·1	9	9	9
α	8	022	$38{\cdot}4_2°$	10·0	5	5	—
$\alpha+\beta$	8+10	022+013	$38{\cdot}4_2°$; $38{\cdot}8_5°$	10·0+3·3	—	—	6
α	10	013	44·00°	9·22	5	5	5
α	12	222	49·55°	1·71	2	2	3
α	14	123	55·30°	6·68	4	4	5
α	16	004	61·50°	0·66	—	—	1
α_1	18	033+114	68·60°	2·33	3	3	4
α_1	20	024	79·00°	2·02	—	1	2

λ=spectral line of Cu–K radiation.
Q=sum of the squares of the indices of the reflexions.
hkl=indices of reflexion.
θ=glancing angle.
$I_{calc.}$=calculated intensities.
$I_{est.}$=estimated intensities from photographs nos. 398, 399, and 384 respectively.

In computing the intensities of α_1- and β-reflexions the proportion $I_\alpha : I_{\alpha_1} : I_\beta = 150 : 100 : 25$ was adopted.

In computing the overlappings of α- and β-reflexions the value $\lambda_\beta = 1{\cdot}38935$ A. and those stated in § 4 for λ_α and λ_{α_1} were used.

6. *Summary of Results.*

Rotation-translation photographs of polycrystalline commercial lithium, containing about $\frac{1}{4}$ atomic per cent. sodium, give diffraction patterns of both lithium and sodium, the reflexions of lithium ranging from $\theta = 18°$ to $\theta = 80°$. From these photographs we may conclude :

(1) That the amount of sodium, if any, soluble in lithium is far less than $\frac{1}{4}$ atomic per cent.

(2) That the lattice constant of commercial lithium at 20° C. is $a = 3{\cdot}501_9 \pm 0{\cdot}000_2$ A.

(3) That the lattice constant of pure lithium is in the interval: $3{\cdot}501_5$ A.$< a \leqslant 3{\cdot}501_9$ A.

(4) That the lattice is a body-centred cubic one, belonging to one of the following space-groups :—T^3—I23, T_d^3—I$\bar{4}$3m, T_h^5—Im3, 0^5—I43, 0_h^9—Im3m.

7. *References.*

(1) Phys. Rev. (2) x. pp. 689–691 (1917).
(2) Proc. Roy. Acad. Amsterdam, xxiii. pp. 1365–1370 (1922) ; *Versl. Akad. Wetens. Amsterdam*, xxix. pp. 1208–1213 (1921) ; *Recueil des Travaux Chimiques des Pays-Bas*, xlii. pp. 862–879 (1923).
(3) *Z. physik. Ch.* (A) cxxxiii. pp. 165–175 (1928).
(4) J. Physic. Ch. xxxii. pp. 358–359 (1928).
(5) A. Schleede and W. Wellmann, *Z. Kristallogr.* (A) lxxxiii. p. 149 (1932).
(6) G. Masing and G. Tammann, *Z. anorg. allg. Ch.* lxvii. pp. 189–190 (1910).
(7) E. Aruja and H. Perlitz, *Z. Kristallogr.* (A) c. p. 196 (1938).
(8) M. Straumanis and A. Jevinš, *Z. Physik*, xcviii. pp. 466–468 (1936).
(9) 'International Tables for the Determination of Crystal Structures,' p. 586. London : Bell and Sons, 1935.
(10) G. Kettmann, *Z. Physik*, liii. p. 206 (1929).
(11) G. Kettmann, *Z. Physik*, liii. p. 198 (1929) ; A. J. Bradley and A. H. Jay, Proc. Phys. Soc. London, xliv. p. 563 (1932).
(12) Landolt-Börnstein, 'Physikalisch-Chemische Tabellen,' 5th ed., 3rd supplementary volume, part 3, p. 2212. Berlin : J. Springer, 1936.
(13) Landolt-Börnstein, 'Physikalisch-Chemische Tabellen,' 5th ed., 3rd supplementary volume, part 1, p. 284. Berlin : J. Springer, 1935.
(14) E. Aruja and H. Perlitz, *Z. Kristallogr.* (A) c. p. 200 (1938).
(15) A. H. Compton and S. K. Allison, 'X-rays in Theory and Experiment,' p. 802. London : Macmillan and Co., 1935.
(16) Phil. Mag. xii. p. 104 (1931).
(17) 'International Tables for the Determination of Crystal Structures,' p. 570. London : G. Bell and Sons, 1935.
(18) N. F. Mott and H. Jones, 'The Theory of the Properties of Metals and Alloys,' p. 14. Oxford : at the Clarendon Press, 1936.

Laboratory of Theoretical & Technical Physics,
University of Tartu.
May 1939.

VI. *The Fabric Theory of Protein Structure.*

By Dorothy Wrinch, M.A., D.Sc., Lecturer in Chemistry, Johns Hopkins University, Research Fellow of Somerville College, Oxford *.

[Received May 1, 1940.]

The molecules of native proteins appear to have structures as definite as any known to chemistry, yet they contain thousands of atoms. This, the central fact in protein chemistry, requires explanation. It has already been shown that it may be explained by postulating, as the essential entity in native proteins, a class of atomic fabrics, since an atomic fabric, capable of forming closed structures by folding, yields certain closed structures only, which are of definite sizes and shapes [1], [2]. That this may be the correct explanation is suggested by the fact that it explains, not only the formation of individual coherent structures containing thousands of atoms, but also the non-random distribution of the molecular weights of such megamolecules. In suggesting that residue number, rather than molecular weight, is the appropriate basis of classification, it, in fact, explains how proteins having very different complements of the various types of residue can yet differ little in their molecular weights [3].

To demonstrate that the idea of a cage structure for proteins is independent of any special class of fabric (*e. g.* the cyclol fabrics [1], [2] or the hydrogen bond fabrics [4]), and as a further suggestion as to a form which the characteristic protein fabric may take, a new class, having the composition appropriate for proteins, is now suggested for consideration. A fragment of the densest member of the class is shown in the figure. The patterns are built upon the hypothesis that some of the amino acid residues in native proteins function, not in the classical keto form

$$-CO-C_{\alpha}HR-NH-$$

but in the enol form

$$\begin{array}{l} \\ \\ \rangle C(OH)-C_{\alpha}R-NH-, \end{array}$$

and that some of the imino acid residues similarly function, not in the classical keto form

$$-CO-C_{\alpha}H-N-$$

---

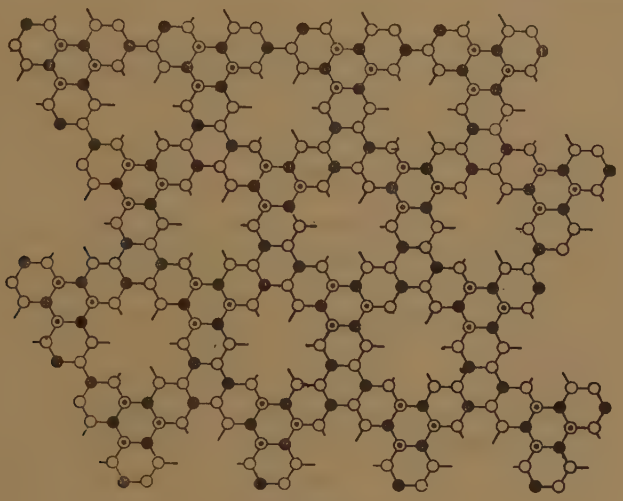
* Communicated by the Author.

(where the C_α and N atoms are members of a five-membered ring) but in the corresponding keto form

$$>C(OH)-\overset{|}{C_\alpha}-N-.$$

In the original cyclol hypothesis, a lactam-lactim transformation was suggested: after the same manner, a keto-enol transformation is suggested in the present hypothesis. As in the original theory, the fabrics may be, but need not be, regarded as prototropic cyclizations[5] and polymerizations of certain linear peptides.

The fabric pictured in the figure has several striking characteristics.



●=C carrying R group.
⊙=C carrying OH group upwards.
○=C carrying OH group downwards.
○—, ○⁻=NH group.

1. Like the original cyclol fabric, it consists of "3-way" hexagons of atoms alternating with "2-way" hexagons. The former consist of carbons carrying R groups and OH groups, alternately: the latter of opposite pairs of carbons carrying R groups, carbons carrying OH groups, and nitrogens carrying hydrogens. At present the postulate of tetrahedral valency angles for the atoms will be adhered to strictly, and only crumpled hexagons will be considered. The trios of OH groups on one 3-way hexagon then emerge normal to its median plane: similarly, the C atoms in the R groups. The first trios lie on one side of the median planes, the second on the other. OH groups in neighbouring 3-way

hexagons lie on opposite sides of the fabric : similarly in the C_β atoms. In the particular fabric shown in the figure, a breakdown of the fabric into linear peptides in which all the C_α atoms have the same geometrical configuration necessarily results, in general, in the formation of hexapeptide rings. This curious finding is reminiscent of the important fact that there appears to be a fundamental difference in the behaviour of the hexapeptide ester of glycine and (*e. g.*) the tetrapeptide ester [6].

2. The distance between oxygens carried on a single 3-way hexagon is $1.63a$ (strictly $\frac{2}{3}\sqrt{6}.a$), if a be the length of the C—C bond in this ring. This distance may be compared with the distance between oxygens in 1, 8-dihydroxy-anthraquinone, which is presumably about 2·5 A. In this compound the existence of hydrogen bridges between oxygens has been inferred from spectroscopic evidence. Evidently if a is 1·53 A. or more (a reasonable supposition for this carbon ring), the oxygen separation is at least as great as in this case, and the formation of intramolecular OHO bridges in the fabric may be anticipated.

3. The disposition of nitrogens around the lacunæ is also favourable for the formation of intramolecular hydrogen bridges. For consider the case of phthalocyanine [7], which has its inner nitrogen atoms spaced on a rectangle of sides 2·65 A. and 2·76 A., those at the shorter distance apart being bridged by hydrogens. The distance between nitrogens around a lacuna is $0{\cdot}94\ (a+b)$, if the C—N bonds (at present assumed to be all of equal length) are of length b. If b is even as small as 1·39 A. (the shortest C—N bond without resonance so far found [8]), and a as small as 1·53 A., the NH . . . N distance is 2·75 A., which is certainly long enough for a hydrogen bridge, and so far as this issue is concerned b can be considerably shorter.

4. All amino acids except glycine have $C_\beta H_2$— or $C_\beta H_3$ group. The arrangement of these groups on a 3-way hexagon in the fabric under discussion is similar to that found for the trios of methyl groups carried by carbons alternating with nitrogens in (trimeric) acetaldehyde ammonia [9]. The distance apart of the C_β atoms bears the same relation to the distance apart of these three methyl groups in this structure as the length of the C—C in the fabric under discussion bears to the C—N bond length in this crystal—it is thus presumably greater.

5. The arrangement of the C_β atoms of R groups is no longer (as in the original cyclol fabric) dorsiventral. This feature of the hypothesis may prove to be of importance in explaining how a protein, which in the native state is soluble, yet forms mono-layers of great insolubility [10].

Details of the cage structure formed by the fabric will shortly be published.

References.

(1) Wrinch, 'Nature,' cxxxvii. p. 411 (1936) ; Proc. Roy. Soc. London, A, clx. p. 59 (1937).
(2) Wrinch, Proc. Roy. Soc. London, A, clxi. p. 505 (1937) ; Cold Spring Harbor Sym. Qual. Biol. p. 6 (1936).
(3) Svedberg, ' Nature,' cxxxix. p. 1051 (1937).
(4) Wrinch and Jordan Lloyd, ' Nature,' cxxxviii. p. 758 (1936).
(5) Astbury and Wrinch, ' Nature,' cxxxix. p. 798 (1937).
(6) Fischer, Ber. *Deutschen Chem. Gs.* xxxix. p. 2893 (1906).
(7) Robertson, J. C. S. 1935, pp. 615 and 1195.
(8) Albrecht and Corey, J. A. C. S. lxi. p. 1087 (1939).
(9) Moerman, *Z. Kristallog.* A, xci. p. 447 (1938).
(10) Langmuir, Cold Spring Harbor Sym. Qual. Biol. p. 6 (1938.)

VII. *Dark Bands in the Spectra of Double Diffraction Gratings.*

By R. A. Houstoun, D.Sc., University of Glasgow *.

[Received August 8, 1939.]

Summary.

When electrical oscillations set up ultrasonic vibrations in a quartz plate, they convert it into a diffraction grating. Such gratings have been called acoustic gratings; they give bright and clear images, but their resolving power is small. When examining the spectrum of monochromatic light that had passed twice through such a grating I noticed that the image of the slit was covered by black bands parallel to its edge. On looking up the literature of the subject I found that P. Cermak and H. Schoeneck (1) had already observed such bands in 1936 and that they had also produced them with optical diffraction gratings. This I was able to do in a novel and very striking manner. Cermak and Schoeneck made no attempt to explain the bands. This was done by R. Bär (2) in 1936, and his explanation is regarded as satisfactory by L. Bergmann (3). I am convinced as a result of my own experiments that Bär's explanation is not always applicable, and I have arrived at another explanation. I give here, first my own explanation, then an account of my experiments, and, finally, a discussion of Bär's views. Incidentally the bands seem important as a means of determining ultrasonic wave-length, and they are very easy to produce with optical gratings. Indeed, their measurement might be used as an exercise on the optical diffraction grating in laboratory courses for students.

Theory.

Fig. 1 represents two equal gratings set up parallel with the clear spaces opposite one another. Monochromatic light is incident in the direction of the arrow. Let l be the distance between the two gratings and consider the beam diffracted from one of the clear spaces at an angle θ. When it reaches the second grating its mid-ray is displaced a distance x given by

$$x = \tan\theta.$$

If this equals ne, where e is the grating space and n any integer, the beam will pass through; otherwise it will be partially or wholly obstructed.

---

* Communicated by the Author.

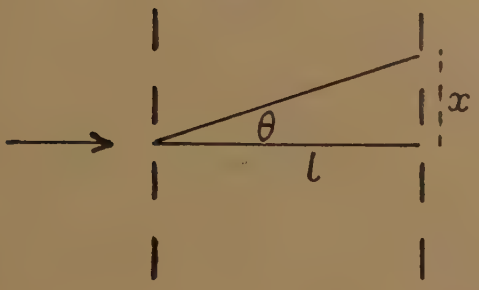
Suppose now that the incident light comes from a collimator with a wide slit; θ will have a range of values for some of which the light will get through and for some of which it will be stopped. Consequently, if the transmitted light is received by a telescope, the observer will see the image of the slit covered by dark bands. These bands are approximately equidistant, and their angular separation is given by

$$e=dx=l\sec^2\theta\,d\theta$$

or

$$d\theta=\frac{e}{l\sec^2\theta}.$$

Fig. 1.



It is not necessary to have two gratings. The rays may be reflected so as to go through the same grating twice, as is shown in fig. 2, which represents a glass plate with the grating ruled on the surface AB. The

Fig. 2.

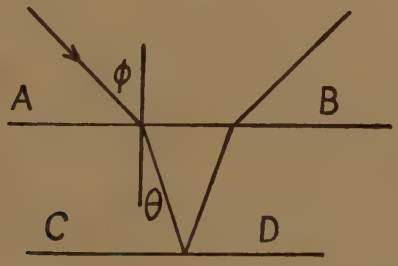


figure shows the formation of the direct image by internal reflexion at the surface CD, the light being simply refracted at the surface AB. Let l be the thickness of the plate and μ its refractive index, and let ϕ be the angle of incidence and θ the angle of refraction. Then we have

$$\sin\phi=\mu\sin\theta, \qquad \cos\phi\,d\phi=\mu\cos\theta\,d\theta,$$
$$x=l\tan\theta, \qquad dx=l\sec^2\theta\,d\theta,$$

and the width of a band is given by $2\,dx=e$. This leads to the expression

$$d\phi=\frac{\mu e}{2l}\frac{\cos^3\theta}{\cos\phi}$$

for the angular separation of the bands.

Experiments with Optical Gratings.

For testing the above formula I had at my disposal six " contact copy " diffraction gratings mounted on glass plates. The number of rulings per inch was in every case about 14,500, but it was not the same for any two of the gratings. Two of the gratings were mounted on optically worked glass and the other four on selected plate glass. One of them is in use in a students' laboratory course ; the others were its predecessors which had been withdrawn from use owing to being too much damaged by thumb-marks, etc. The telescope of a spectrometer was moved round until it was almost touching the collimator ; their directions were then about 23° apart. The slit was then illuminated with Na light, the six gratings placed in succession on the prism table, and a search made for the bands. They were obtained in every case almost immediately and, as one of the gratings was peeling off its glass mount, it may be stated emphatically that the bands are easy to obtain in this way. The grating rulings do not require to be parallel to the slit, but the angle between the directions of the telescope and collimator must not be greater than 45°, if the bands are to be distinct. The procedure was to keep telescope and collimator fixed and rotate the prism table so as to bring the direct image and the diffraction images on each side of it into the centre of the field. In one case the direct image and the first- and second-order images on each side of it were covered with perfect bands ; in the case of the other gratings one or more of the images were blurred or failed to appear. This I ascribed to irregularity in the grating space, because the bands could generally be improved by a little displacement of the grating on the prism table or by screening off part of its surface.

The angular separation of the bands was measured and found to be in agreement with theory in the case of the direct image. As an example of the nature of the agreement the following case may be taken :—

The measurements were made on the direct image. μ was found to be 1·529 by a total reflexion method. The thickness of the glass plate varied from 0·266 to 0·267 cm. The thickness of glass+film was 0·2765 cm. This was taken as l. $\phi=11°\ 44{\cdot}7'$, $\theta=7°\ 39'$. There were 14,543 rulings to the inch. When these numbers are inserted in the formula we obtain

$$d\phi=1{\cdot}643'.$$

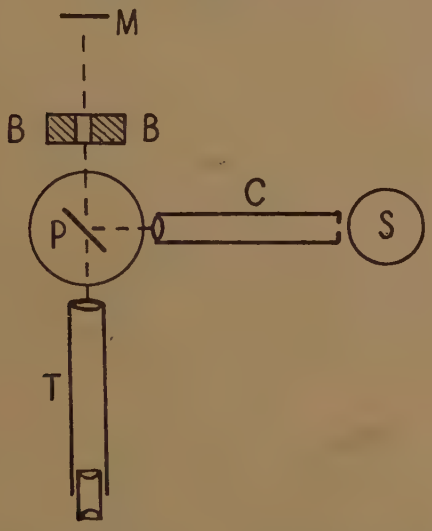
By actual measurement of twenty consecutive bands the mean value obtained was

$$d\phi = 1{\cdot}63'.$$

Experiments with an Ultrasonic Grating.

Fig. 3 shows the arrangement. The source of light S was a Hg arc. T and C are respectively the telescope and collimator of a spectrometer on the table of which is mounted a half-silvered plate P. The light is rendered parallel by the collimator and is reflected by the plate so as to pass through the piezoquartz to the mirror M. It is then reflected

Fig. 3.



back on its path to the telescope. The piezoquartz is placed between the brass blocks BB which form the condenser of the oscillator.

The dimensions of the quartz were $3{\cdot}52 \times 12{\cdot}975 \times 21{\cdot}95$ mm.[3] The light passed through it in the direction of the long side which was the direction of the optic axis ; the side of length 3·52 mm. was parallel to the electric axis. The large faces were parallel to 3″ of arc. The oscillator was a " push-pull " one of fixed capacity and the inductance was varied by sliding rods in parallel tubes. In this way the frequency could be changed continuously from 87 to 109 megacycles per second.

The light was diffracted on its first passage through the quartz, giving a first- and second-order image on each side, but under the conditions of the experiment these were faint and could be neglected, and it was necessary to consider only the diffraction images formed on the return

journey by the light that was undeviated on its first passage through the quartz. There were two of these on each side. The mirrors and surfaces of the quartz were of good optical quality, so all five images of the slit had sharp edges. Owing to the dispersion of the quartz, yellow and violet images were superposed on the green ; these could be eliminated by the use of a filter, but this was hardly necessary, as the green image was much brighter than the others. When the mirror M was close to the quartz, well-defined black bands appeared on the four diffraction images, but never on the central one. These bands were parallel to the edge of the image, and they became narrower when the mirror was moved away from the quartz. Their angular separation was the same on all four images.

According to the theory of the formation of these images given by Raman and Nath, which has been found to agree very well with experiment

Fig. 4.

on every point on which it has been tested, the effect of the first passage through the quartz is to impose on the wave-front a periodic change of phase. The crests and troughs are no longer straight lines but corrugated, as shown in fig. 4. If the wave returns exactly on its path, the changes of phase are doubled ; this has the effect of increasing the intensity of the diffraction images. If, however, the return wave is displaced half a grating space, the corrugations are removed from the wave-front and the diffraction images disappear. If the direction of the incident light is varied continuously, the diffraction image appears and disappears, or, in other words, if the image of the slit is a wide one it is covered with dark bands. The angular separation of these is given by the formula already proved,

$$d\theta = \frac{e}{l \sec^2 \theta},$$

if we put $\theta=0$ and replace l by $2l$, where l is the distance between the quartz and the mirror. The formula then reduces to

$$d\theta=\frac{e}{2l}.$$

First of all the variation of $d\theta$ with l was tested. The slit-width was kept constant and the distance of the mirror M from the crystal increased, the number of bands being measured for each position. The following table gives a typical set of observations, each result being the mean of two measurements.

Distance of mirror from centre of quartz (cm.).	Number of bands.	$\frac{\text{Number}}{\text{Distance}}$.
3·1	6·0	1·94
5·1	10·5	2·01
7·1	13·5	1·90
9·1	17·3	1·90

The agreement is good when we consider that the quartz is 2·195 cm. long.

Next the slit-width and position of the mirror M were kept constant and the frequency of the oscillator varied. This has the effectof changing e, the grating space. The value of e can be obtained very simply by the formula

$$\lambda=e\sin\theta,$$

where θ is the deviation of the first-order image and λ the wave-length of the Hg green line. For the resonance positions nearest the ends of the range θ had the values 35·2′ and 28·5′. The number of bands visible in these positions was 11·35 and 8·92, each number being the mean of three determinations. We can put angles for sines since they are so small. Theory requires that

$$\frac{35\cdot2}{11\cdot35} \text{ should equal } \frac{28\cdot5}{8\cdot92}.$$

The values of these ratios are 3·101 and 3·195.

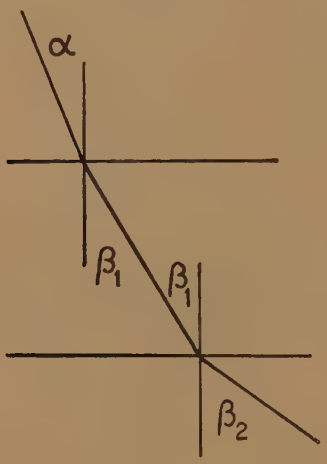
As a final test the value of e was calculated for one position of resonance from the formulæ $\lambda=e\sin\theta$ and $d\theta=e/2l$. The first method gave $e=5\cdot61\times10^{-3}$ cm. The second gave $5\cdot94\times10^{-3}$ cm., $5\cdot90\times10^{-3}$ cm., and $5\cdot35\times10^{-3}$ cm., different values of l being used in each case. Also the whole apparatus was thoroughly readjusted between the second and third determinations. The value obtained from the first formula is undoubtedly the more accurate one, but the second formula will be the better one in

the case of lower frequencies where θ is smaller and consequently more difficult to measure. In this case also l will be larger in comparison with the length of the crystal. I had the impression that the reference point from which l should be measured was moving about a little, but could not get certainty on this point.

Discussion of the Theory.

Bär considers two parallel gratings of equal grating space (fig. 5). Monochromatic light is incident on the first at angle α. It is diffracted at angle β_1 ; it then falls on the second grating and is diffracted at angle β_2.

Fig. 5.



For the first diffraction

$$\sin \beta_1 - \sin \alpha = n_1 \lambda / e ;$$

for the second

$$\sin \beta_2 - \sin \beta_1 = n_2 \lambda / e.$$

Therefore

$$\sin \beta_2 - \sin \alpha = (n_1 + n_2) \lambda / e,$$

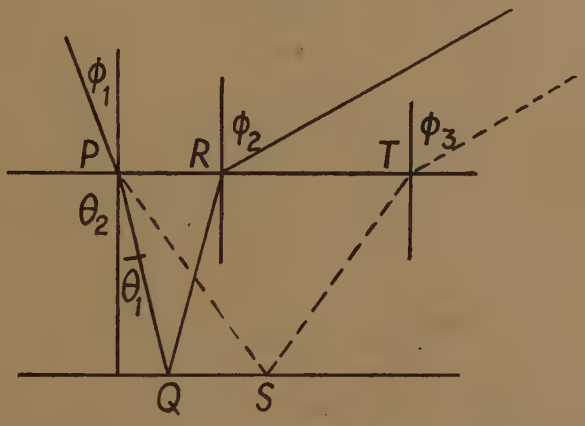
and all the beams for which $n_1 + n_2 = n$ arrive in the same spectrum of the nth order ; but they travel along different paths. They are consequently in a condition to interfere, the bands being of the same class as those referred to as Haidinger's or Mascart's fringes or the colours of thick plates. From this it will be seen that Bär ignores completely the function of the rulings as obstacles.

In my experiments with the optical grating in the direct-image position two images were formed, one by reflexion at the front surface and the

other by refraction at the front surface, reflexion at the back, and refraction at the front again. The two surfaces were slightly out of parallelism, consequently the images only partially overlapped. The bands appeared only on the image formed by internal reflexion, and they were naturally much clearer on the non-overlapping portion of this image. They cannot be due to interference in this case, for there is nothing for the image to interfere with, and I have no doubt whatever they are fully accounted for by my explanation.

We could, of course, get the same direction of emergence by giving the light a diffraction to the one side on its first meeting the grating and a diffraction to the other on its meeting the grating a second time, so as to make $n_1+n_2=0$ in Bär's formula. But the image produced in this manner would be faint ; I have found no indication of it whatever.

Fig. 6.



The bands produced by the grating in the direct-image position are very interesting, because they indicate that the disturbance to the wave-front due to its passage through the grating has not passed off before the wave meets the grating a second time. We have here two gratings interfering with one another. It has, of course, always been obvious that one grating would interfere with the action of another, because, if two equal gratings are in contact, they act as a single grating. But I know of no attempt to measure the distance within which this interaction takes place.

Let us now consider the optical grating in the first-order position (fig. 6). ϕ_1 is the angle of incidence. A ray is diffracted externally so as to form part of a first-order image. It is not shown in the figure. Light is refracted at P, reflected at Q, and diffracted at R, and light

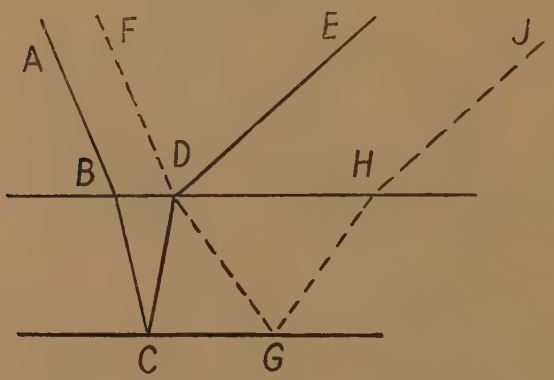
is diffracted at P, reflected at S, and refracted at T. If we write down the equations between the angles

$$\sin \phi_1 = \mu \sin \theta_1, \qquad \mu \sin \theta_2 - \sin \phi_1 = \lambda/e,$$
$$\sin \phi_2 - \mu \sin \theta_1 = \lambda/e, \quad \sin \phi_3 = \mu \sin \theta_2,$$

from which it follows that $\phi_2 = \phi_3$. Consequently the two emergent beams shown in the figure are in a condition to interfere and might be expected to give bands.

As a matter of fact well-defined bands are given both by the first-order and second-order diffraction images produced in this way. Owing to the non-parallelism of the surfaces we see the image formed by diffraction at the front and the image formed as indicated in fig. 6 partially superposed

Fig. 7.



on each other; the image diffracted at the front is easy to recognize, as it is always free from bands. The other always shows bands, and the separation of these bands in all four diffraction images is not much different from its value for the corresponding direct image. But these bands cannot be explained without taking into account the function of the rulings as obstacles.

In the passage already cited Bergmann states that Bär's explanation has been anticipated by Lommel[1], and that Bär makes no reference to Lommel's work. But Lommel's explanation is different from Bär's. It is illustrated in fig. 7; BH is the grating and CG a parallel reflecting surface. We consider a diffracting particle on the surface of the grating at D. A ray AB is refracted at B, reflected at C, and diffracted at D along DE; a parallel ray FD is diffracted along DG, reflected, at G, and refracted at H. The two parallel emergent rays DE and HJ are in a condition to interfere. The path differences are the same as in fig. 6, though

the insistence on the same diffracting particle at D recalls the explanation of Newton's diffusion rings. But Lommel, in addition, considers the effect of the rulings as obstacles; this is not easy, and he requires thirty pages in the *Annalen* to show that the theory is in agreement with his experiments.

Curiously enough, neither Lommel nor Cermak and Schoeneck studied the easy case—that of the direct image. Nor did they use the back surface of the grating as reflecting surface, but employed a separate mirror. In this way they added greatly to the difficulty of their experiments without any corresponding gain, for the bands are bright enough without the assistance of a silver reflecting surface, at least if one of the modern sodium laboratory lamps is used.

I am in agreement with Lommel's theoretical treatment, but consider it too involved to afford an easy comparison with experiment. It is possible, however, to treat the matter from a general standpoint. The chief results of experiment are that on each image the bands are always equally spaced, and that on each image they are all nearly equally sharp; there is no periodic variation in the sharpness as we travel across the image. Also the angular separation of the bands in the first- and second-order images does not vary much, not more than 10 per cent. either way in any of the cases investigated, from what it is in the direct image. There are two factors involved in the theory, the obstacle effect and the path difference. I think that the experiments indicate that the obstacle effect is the dominating one, because the results in all cases do not differ much from the results for the direct image, where there is no path difference at all.

Consider now fig. 7. Let DH$=x$ and BD$=x'$. Let us suppose that the inclination of the ray FD alters, D being kept fixed. Then the points H and B move along BH and x and x' change. As these points move from clear spaces to opaque intervals, the fraction of light transmitted will change. We can represent it approximately by $\cos^2 nx$ and $\cos^2 n'x$, where n and n' are constant, provided x and x' do not vary much.

Consider now the two rays DE and HJ. They have amplitudes $\cos^2 nx$ and $\cos^2 n'x$. Let us assume that the phase difference is δ. The intensity of their resultant is

$$\cos^4 nx+\cos^4 n'x+2\cos^2 nx\cos^2 n'x\cos\delta.$$

We assume now that the point D is moved along BH, keeping nx and $n'x'$ fixed. The value of δ will alter, since BH and CG are by no means plane parallel, taking all possible values. We can therefore write the intensity as

$$\cos^4 nx+\cos^4 n'x.$$

Now in all the cases investigated these two cosines vary at nearly the same rate with change in the inclination of FD, and $n'x'$ gives also the

rate of variation for the case of the direct image. So the fact that the angular separation of the bands remains always much about the same is explained. But the formula above should give a periodic fluctuation in the brightness of the bands, as the cosines go in and out of phase. There is some fluctuation, and it is masked to some extent by stray light, but there is not as much as we would expect on this line of reasoning. But I have not been able to devise any simple method of treating the problem more accurately.

Bär obtains the same formula as I do, namely, $d\phi = e/2l$, for the angular separation of the bands on the first-order diffraction image formed by the ultrasonic grating. Cermak and Schoeneck also showed experimentally that in this case $d\phi$ was proportional to $e/2l$. Bär only obtains half the separation that I do, and that I found, for the second-order diffraction image formed by the ultrasonic grating. But by choosing different experimental conditions it may be possible to detect the whole series of bands indicated by his theory.

The expenses of this investigation were defrayed by the Carnegie Trust for the Universities of Scotland.

References.

(1) P. Cermak und H. Schoeneck, *Ann. d. Phys.* xxvi. p. 465 (1936).
(2) R. Bär, *Helv. Phys. Acta*, x. p. 130 (1937).
(3) L. Bergmann, ' Ultrasonics and their Scientific and Technical Applications,' p. 75. English translation.
(4) E. Lommel, *Ann. d. Phys.*, Ergänzungsband 8, p. 82 (1878).

August 1939.

VIII. *Notices respecting New Books.*

American Mathematical Society Semicentennial Publications (in two volumes). (The Collegiate Press, George Banta Publishing Company, Menasha, Wisconsin.)

The present handsome volumes bear the sub-titles respectively :—

Vol. I. " A Semicentennial History of the American Mathematical Society, 1888–1938, with Biographies and Bibliographies of the Past Presidents," by Raymond Clare Archibald (262 pages).

Vol. II. " Semicentennial Addresses of the American Mathematical Society " (315 pages).

The detailed titles of the various chapters in Vol. I. are as follows :—

I. Introductory. The Society, 1888–1930.
II. The Society, 1921–1938.
III. Financial Affairs. Special Funds.
IV. The Bulletin.
V. The Transactions.
VI. Colloquium Lectures and Colloquium Publications.
VII. The Chicago Group and the Chicago Section.
VIII. Annual Meetings.
IX. Summer Meetings.
X. Josiah Willard Gibbs Lectureship.
XI. The Library.
XII. The Council and Board of Trustees.
XIII. The Secretaries.
XIV. The Vice-Presidents.
XV. The Presidents.

The following gives the contents of Vol. II. :—

I. Fifty years of algebra in America, 1888–1938, by E. T. Bell.
II. Algebraic aspects of the theory of differential equations, by J. F. Ritt.
III. The historical background of harmonic analysis, by Norbert Wiener.
IV. Recent developments in the calculus of variations, by E. J. McShane.
V. Recent trends in geometry, by T. Y. Thomas.
VI. The sphere in topology, by R. L. Wilder.
VII. Dirichlet problems, by G. C. Evans.
VIII. Hydrodynamic stability, by J. L. Synge.
IX. Fifty years of American mathematics, by G. D. Birkhoff.

Fifty years of organized and effective mathematical work in America have been accomplished, and the American Mathematical Society has celebrated this accomplishment by the issue of these two large volumes, the contents of which with the respective authors have been set forth in detail above. The dominant keynote throughout both the books is that the cultivation and

development of the higher mathematics in all its aspects is a work of national importance. It has therefore been thought worth while to set forth the history of the Society, its officers and activities in almost complete detail, and to give some account of the reactions of mathematical thought as between the Old World and the New. It is no exaggeration to say that the books throb with vital interest from start to finish, whether the subject under discussion be the development of some mathematical branch, or the personal career of a president. There is no uniform method of treatment, but each author has been allowed to present his theme according to his own personality and outlook.

The idea of the foundation of a mathematical society in America came directly from England, and the sequence of ideas that led Thomas S. Fiske to initiate the movement are of great general interest. Fiske had been taught in America by G. L. Rives, a Cambridge fifth wrangler, and acting on his advice he came to England to complete his studies. He was armed with letters of introduction from Rives to Cayley, Glaisher, Frost, Forsyth, and George Darwin. He has himself put it on record that " scientifically I benefited most from the instruction and advice of Forsyth and from my reading with Dr. H. W. Richmond, who consented to give me private lessons. However, from Dr. J. W. L. Glaisher, who made of me an intimate friend, who spent with me many evenings in heart to heart conversations, who took me with him to meetings of the London Mathematical Society, and who entertained me with gossip about scores of contemporary and earlier mathematicians, I gained more in a general way than from anyone else . . . . On my return to New York I was filled with the thought that there should be a stronger feeling of comradeship among those interested in mathematics, and I proposed to my classmates and friendly rivals, Jacoby and Stabler, that we should try to organize a local mathematical society."

The following document was issued over the signatures of these three young men, all born in 1865 : " It is proposed by some recent students of the graduate school of Columbia College to establish a mathematical society for the purpose of preserving, supplementing, and utilizing the results of their mathematical studies. It is believed that the meetings of the society may be rendered interesting by the discussion of mathematical subjects, the criticism of current mathematical literature, and the solution of problems proposed by its members and correspondents. It is also intended that original investigations to which members may be led shall be brought before the society at its meetings . . . ."

On November 24th, 1888, the following six persons responded to the call : Professors Van Amringe and Rees, Messrs. Fiske, Jacoby, Stabler, MacLay, and the New York Mathematical Society was born. Out of this, in due course, developed the American Mathematical Society as we know it to-day.

The course was not always smooth. It had its financial difficulties, and it had even its political difficulties in connexion with international meetings organized by the Society. In 1903 the mathematicians of America rated in order their eighty leading men of research. It is interesting to note that our English mathematician, Frank Morley, whose professional life was spent in America, was placed seventh on the list. The debt owed by America to the English J. J. Sylvester during his brief sojourn in the United States is everywhere handsomely acknowledged. He founded the 'American Journal of Mathematics.'

America's interest in her distinguished mathematicians is certainly not impersonal. Detailed lives of the presidents of the Society are given in Vol. I. and form very interesting reading. Not only are the private and personal

biographies given, but sidelights into the social history of the United States at the various epochs are also set forth. Thus we are enabled to assess the progress of mathematics against the social background.

The final section in Vol. II. by Professor G. D. Birkhoff is particularly interesting, in that he passes in review the organized mathematical strength of the United States as a whole. He says, "there must be between two and three thousand mathematical teaching positions in our higher institutions, with an average salary which must certainly lie between two to three thousand dollars. We see in this way that there is probably a sum of about six millions of dollars which goes each year to the support of higher mathematics." The following by Birkhoff is also of such general interest that it deserves quotation verbatim: "Since the Great War salaries have been increased and the teaching burden has been reduced, at least in the better institutions. I remember talking some twenty years ago with the late J. C. Fields of Toronto about the status of professors throught the world; it will be recalled that Fields did more than anyone else to bring about the important International Mathematical Congress held at Toronto in 1924. He told me that, after making a special study of the facts, he had come to the conclusion that the American professor was the worst treated of all. At that time there was much in his contention, even though there were already in existence a number of American professorial chairs where the salary was good and the teaching duties not excessive. To-day there are many such positions. In this connection it may be well to mention the fact that Harvard University has been able to reduce the amount of teaching and tutorial routine of the regular mathematical staff to six hours a week, of which only three hours are devoted to more or less elementary mathematical instruction. Such a schedule gives to all concerned a notable opportunity to carry on mathematical research, and would be socially unjustifiable unless the highest standards of achievement were being maintained. Although such ideal conditions are impracticable at present except in a few fortunate institutions, it should be strongly emphasized that twelve hours of instruction a week (including at least one course of advanced grade) is about all that can be required if the best standards of scholarship are expected. Indeed, wherever possible, the hours of instruction should be reduced to not more than nine, and if there are heavy outside duties there should be a compensating diminution in teaching.

"But the situation has very definitely a complementary aspect. On our part there is an unescapable, deep responsibility to the nation which, somewhat unwittingly perhaps, has afforded us such splendid support. It is our duty to take an active and thoughtful part in the elementary mathematical instruction of our colleges, universities, and technical schools, as well as to participate in the higher phases. To these tasks we must bring a broad mathematical point of view and a fine enthusiasm. In so far as possible we must actively continue as competent scholars and research workers. Only by so doing can we play our proper part."

If this review has dealt mainly with the biographical and administrative aspects of the development of the Society, as set forth in these volumes, this does not underrate the great interest and value of the historical surveys of some of the most important branches of mathematics by leading American experts. It is to be hoped that other mathematical and scientific societies will follow the excellent example set by these volumes and issue from time to time an authoritative account of their own histories and developments. W. P. M.

Internal Constitution of the Earth. Edited by BENO GUTENBERG. [413 pp.] (McGraw-Hill, 1939. Price 30*s.*)

THE volume under review is No. VII. of a series of monographs on the physics of the earth ; these have been prepared by subcommittees of a large committee appointed in 1926 by the Division of Physical Sciences of the National Research Council, Washington, D.C., and the first six were published as Bulletins of that Council. The work has been shared among nine distinguished contributors, L. H. Adams, R. A. Daly, B. Gutenberg, H. Jeffreys, W. D. Lambert, J. B. Macelwane, C. F. Richter, C. E. van Orstrand, and the late H. S. Washington. The sixteen chapters that make up the book are independent contributions by their respective authors.

The data of geophysics are drawn from a large number of different branches of knowledge, such as physics, chemistry, geology in its various aspects, astronomy, geodesy and seismology. Of late years there has been a notable tendency for the workers in different parts of the subject to collaborate, and it has accordingly become increasingly difficult to divide up the exposition into separate chapters. Broadly speaking, it may be said that the writers of this monograph have raised far more difficulties than they have solved ; this, and the occasional divergences of opinion among the authors, indicate a healthy state of development. The fair-minded tone of the book bespeaks a creditable avoidance of undue interference on the part of the editor, and the occasional overlapping serves merely to emphasize the close interrrelation of the various topics dealt with.

The way in which the subject-matter is set out is noticeably different from chapter to chapter. This depends partly on the author, but mainly on the matter. The opening article, by Dr. Jeffreys, on the origin of the solar system, makes full use of the laws of physics, especially in the rejection of alternatives ; the difficulty here is not so much to decide which is the most likely theory as to find one single theory that will fit all the facts, and so far the only tenable hypothesis seems to be that the sun was originally a double star, and that a passing star collided with the sun's companion, carrying it away into space and leaving the sun with a family of planets. The relevance of this discussion to the subject of the monograph is that the present thermal state of the interior of the earth cannot be calculated without the knowledge of the temperature distribution at some known epoch, such as that of the solidification of the crust of the earth; it arises, too, when one seeks to explain the present configuration and structure of the continents and oceans.

Prof. Daly's chapter on the relevant facts and inferences from field geology illustrates how closely the interests of the geodesist, vulcanologist, petrologist, and laboratory experimenter are interwoven with those of the geologist and geophysicist, and it leads naturally to the discussion by L. H. Adams, Director of the Geophysical Laboratory, of the elastic properties of materials of the earth's crust. The compressibility of a number of different rocks has been determined at pressures up to about 12,000 atmospheres, and the measures of the elastic constants and the waves velocities calculated from them, at pressures up to 4000 atmospheres, are most valuable in a comparison with the velocities determined seismologically ; the reconciliation is by no means complete, for although seismology has confirmed the existence of a continental granitic layer as inferred by geologists, the composition of the subjacent layer (or layers), often called the " intermediate layer," is still a matter for further investigation. The chapter by the late H. S. Washington has been revised and edited by L. H.

Adams ; it deals with the crust of the earth in relation to its interior and is notable for its section on the composition of meteorites, which are held by many astronomers and geophysicists to provide samples of the sort of matter that forms the interior of the earth.

The rate at which heat is reaching the surface of the earth from the interior is at once found if the temperature gradient and conductivity of the rock are known ; a large amount of information concerning temperature gradients in wells and borings in all parts of the world has been collected by Van Orstrand in his article in this book, and this is suitably followed by an account by Gutenberg of theoretical researches on the cooling of the earth. The various attempts to calculate the temperature distribution in the earth have suffered for lack of a trustworthy estimate of the rate of conduction of heat to the surface ; conductivities of rocks vary so widely that it is essential for the conductivity to be known at the very place at which the temperature gradient is measured. Reliable measurements, by Benfield in Great Britain and by Bullard in Africa, have only recently been made—too recently for inclusion in this book ; four different curves obtained by different authors are given to show the variation of temperature with depth, and it would have facilitated comparison if, in addition to quoting references, explicit mention had been made of the data, including heat flow near the surface, on which the curves are based.

There are two interesting chapters by Prof. Gutenberg on " Forces in the Earth's Crust " and " Hypotheses on the Development of the Earth's Crust." These are rather controversial topics, and the summary is a fair one ; Gutenberg personally inclines to a modification of Wegener's hypothesis, and supposes that the continents did not break, but flowed apart, and that a connexion of continental material still exists between them across the bottom of the Atlantic and Indian Oceans. A feature of this book is the copious list of references given at the end of each chapter, and throughout the book one is filled with admiration for the wonderful grasp that Prof. Gutenberg has of the literature of the many topics in which he is interested.

Father Macelwane's chapter on the evidence derived from seismic sources is a long one, over 70 pages, and the bibliography runs to 222 entries ; without some classification such a list is rather unwieldy, but in many cases the title of the paper is a guide. The author has made an excellent digest of the enormous amount of seismological work done in comparatively recent years, and in this respect the chapter is not unduly long. What seems quite unnecessary is the inclusion of many pages of tables of velocities of surface waves across various regions of the earth's surface ; as they are collected from a small number of sources there is no obvious reason for reprinting the values in a book where important researches have often to be touched on in a short paragraph only. Macelwane is inclined to object to the use of the term " Rayleigh waves " as applied to one constituent of the surface waves of an earthquake on the ground that " the theory proposed by Lord Rayleigh is not adequate to describe them "; this is rather a trivial objection to make. The outstanding importance of Rayleigh's work was his discovery that, in addition to the two types of wave already known to be propagated through the interior of a solid, there could exist a wave whose amplitude is sensible mainly near the surface of a solid ; the motion of the particles in such a wave is elliptic, much as in a wave on water. The extension of the theory from Rayleigh's ideal case of a uniform solid with a plane boundary to cover heterogeneous media is quite straightforward, although mathematically complicated, and the only new feature is that heterogeneity introduces dispersion. On the subject of surface waves the writer is rather

pessimistic, and he does not point out that to separate the two types of surface waves it is necessary to select shocks from which the surface waves arrive in azimuths corresponding to the orientations of the instruments ; nor does he even mention group-velocity, although it is to dispersion that the form of the long wave phase of a seismogram is chiefly due. Yet to dismiss the chapter with these criticisms would convey an unfair picture of the author's work. To arrange and classify such a vast bulk of research is a forbidding task, calling for much labour added to a wide knowledge ; the arrangement of the sub-sections, such as evidence from near earthquakes, from surface waves, from deep-focus earthquakes, conclusions concerning the rock mantle, shows good balance and discrimination, and the evidence for such controversial matters as the alleged propagation of S waves through the core (which, in fact, the author's own researches have sought to establish) is presented fairly and with due caution.

A short chapter dealing with the geographical distribution and mechanism of deep-focus earthquakes, by Gutenberg and Richter, follows logically on the seismological chapter, and the same authors contribute a chapter on " Structure of the Crust, Continents and Oceans," in which the evidence is again mainly seismological, but to some extent gravitational. There follows a chapter on the density, gravity, pressure and ellipticity in the interior of the earth, written by Walter D. Lambert, who is particularly well qualified for the task. The monograph concludes with chapters by Gutenberg on the elastic constants in the interior of the earth, and on viscosity, strength, and internal friction in the interior, and, finally, a general summary of the state of the earth's interior.

As an example of collaboration the book reaches a high standard ; it achieves coherence without sacrificing the individuality of the experts who constitute the editorial panel. To them, and in particular to the editor, geophysicists are greatly indebted. It is safe to say that there is no reader who will not learn something from this book : there are many who will learn much.

ROBERT STONELEY.

Vowel Sounds in Poetry. By M. M. MACDERMOTT. (Kegan Paul, Trench, Trubner & Co. Price 5*s*.)

SCIENCE is gradually expressing the technique of the arts in terms of its own symbolism, and the spoken language cannot escape being picked to pieces in this way. The author of this book (one of the Psyche monographs) sets out to do this for poetry, with some success. Of course, it would be impossible within the limits of a text which a publisher would consider, to set down all the analyses which this painstaking author has made, but he gives us enough to make a good case for his main thesis, which is, that the timbre of the assemblage of vowels, which gives to a poem its characteristic music, is a matter primarily of the relative occurrence of front and back vowels. Further, that the tone-colour is greatly affected by the relative frequency of vowels and consonants.

The analyses of poems and the methods adopted in the analysis occupy roughly half the book. The rest is devoted to a useful summary of modern work on the phonetics of the English vowels.

E. G. R.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 2.

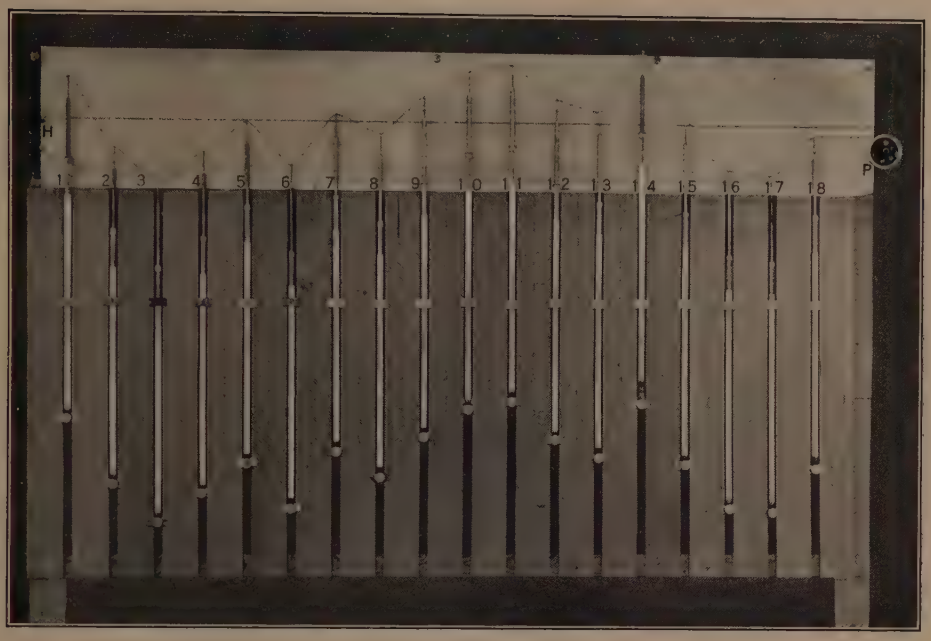


Fig. 3.

Fig. 4.

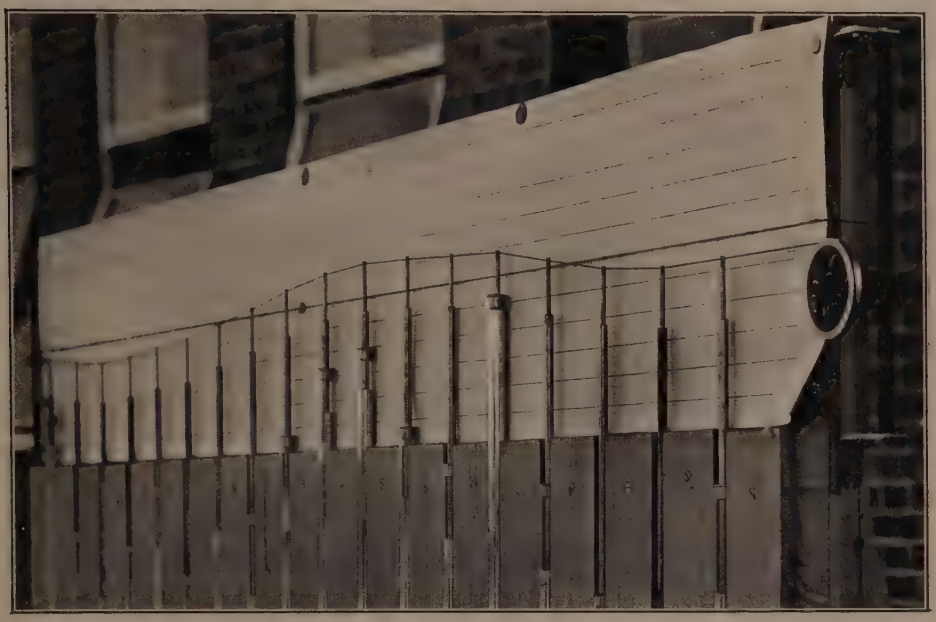
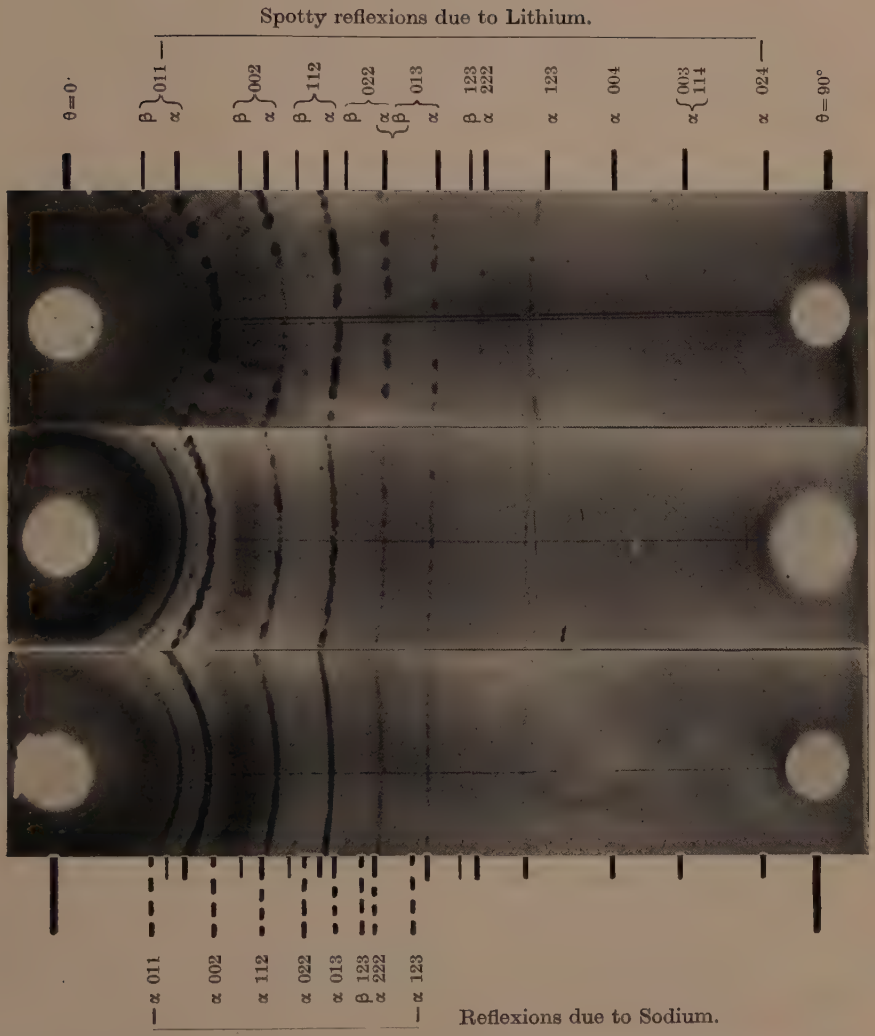



Fig. 5.



Photographs of polycrystalline commercial lithium containing about $\frac{1}{4}$ atomic per cent. sodium. The middle and lower photograph are of the same specimen, but in the middle photograph the specimen was rotated only during exposure (rotation-photograph), whilst in the lower photograph it was rotated and shifted along the axis of rotation (rotation-translation photograph).

CONTENTS OF No. 199.

Page

IX. Three-dimensional Vortex Motion in a Viscous Fluid. By Sydney Goldstein 85

X. Finite Complexes in Crystals: a Classification and Review. By A. F. Wells, M.A., Ph.D., Crystallographic Laboratory, Cambridge..... 103

XI. Lorentz Transformation and Hamilton's Quaternions. By Otto F. Fischer..... 135

XII. The Sine Relation and Minimum Ellipticity in Foucault's Pendulums with Copper Suspensions. By B. Dasannacharya, M.A., Dr.Phil., F.Inst.P., Professor of Physics, Benares Hindu University, Benares, India, and Balram Singh Gautom, M.Sc. (Benares) 151

XIII. An Analogy between the Theories of Potential and Vibrations. By Harold Jeffreys 161

XIV. Some Integral Representations of the Associated Legendre Functions. By A. Erdélyi 168

XV. Notices respecting New Books :—

Mr. Gerhard Tintner's The Variate Difference Method 172

Messrs. H. Le Boiteux and R. Boussard's Elasticité et Photoélasticimétrie 172

Dr. E. G. Richardson's Sound 172

*** It is requested that all Communications for this Work may be addressed, post-paid, to the care of Messrs. Taylor and Francis, Ltd., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

IX. *Three-dimensional Vortex Motion in a Viscous Fluid.*

By Sydney Goldstein *.

[Received July 20, 1939.]

Largely in order to illustrate the mechanism of the increase of vorticity in turbulent flow due to extension of the vortex-lines, Taylor and Green † calculated the development with time of the motion of an infinite incompressible fluid with velocity components u, v, w, given initially by

$$\left.\begin{aligned} u &= A\cos ax \sin ay \sin az, \\ v &= -A\sin ax \cos ay \sin az, \\ w &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

In their solution, u, v, w, and the pressure p are found as power-series in the time t. The motion is periodic in x, y, and z for all time, with wave-lengths $2\pi/a$ in each of the x, y, and z directions. Average values are taken throughout a fundamental parallelepiped, and the mean square vorticity $\overline{\omega^2}$ was found by Taylor and Green as far as the term in t^5.

In order that the effect in turbulent flow of the extension of the vortex-lines should be illustrated, the vortex motion considered must be three-dimensional and possess a definite scale. It is difficult to find a motion possessing these characteristics for which the calculations would be easier than for the motion (1). On the other hand, examination of the equations suggests that a solution in powers of the Reynolds number is preferable to one in powers of the time; and the solution obtained in this way is found to contain all the information that is obtained from the solution in powers of the time, and also additional information, especially for small Reynolds numbers.

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0; \quad \ldots\ldots\ldots \quad (2)$$

the equations of motion are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u, \quad \ldots \quad (3)$$

---

* Communicated by the Author.

† Proc. Roy. Soc. A, clviii. pp. 499–521 (1937).

and two similar equations, ρ being the density, and ν the kinematic viscosity, assumed constant. When we differentiate the equations of motion with respect to x, y, and z in order, add the results, and use the equation of continuity, we find the following equation for the pressure:

$$-\frac{1}{\rho}\nabla^2 p=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right) \quad . \quad . \quad . \quad (4)$$

In solving the equations with the initial conditions (1) we may either, following Taylor and Green, use (4) to find the pressure at each stage, or we may eliminate the pressure from the equations of motion and so obtain equations for the vorticity components, solve for the vorticity components, and at each stage find the components of the vector potential from the vorticity components and find the velocity components from the vector potential. The amount of work involved seems to be about the same in the two methods, and the former method has been adopted, since it provides also a formula for the pressure.

We denote by ξ, η, ζ the vorticity components, and reduce all the equations to non-dimensional form by writing

$$\left.\begin{aligned} &u'=\frac{u}{\mathrm{A}}, \quad v'=\frac{v}{\mathrm{A}}, \quad w'=\frac{w}{\mathrm{A}}, \quad p'=\frac{p}{\rho \mathrm{A}^2},\\ &x'=ax, \quad y'=ay, \quad z'=az, \quad t'=\mathrm{A}at,\\ &\xi'=\frac{\xi}{a\mathrm{A}}, \quad \eta'=\frac{\eta}{a\mathrm{A}}, \quad \zeta'=\frac{\zeta}{a\mathrm{A}},\\ &\mathrm{R}=\frac{\mathrm{A}}{\nu a}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Then equations (2), (3), and (4) become

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+w'\frac{\partial u'}{\partial z'}=-\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{R}}\nabla^2 u', \quad . \quad . \quad . \quad (7)$$

and two similar equations, with

$$\nabla^2=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$-\nabla^2 p'=\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial v'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2+2\left(\frac{\partial v'}{\partial x'}\frac{\partial u'}{\partial y'}+\frac{\partial w'}{\partial y'}\frac{\partial v'}{\partial z'}+\frac{\partial u'}{\partial z'}\frac{\partial w'}{\partial x'}\right). \quad . \quad . \quad . \quad (9)$$

The initial conditions (1) are

$$\left.\begin{aligned} u' &= \cos x' \sin y' \sin z', \\ v' &= -\sin x' \cos y' \sin z', \\ w' &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (10)$$

at $t'=0$.

For small values of the Reynolds number the quadratic terms may be neglected, and we have to solve the equations

$$\frac{\partial u'}{\partial t'} = \frac{1}{\mathrm{R}} \nabla^2 u' \quad \ldots\ldots\ldots \quad (11)$$

and two similar equations, with the initial conditions (10). The solution is

$$\left.\begin{aligned} u_0' &= e^{-3t'/\mathrm{R}} \cos x' \sin y' \sin z', \\ v_0' &= -e^{-3t'/\mathrm{R}} \sin x' \cos y' \sin z', \\ w_0' &= 0. \end{aligned}\right\} \quad \ldots\ldots \quad (12)$$

The solution of the equations of motion is represented by (12) for any Reynolds number for sufficiently small values of t', since at the beginning of the motion the local rate of change and the diffusion preponderate over the convection and the effect of the pressure gradient.

For the second approximation we substitute from (12) into the quadratic terms in (7) and into (9), solve (9) for p', substitute the value found for p' into (7), and then solve (7) again for u', v', w'. This process is equivalent to substituting *

$$t'/\mathrm{R} = \mathrm{T}, \quad \ldots\ldots\ldots \quad (13)$$

expanding u', p' in series

$$\left.\begin{aligned} u' &= u_0' + u_1'\mathrm{R} + u_2'\mathrm{R}^2 + u_3'\mathrm{R}^3 + \ldots, \\ p' &= p_0' + p_1'\mathrm{R} + p_2'\mathrm{R}^2 + p_3'\mathrm{R}^3 + \ldots, \end{aligned}\right\} \quad \ldots\ldots \quad (14)$$

with similar series for v' and w', substituting into (7) and (9) and equating coefficients of powers of R. In this way we find the equations

$$\left.\begin{aligned} -\nabla^2 p_0' &= \left(\frac{\partial u_0'}{\partial x'}\right)^2 + \left(\frac{\partial v_0'}{\partial y'}\right)^2 + \left(\frac{\partial w_0'^2}{\partial z'}\right) \\ &\quad + 2\left(\frac{\partial v_0'}{\partial x'}\frac{\partial u_0'}{\partial y'} + \frac{\partial w_0'}{\partial y'}\frac{\partial v_0'}{\partial z'} + \frac{\partial u_0'}{\partial z'}\frac{\partial w_0'}{\partial x'}\right), \\ -\nabla^2 p_1' &= 2\frac{\partial u_0'}{\partial x'}\frac{\partial u_1'}{\partial x'} + 2\frac{\partial v_0'}{\partial y'}\frac{\partial v_1'}{\partial y'} + 2\frac{\partial w_0'}{\partial z'}\frac{\partial w_1'}{\partial z'} \\ &\quad + 2\left(\frac{\partial v_0'}{\partial x'}\frac{\partial u_1'}{\partial y'} + \frac{\partial v_1'}{\partial x'}\frac{\partial u_0'}{\partial y'} + \frac{\partial w_0'}{\partial y'}\frac{\partial v_1'}{\partial z'} + \frac{\partial w_1'}{\partial y'}\frac{\partial v_0'}{\partial z'}\right. \\ &\quad \left. + \frac{\partial u_0'}{\partial z'}\frac{\partial w_1'}{\partial x'} + \frac{\partial u_1'}{\partial z'}\frac{\partial w_0'}{\partial x'}\right), \end{aligned}\right\} \quad (15)$$

* Taylor and Green use T to denote Aat, denoted here by t'.

and so on, together with

$$\left.\begin{aligned}\nabla^2 u_1' - \frac{\partial u_1'}{\partial T} &= u_0' \frac{\partial u_0'}{\partial x'} + v_0' \frac{\partial u_0'}{\partial y'} + w_0' \frac{\partial u_0'}{\partial z'} + \frac{\partial p_0'}{\partial x'},\\ \nabla^2 u_2' - \frac{\partial u_2'}{\partial T} &= u_0' \frac{\partial u_1'}{\partial x'} + v_0' \frac{\partial u_1'}{\partial y'} + w_0' \frac{\partial u_1'}{\partial z'}\\ &\quad + u_1' \frac{\partial u_0'}{\partial x'} + v_1' \frac{\partial u_0'}{\partial y'} + w_1' \frac{\partial u_0'}{\partial z'} + \frac{\partial p_1'}{\partial x'},\end{aligned}\right\} \quad . \quad (16)$$

and so on, with two similar sets of equations for v' and w'. The solutions required for p_0', p_1', etc., are the periodic solutions; the solutions required for u'_1, v_1', w_1', u_2', v_2', w_2', etc., are periodic, and are zero when $T=0$. From the equations for w_1', w_2', w_3', etc., it may be seen (when the corresponding stage in the calculations has been reached) that the expressions for the w' contain no terms independent of z'; consequently, the w' are most easily found from the equation of continuity (6) when the corresponding u' and v' have been found.

Substituting from (12) into the first of equations (15) we find that

$$-\nabla^2 p_0' = -\tfrac{1}{2} e^{-6T}(\cos 2x' + \cos 2y' - \cos 2x' \cos 2z' - \cos 2y' \cos 2z'). \quad . \quad . \quad . \quad (17)$$

Hence

$$p_0' = -\tfrac{1}{8} e^{-6T}(\cos 2x' + \cos 2y' - \tfrac{1}{2}\cos 2x' \cos 2z' - \tfrac{1}{2}\cos 2y' \cos 2z'). \quad (18)$$

From the first of equations (16) it may now be found that

$$\nabla^2 u_1' - \frac{\partial u_1'}{\partial T} = \tfrac{1}{8} e^{-6T} \sin 2x' \cos 2z'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Also $u_1' = 0$ when $T=0$. Hence

$$u_1' = f_1 \sin 2x' \cos 2z', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$f_1 = \frac{1}{16}(e^{-8T} - e^{-6T})$$

$$= -\left(\frac{T}{8} - \frac{7}{8}T^2 + \frac{37}{12}T^3 - \frac{175}{24}T^4 + \ldots\right), \quad . \quad . \quad . \quad . \quad (21)$$

v_1' may be found similarly. It is easy to see from the expressions for u_0', v_0', w_0', p_0' that the expression for v_1' may be obtained from that for u_1' by interchanging x' and y'. Hence

$$v_1' = f_1 \sin 2y' \cos 2z'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

After verifying that w_1' contains no term independent of z', we find w_1' from the equation of continuity, the result being

$$w_1' = -f_1(\cos 2x' \sin 2z' + \cos 2y' \sin 2z'). \quad . \quad . \quad . \quad . \quad (23)$$

Proceeding in the same way, we find the following expressions for p_1', u_2', v_2', w_2', p_2', u_3', v_3', w_3'. We note that the expression for v_2' is obtained from that for u_2' by interchanging x' and y' and changing the sign, and that the expression for v_3' is obtained from that for u_3' by interchanging x' and y', and so on.

$$p_1' = \frac{1}{88}(e^{-9T} - e^{-11T})(\sin x' \sin 3y' \sin z' - \sin 3x' \sin y' \sin z'). \quad (24)$$

$$\left.\begin{aligned} u_2' &= g_1 \cos x' \sin y' \sin z' + g_2(10 \cos 3x' \sin y' \sin z' \\ &\quad + 15 \cos x' \sin 3y' \sin z' - 11 \cos x' \sin y' \sin 3z') \\ &\quad + g_3 \cos 3x' \sin y' \sin 3z', \\ v_2' &= -g_1 \sin x' \cos y' \sin z' - g_2(10 \sin x' \cos 3y' \sin z' \\ &\quad + 15 \sin 3x' \cos y' \sin z' - 11 \sin x' \cos y' \sin 3z') \\ &\quad - g_3 \sin x' \cos 3y' \sin 3z', \\ w_2' &= 15 g_2(\sin x' \sin 3y' \cos z' - \sin 3x' \sin y' \cos z') \\ &\quad + g_3(\sin x' \sin 3y' \cos 3z' - \sin 3x' \sin y' \cos 3z'), \end{aligned}\right\} \quad (25)$$

where

$$\left.\begin{aligned} g_1 &= \frac{1}{768}(-e^{-3T} + 4e^{-9T} - 3e^{-11T}) \\ &= -\frac{T^2}{32} + \frac{23}{96}T^3 - \frac{185}{192}T^4 + \frac{515}{192}T^5 - \dots, \\ g_2 &= \frac{1}{704}(-e^{-9T} + e^{-11T} + 2Te^{-11T}) \\ &= -\frac{1}{11}\left(\frac{T^2}{32} - \frac{31}{96}T^3 + \dots\right), \\ g_3 &= \frac{1}{1280}(4e^{-9T} - 5e^{-11T} + e^{-19T}) \\ &= \frac{T^2}{32} - \frac{13}{32}T^3 + \dots, \end{aligned}\right\} \quad (26)$$

$$\begin{aligned} p_2' = &\left(\frac{e^{-6T}}{3072} + \frac{19e^{-12T}}{8448} - \frac{29e^{-14T}}{11264} - \frac{5Te^{-14T}}{704}\right)(\cos 2x' + \cos 2y') \\ &+ \left(-\frac{e^{-6T}}{6144} - \frac{7e^{-12T}}{16896} + \frac{13e^{-14T}}{22528} + \frac{3Te^{-14T}}{1408}\right) \\ &\qquad \times (\cos 2x' \cos 2z' + \cos 2y' \cos 2z') \\ &+ \left(\frac{31e^{-12T}}{5632} - \frac{21e^{-14T}}{2816} - \frac{5Te^{-14T}}{704} + \frac{e^{-16T}}{512}\right)\cos 2x' \cos 2y' \\ &+ 5\left(\frac{-e^{-12T} + e^{-14T} + 2Te^{-14T}}{528}\right)\cos 2x' \cos 2y' \cos 2z' \end{aligned}$$

$$+\left(\frac{51e^{-12T}}{11264}-\frac{31e^{-14T}}{5632}-\frac{5Te^{-14T}}{704}+\frac{e^{-16T}}{1024}\right)(\cos 4x'+\cos 4y')$$

$$+\frac{(e^{-6T}-e^{-8T})^2}{512}\cos 4z'$$

$$+\left(\frac{e^{-12T}}{1600}-\frac{3e^{-14T}}{5120}-\frac{Te^{-14T}}{640}-\frac{e^{-22T}}{25600}\right)$$

$$\times(\cos 2x'\cos 4z'+\cos 2y'\cos 4z')$$

$$+\left(-\frac{9e^{-12T}}{8800}+\frac{31e^{-14T}}{28160}+\frac{Te^{-14T}}{704}-\frac{e^{-22T}}{128800}\right)$$

$$\times(\cos 4x'\cos 2z'+\cos 4y'\cos 2z')$$

$$+\left(\frac{-e^{-12T}+e^{-14T}+2Te^{-14T}}{2816}\right)(\cos 4x'\cos 2y'+\cos 2x'\cos 4y')$$

$$+\left(\frac{-13e^{-12T}}{28160}+\frac{29e^{-14T}}{67584}+\frac{5Te^{-14T}}{4224}+\frac{e^{-22T}}{30720}\right)$$

$$\times(\cos 4x'\cos 2y'\cos 2z'+\cos 2x'\cos 4y'\cos 2z')$$

$$+\left(\frac{3e^{-12T}}{2560}-\frac{e^{-14T}}{512}+\frac{e^{-16T}}{1536}+\frac{e^{-22T}}{7680}\right)\cos 2x'\cos 2y'\cos 4z'. \quad (27)$$

$$\left.\begin{aligned}
u_3' = {} & h_1\sin 2x'\cos 2z'+h_2\sin 2x'\cos 2y'\cos 2z'+h_3\sin 4x'\cos 2y' \\
& -2h_3\sin 2x'\cos 4y'+h_4\sin 2x'\cos 4z'+h_5\sin 4x'\cos 2z' \\
& +h_6\sin 2x'\cos 2y'\cos 4z'+h_7\sin 4x'\cos 2y'\cos 2z' \\
& +h_8\sin 2x'\cos 4y'\cos 2z'+h_9\sin 4x'\cos 4z' \\
& +h_{10}\sin 4x'\cos 2y'\cos 4z'; \\
v_3' = {} & h_1\sin 2y'\cos 2z'+h_2\cos 2x'\sin 2y'\cos 2z'+h_3\cos 2x'\sin 4y' \\
& -2h_3\cos 4x'\sin 2y'+h_4\sin 2y'\cos 4z'+h_5\sin 4y'\cos 2z' \\
& +h_6\cos 2x'\sin 2y'\cos 4z'+h_7\cos 2x'\sin 4y'\cos 2z' \\
& +h_8\cos 4x'\sin 2y'\cos 2z'+h_9\sin 4y'\cos 4z' \\
& +h_{10}\cos 2x'\sin 4y'\cos 4z'; \\
w_3' = {} & -h_1(\cos 2x'+\cos 2y')\sin 2z'-2h_2\cos 2x'\cos 2y'\sin 2z' \\
& -\tfrac{1}{2}h_4(\cos 2x'+\cos 2y')\sin 4z'-2h_5(\cos 4x'+\cos 4y')\sin 2z' \\
& -h_6\cos 2x'\cos 2y'\sin 4z' \\
& -(2h_7+h_8)(\cos 4x'\cos 2y'+\cos 2x'\cos 4y')\sin 2z' \\
& -h_9(\cos 4x'+\cos 4y')\sin 4z'-h_{10}(\cos 4x'\cos 2y' \\
& \qquad +\cos 2x'\cos 4y')\sin 4z',
\end{aligned}\right\} \quad (28)$$

where

$$
\begin{aligned}
h_1 &= \frac{e^{-6T}}{6144} - \frac{17}{236544}e^{-8T} - \frac{293}{168960}e^{-12T} + \frac{7}{4224}e^{-14T} \\
&\qquad + \frac{7}{2816}Te^{-14T} - \frac{e^{-22T}}{71680} \\
&= \frac{43}{4224}T^3 - \frac{43}{352}T^4 + \dots \\
h_2 &= \frac{7}{3300}e^{-12T} - \frac{29}{10560}Te^{-12T} - \frac{73}{33792}e^{-14T} - \frac{5}{4224}Te^{-14T} + \frac{e^{-22T}}{25600} \\
&= -\frac{19}{3168}T^3 + \dots \\
h_3 &= \frac{e^{-12T}}{2048} - \frac{e^{-14T}}{2304} - \frac{Te^{-14T}}{768} - \frac{e^{-20T}}{18432} \\
&= \frac{T^3}{384} + \dots \\
h_4 &= -\frac{21}{25600}e^{-12T} + \frac{7}{9216}e^{-14T} + \frac{Te^{-14T}}{480} + \frac{e^{-20T}}{46080} + \frac{e^{-22T}}{25600} \\
&= -\frac{T^3}{192} + \dots \\
h_5 &= \frac{251}{281600}e^{-12T} - \frac{911}{1013760}e^{-14T} - \frac{17}{8448}Te^{-14T} \\
&\qquad + \frac{73}{506880}e^{-20T} - \frac{7}{51200}e^{-22T} \\
&= \frac{27}{3520}T^3 + \dots \\
h_6 &= \frac{e^{-12T}}{1920} - \frac{3}{2560}e^{-14T} + \frac{e^{-16T}}{1536} + \frac{e^{-22T}}{7680} - \frac{e^{-24T}}{7680} \\
&= \frac{T^3}{96} + \dots \\
h_7 &= -\frac{3}{10560}e^{-12T} + \frac{27}{84480}e^{-14T} + \frac{Te^{-14T}}{2112} - \frac{e^{-22T}}{7680} + \frac{e^{-24T}}{10560} \\
&= -\frac{T^3}{198} + \dots \\
h_8 &= \frac{23}{168960}e^{-12T} - \frac{e^{-14T}}{5632} - \frac{Te^{-14T}}{8448} + \frac{e^{-22T}}{7680} - \frac{e^{-24T}}{11264} \\
&= \frac{49}{12672}T^3 + \dots \\
h_9 &= -\frac{e^{-12T}}{12800} + \frac{e^{-14T}}{9216} - \frac{e^{-22T}}{25600} + \frac{e^{-32T}}{115200} \\
&= -\frac{T^3}{192} + \dots
\end{aligned}
\qquad \text{. (29)}
$$

$$h_{10}=\frac{e^{-12T}}{30720}-\frac{e^{-14T}}{22528}+\frac{e^{-22T}}{71680}-\frac{e^{-36T}}{473088}$$
$$=\frac{T^3}{384}+\ldots \qquad \left.\begin{matrix}\\ \\ \\ \\ \\ \end{matrix}\right\} . \quad (29)\ (cont.)$$

We see that when u' is expressed as a triple Fourier series the coefficients of the various terms are expressible as series of ascending powers of R multiplied by functions of T, the coefficient of $\cos x' \sin y' \sin z'$ being an even series commencing with a term in R^0, that of $\sin 2x' \cos 2z'$ being an odd series commencing with a term in R^1, those of

$$\cos 3x' \sin y' \sin z', \quad \cos x' \sin 3y' \sin z',$$
$$\cos x' \sin y' \sin 3z', \quad \cos 3x' \sin y' \sin 3z'$$

being even series commencing with terms in R^2, and so on. Similar results hold for v' and w'. (See equations (12), (20), (22), (23), (25), (28).) Hence, if we write

$$q^2=u^2+v^2+w^2, \quad q'^2=u'^2+v'^2+w'^2=q^2/A^2, \quad . \quad . \quad . \quad (30)$$

$$\omega^2=\xi^2+\eta^2+\zeta^2, \quad \omega'^2=\xi'^2+\eta'^2+\zeta'^2=\omega^2/a^2A^2, \quad . \quad . \quad . \quad (31)$$

and denote by $\overline{q'^2}$ and $\overline{\omega'^2}$ the space-average values of q'^2 and ω'^2 taken throughout a fundamental parallelepiped, it follows from the orthogonal property of the terms in the triple Fourier series that $\overline{q'^2}$ and $\overline{\omega'^2}$ will be expressible as even power-series in R and can be obtained correct to the term in R^4 if in u' the coefficient of $\cos x' \sin y' \sin z'$ is found to R^4, that of $\sin 2x' \cos 2z'$ to R^3, and the remaining coefficients to R^2, with corresponding results for v' and w'. Hence to find $\overline{q'^2}$ and $\overline{\omega'^2}$ to R^4 it remains only to find the coefficients of $\cos x' \sin y' \sin z'$ in u_4' and of $\sin x' \cos y' \sin z'$ in v_4'. Further, of all the terms in u_3', v_3', w_3' in equation (28) only those in h_1 are required in finding $\overline{q'^2}$ and $\overline{\omega'^2}$ to R^4: the remaining terms in u_3', v_3', w_3' affect only the terms in R^6 in $\overline{q'^2}$ and $\overline{\omega'^2}$, and need not have been found if we are finding $\overline{q'^2}$ and $\overline{\omega'^2}$ only up to R^4. (They have been reported here since they were found as a beginning of the calculation of the terms in R^6 in $\overline{q'^2}$ and $\overline{\omega'^2}$.)

In the same way when p' is expressed as a triple Fourier series, $\cos 2x'$, $\cos 2y'$, $\cos 2x' \cos 2z'$, $\cos 2y' \cos 2z'$ occur with coefficients which are even power-series in R beginning with terms in R^0, $\sin x' \sin 3y' \sin z'$, $\sin 3x' \sin y' \sin z'$ with coefficients which are odd power-series in R beginning with terms in R^1, and so on, as can be seen from equations (18), (24), and (27). Hence the average values of the pressure and of the pressure-gradient are expressible as even power-series in R, with coefficients which are functions of T, and from equations (18), (24), and (27) the terms in R^0 and R^2 can be found. The terms in R^3 in the

coefficients of $\sin x' \sin 3y' \sin z'$, $\sin 3x' \sin y' \sin z'$ in p_3' have been found to be

$$\frac{1}{11}\{e^{-3T}(-2h_1+2h_2-h_3-2h_5+\tfrac{1}{2}h_7)-f_1(2g_1+44g_2+2g_3)\},$$

and the same expression with the sign changed, respectively; whilst those of $\sin 3x' \sin y' \sin 3z'$ and $\sin x' \sin 3y' \sin 3z'$, which are required for the calculation of u_4', v_4', w_4' if the terms in R^6 in q'^2 and $\overline{\omega'^2}$ are sought, are found to be

$$\frac{e^{-3T}}{19}(-2h_2+\tfrac{3}{2}h_4+6h_5-h_6-\tfrac{3}{2}h_7-h_8),$$

and the same expression with the sign changed, respectively. Also no terms in $\sin x' \sin y' \sin z'$, $\sin x' \sin y' \sin 3z'$ occur in p_3'. No further calculations of the expressions for p' have, however, been carried out, so the average values of the pressure and the pressure gradient have been found only as far as the term in R^2.

The coefficients of $\cos x' \sin y' \sin z'$ in u_4' and of $-\sin x' \cos y' \sin z'$ in v_4' were found to be k_1 (in each case), where

$$\begin{aligned} k_1 = {} & \frac{1009}{423886848}e^{-3T} - \frac{e^{-9T}}{49152} + \frac{145}{15138816}e^{-11T} + \frac{211}{1351680}e^{-15T} \\ & - \frac{5695}{26492928}e^{-17T} - \frac{3}{14336}Te^{-17T} + \frac{193}{2883584}e^{-19T} \\ & + \frac{19}{180224}Te^{-19T} + \frac{1}{3153920}e^{-25T} \\ = {} & \frac{31}{33792}T^4 - \frac{111}{11264}T^5 + \ldots\ldots \end{aligned} \qquad (32)$$

The expressions for u', v', w' required to calculate $\overline{q'^2}$ and $\overline{\omega'^2}$ to the terms in R^4 are therefore (see equations (12), (13), (14), (20), (22), (23), (25), (28)

$$\left.\begin{aligned} u' = {} & (e^{-3T}+R^2g_1+R^4k_1)\cos x' \sin y' \sin z' + (Rf_1+R^3h_1)\sin 2x' \cos 2z' \\ & + 10R^2g_2 \cos 3x' \sin y' \sin z' + 15\,R^2g_2 \cos x' \sin 3y' \sin z' \\ & - 11R^2g_2 \cos x' \sin y' \sin 3z' + R^2g_3 \cos 3x' \sin y' \sin 3z', \\ v' = {} & -(e^{-3T}+R^2g_1+R^4k_1)\sin x' \cos y' \sin z' + (Rf_1+R^3h_1)\sin 2y' \cos 2z' \\ & - 10R^2g_2 \sin x' \cos 3y' \sin z' - 15R^2g_2 \sin 3x' \cos y' \sin z' \\ & + 11R^2g_2 \sin x' \cos y' \sin 3z' - R^2g_3 \sin x' \cos 3y' \sin 3z', \\ w' = {} & -(Rf_1+R^3h_1)(\cos 2x' + \cos 2y') \sin 2z' \\ & + 15R^2g_2(\sin x' \sin 3y' - \sin 3x' \sin y') \cos z' \\ & + R^2g_3(\sin x' \sin 3y' - \sin 3x' \sin y') \cos 3z'. \end{aligned}\right\} \qquad (33)$$

Hence

$$\overline{q'^2}=\tfrac{1}{4}(e^{-3T}+R^2g_1+R^4k_1)^2+(Rf_1+R^3h_1)^2+\tfrac{1}{4}R^4(671g_2{}^2+2g_3{}^2)+\dots$$
$$=\tfrac{1}{4}(e^{-6T}+R^2F_1+R^4F_2+\dots), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where

$$F_1=2e^{-3T}g_1+4f_1{}^2$$
$$=\frac{1}{384}(-e^{-6T}+10e^{-12T}-15e^{-14T}+6e^{-16T})$$
$$=-\frac{5}{24}T^3+\frac{5}{2}T^4-\frac{367}{24}T^5+\frac{761}{12}T^6-\frac{7207}{36}T^7+\frac{2049}{4}T^8-\frac{5012659}{4536}T^9$$
$$+\frac{7805347}{3780}T^{10}-\frac{38647727}{11340}T^{11}+\frac{14297023}{2835}T^{12}-\dots \quad . \quad . \quad . \quad (35)$$

$$F_2=g_1{}^2+2e^{-3T}k_1+8f_1h_1+671g_2{}^2+2g_3{}^2$$
$$=\frac{4105}{635830272}e^{-6T}-\frac{5}{36864}e^{-12T}+\frac{185}{1261568}e^{-14T}-\frac{17}{473088}e^{-16T}$$
$$+\frac{26153}{10137600}e^{-18T}-\frac{27171}{5519360}e^{-20T}-\frac{279}{39424}Te^{-20T}+\frac{10217}{4325376}e^{-22T}$$
$$+\frac{619}{90112}Te^{-22T}+\frac{61}{11264}T^2e^{-22T}+\frac{137}{7884800}e^{-28T}-\frac{11}{573440}e^{-30T}$$
$$+\frac{1}{819200}e^{-38T}$$
$$=-\frac{5}{528}T^5+\frac{361}{1408}T^6-\frac{24139}{7392}T^7+\frac{1504561}{55440}T^8-\frac{112221863}{665280}T^9$$
$$+\frac{803007817}{950400}T^{10}-\frac{49020957457}{13721400}T^{11}+\frac{722096296337}{54885600}T^{12}$$
$$-\frac{1709850660779}{39639600}T^{13}+\frac{547751178043483}{4281076800}T^{14}$$
$$-\frac{13017506311230613}{37459422000}T^{15}+\dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36$$

In the same way, if we write down the formulæ for ξ, η', ζ' corresponding to (33) and take the mean value of the sum of the squares, we find that

$$\overline{\omega'^2}=\tfrac{3}{4}(e^{-6T}+R^2G_1+R^4G_2+\dots), \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where

$$G_1=2e^{-3T}g_1+\frac{32}{3}f_1{}^2$$
$$=\frac{1}{384}(-e^{-6T}+20e^{-12T}-35e^{-14T}+16e^{-16T})$$

$$= \frac{5}{48}T^2 - \frac{5}{3}T^3 + \frac{1835}{144}T^4 - \frac{761}{12}T^5 + \frac{50449}{216}T^6 - 683T^7 + \frac{5012659}{3024}T^8$$
$$- \frac{7805347}{2268}T^9 + \frac{425124997}{68040}T^{10} - \frac{28594046}{2835}T^{11} + \ldots \quad . \quad . \quad . \quad (38)$$

$$G_2 = {g_1}^2 + 2e^{-3T}k_1 + \frac{64}{3}f_1h_1 + \frac{7381}{3}{g_2}^2 + \frac{38}{3}{g_3}^2$$
$$= \frac{4105}{635830272}e^{-6T} - \frac{5}{18432}e^{-12T} + \frac{185}{540672}e^{-14T} - \frac{17}{177408}e^{-16T}$$
$$+ \frac{26153}{3379200}e^{-18T} - \frac{4203}{275968}e^{-20T} - \frac{465}{19712}Te^{-20T} + \frac{97531}{12976128}e^{-22T}$$
$$+ \frac{2107}{90112}Te^{-22T} + \frac{61}{3072}T^2e^{-22T} + \frac{137}{1689600}e^{-28T} - \frac{11}{114688}e^{-30T}$$
$$+ \frac{19}{2457600}e^{-38T}$$
$$= \frac{25}{3168}T^4 - \frac{361}{1408}T^5 + \frac{24139}{6336}T^6 - \frac{1504561}{41580}T^7 + \frac{112221863}{443520}T^8$$
$$- \frac{803007817}{570240}T^9 + \frac{49020957457}{7484400}T^{10} - \frac{722096296337}{27442800}T^{11}$$
$$+ \frac{1709850660779}{18295200}T^{12} - \frac{547751178043483}{1834747200}T^{13}$$
$$+ \frac{13017506311230613}{14983768800}T^{14} - \ldots \quad . \quad . \quad .

and $$\frac{d}{dt}(\tfrac{1}{2}\overline{q^2}) = -\nu\overline{\omega^2}, \quad \ldots\ldots\ldots \quad (43)$$

i. e., $$\frac{d}{dt'}(\tfrac{1}{2}\overline{q'^2}) = -\frac{\overline{\omega'^2}}{\mathrm{R}}. \quad \ldots\ldots\ldots \quad (44)$$

Hence $$\frac{d}{d\mathrm{T}}(\tfrac{1}{2}\overline{q'^2}) = -\overline{\omega'^2}. \quad \ldots\ldots\ldots \quad (45)$$

From (34) and (37) it follows that the F and G are connected by the relations

$$\mathrm{G}_1 = -\frac{1}{6}\frac{d\mathrm{F}_1}{d\mathrm{T}}, \quad \mathrm{G}_2 = -\frac{1}{6}\frac{d\mathrm{F}_2}{d\mathrm{T}}, \quad \ldots\ldots \quad (46)$$

which are seen from (35), (36), (38), (39) to be satisfied.

Our formulæ include those obtained by Taylor and Green. Their formulæ for u', v', w' are obtained if we replace $e^{-3\mathrm{T}}$ in our formulæ by its expansion as far as the term in T^5, and replace f_1, g_1, g_2, g_3, and the h by their expansions as far as the terms stated explicitly in equations (21), (26), (29). Their formula for $\overline{q'^2}$ is obtained from ours if we replace $e^{-6\mathrm{T}}$, F_1, and F_2 by their expansions as far as the terms in T^6, and their formula for $\overline{\omega'^2}$ by replacing $e^{-6\mathrm{T}}$, G_1, and G_2 by their expansions as far as the terms in T^5. (The results of Taylor and Green, as far as they go, are verified by work reported here with the exception that on p. 521 of their paper the signs of z_1, z_2, z_3 should be changed.) The formula which they obtained for the pressure was not stated by Taylor and Green (with the exception of the expression for p at $t=0$).

The mean square values of the pressure and the pressure-gradient can be calculated to the terms in R^2 from the following expression for p' (see equations (14), (18), (24), (27)) :—

$$\begin{aligned} p' = {} & [\cos 2x' + \cos 2y']\left[-\frac{e^{-6\mathrm{T}}}{8} + \mathrm{R}^2\left(\frac{e^{-6\mathrm{T}}}{3072} + \frac{19}{8448}e^{-12\mathrm{T}} - \frac{29}{11264}e^{-14\mathrm{T}}\right.\right. \\ & \left.\left. - \frac{5}{704}\mathrm{T}e^{-14\mathrm{T}}\right)\right] + [\cos 2x' \cos 2z' + \cos 2y' \cos 2z'] \\ & \times\left[\frac{e^{-6\mathrm{T}}}{16} + \mathrm{R}^2\left(-\frac{e^{-6\mathrm{T}}}{6144} - \frac{7}{16896}e^{-12\mathrm{T}} + \frac{13}{22528}e^{-14\mathrm{T}} + \frac{3}{1408}\mathrm{T}e^{-14\mathrm{T}}\right)\right] \\ & + \frac{\mathrm{R}}{88}(e^{-9\mathrm{T}} - e^{-11\mathrm{T}})(\sin x' \sin 3y' \sin z' - \sin 3x' \sin y' \sin z'). \quad (47) \end{aligned}$$

Hence

$$\begin{aligned} \overline{p'^2} = {} & \frac{9}{512}\left\{e^{-12\mathrm{T}} - \mathrm{R}^2\left(\frac{e^{-12\mathrm{T}}}{192} + \frac{551}{17424}e^{-18\mathrm{T}} - \frac{271}{7744}e^{-20\mathrm{T}}\right.\right. \\ & \left.\left. - \frac{43}{396}\mathrm{T}e^{-20\mathrm{T}} - \frac{2}{1089}e^{-22\mathrm{T}}\right) + \ldots\right\} \\ = {} & \frac{9}{512}\left\{e^{-12\mathrm{T}} - \mathrm{R}^2\left(\frac{219}{968}\mathrm{T}^2 - \frac{52739}{13068}\mathrm{T}^3 + \ldots\right) + \ldots\right\}. \quad \ldots\ldots \quad (48) \end{aligned}$$

and

$$\overline{\left(\frac{\partial p'}{\partial x'}\right)^2}+\overline{\left(\frac{\partial p'}{\partial y'}\right)^2}+\overline{\left(\frac{\partial p'}{\partial z'}\right)^2}$$

$$=\frac{5}{64}\left\{e^{-12T}-R^2\left(\frac{e^{-12T}}{192}+\frac{71}{2640}e^{-18T}-\frac{97}{3520}e^{-20T}\right.\right.$$

$$\left.\left.-\frac{23}{220}Te^{-20T}-\frac{1}{220}e^{-22T}\right)+\ldots\right\}$$

$$=\frac{5}{64}\left\{e^{-12T}-R^2\left(\frac{93}{440}T^2-\frac{823}{220}T^3+\ldots\right)+\ldots\right\}. \quad . \quad (49)$$

TABLE I.

T.	-10^4F_1.	-10^8F_2.	10^4G_1.	10^8G_2.
0·001	0·000 00	0·000 000	0·001 03	0·000 001
0·0016	0·000 01	0·000 000	0·002 82	0·000 006
0·002	0·000 02	0·000 000	0·004 04	0·000 012
0·003	0·000 07	0·000 000	0·010 97	0·000 087
0·005	0·000 24	0·000 003	0·024 04	0·000 419
0·006	0·000 42	0·000 006	0·034 06	0·000 840
0·006	0·000 57	0·000 010	0·041 60	0·001 253
0·0075	0·000 80	0·000 018	0·051 95	0·001 952
0·0083	0·001 09	0·000 030	0·063 28	0·002 893
0·01	0·001 85	0·000 072	0·088 71	0·005 675
0·0116	0·002 88	0·000 149	0·117 55	0·009 941
0·012	0·003 12	0·000 170	0·123 69	0·011 002
0·0125	0·003 50	0·000 205	0·133 13	0·012 737
0·013	0·004 21	0·000 277	0·149 45	0·016 029
0·015	0·005 87	0·000 476	0·184 12	0·024 262
0·016	0·007 90	0·000 770	0·221 26	0·034 929
0·02	0·013 12	0·001 746	0·301 83	0·064 550
0·025	0·024 14	0·004 626	0·434 74	0·132 184
0·03	0·039 30	0·009 969	0·576 87	0·228 946
0·031	0·042 84	0·011 411	0·605 95	0·251 663
0·032	0·046 57	0·012 992	0·635 17	0·275 430
0·036	0·063 22	0·020 829	0·752 68	0·380 073
0·04	0·082 69	0·031 335	0·869 83	0·497 209
0·048	0·129 94	0·061 2	1·096 65	0·749 5
0·05	0·143 43	0·070 6	1·150 84	0·812 2
0·075	0·360 2	0·234 4	1·694 0	1·204 5
0·1	0·636 6	+ 0·354 8	1·940 7	+ 0·090 9
0·15	1·201 0	− 0·547 9	1·690 6	− 6·865 1
0·2	1·604	− 3·810	0·964	−14·339
0·25	1·779	− 8·616	+0·228	−16·667
0·3	1·761	−13·255	−0·312	−13·534
0·35	1·615	−16·455	−0·625	− 7·584
0·4	1·404	−17·795	−0·754	− 1·494
0·45	1·175	−17·494	−0·760	+ 3·206
0·5	0·955	−16·056	−0·697	+ 6·083
1·0	0·063 3	− 1·53	−0·062 1	+ 1·47

No numerical computations have been carried out for the mean square pressure and pressure gradient, but numerical values have been computed for $\overline{q'^2}$ and $\overline{\omega'^2}$ for comparison with the values computed by Taylor and Green. The formulæ used are those in equations (34) and (37). Table I.

contains values of F_1, F_2, G_1, G_2 for certain values of T, several of which were chosen specially to enable the comparison of computed values of $\overline{q'^2}$ and $\overline{\omega'^2}$ to be made with the values computed by Taylor and Green

TABLE II. R=300.

T.	t'.	$e-6T$.	$-R^2F_1$.	$-R^4F_2$.	$4\overline{q'^2}$.	R^2G_1.	R^4G_2.	$4\overline{\omega'^2}/3$.	$3a^2\lambda^2/5$
0·001	0·3	0·9950	0·0000	0·0000	0·995	0·0093	0·0001	1·004	0·991
0·0016	0·5	0·9900	0·0001	0·0000	0·990 (0·990)	0·0254	0·0005	1·016 (1·016)	0·974
0·003	1·0	0·9802	0·0006	0·0000	0·980 (0·980)	0·0987	0·0070	1·086 (1·086)	0·902
0·005	1·5	0·9704	0·0022	0·0002	0·968 (0·968)	0·2164	0·0339	1·221 (1·220)	0·793
0·006	2·0	0·9608	0·0051	0·0008	0·955 (0·955)	0·3744	0·1015	1·44 (1·43)	0·66
0·0083	2·5	0·9512	0·0098	0·0024	0·939 (0·939)	0·5695	0·2343	1·76 (1·75)	0·53
0·01	3·0	0·9418	0·0166	0·0058	0·919 (0·919)	0·7984	0·4597	2·20 (2·17)	0·42
0·0116	3·5	0·9324	0·0259	0·0121	0·89	1·0579	0·8052	2·80 (2·73)	0·32
0·013	4·0	0·9231	0·0379	0·0224	0·86 (0·86)				
0·016	5·0	0·9048	0·0711	0·0624	0·77 (0·78)				

TABLE III. R=200.

T.	t'.	$e-6T$.	$-R^2F_1$.	$-R^4F_2$.	$4\overline{q'^2}$.	R^2G_1.	R^4G_2.	$4\overline{\omega'^2}/3$.	$3a^2\lambda^2/5$
0·001	0·2	0·9950	0·0000	0·0000	0·995	0·0041	0·0000	0·999 (0·998)	0·996
0·002	0·4	0·9881	0·0001	0·0000	0·988	0·0162	0·0002	1·005 (1·004)	0·983
0·005	1·0	0·9704	0·0010	0·0000	0·969 (0·969)	0·0962	0·0067	1·073 (1·073)	0·903
0·0075	1·5	0·9560	0·0032	0·0003	0·952 (0·953)	0·2078	0·0312	1·195 (1·194)	0·796
0·01	2·0	0·9418	0·0074	0·0012	0·933 (0·933)	0·3548	0·0908	1·39 (1·38)	0·67
0·0125	2·5	0·9277	0·0140	0·0033	0·910 (0·911)	0·5325	0·2038	1·66 (1·64)	0·55
0·015	3·0	0·9139	0·0235	0·0076	0·883 (0·884)	0·7365	0·3882	2·04 (1·98)	0·43
0·02	4·0	0·8869	0·0525	0·0279	0·81 (0·81)				
0·025	5·0	0·8607	0·0966	0·0740	0·69 (0·72)				

TABLE IV. R=100.

T.	t'.	$e-6T$.	$-R^2F_1$.	$-R^4F_2$.	$4\overline{q'^2}$.	R^2G_1.	R^4G_2.	$4\overline{\omega'^2}/3$.	$3a^2\lambda^2/5$
0·002	0·2	0·9881	0·0000	0·0000	0·988	0·0040	0·0000	0·992	0·996
0·006	0·6	0·9646	0·0004	0·0000	0·964	0·0341	0·0008	1·000 (1·000)	0·964
0·01	1·0	0·9418	0·0018	0·0001	0·940 (0·940)	0·0887	0·0057	1·036 (1·036)	0·907
0·015	1·5	0·9139	0·0059	0·0003	0·908 (0·908)	0·1841	0·0243	1·122 (1·119)	0·809
0·02	2·0	0·8869	0·0131	0·0017	0·872 (0·873)	0·3018	0·0645	1·25 (1·23)	0·70
0·025	2·5	0·8607	0·0241	0·0046	0·832 (0·834)	0·4347	0·1322	1·43 (1·35)	0·58
0·031	3·1	0·8303	0·0428	0·0114	0·776 (0·784)	0·6059	0·2517	1·69 (1·43)	0·46
0·036	3·6	0·8057	0·0632	0·0208	0·72 (0·74)	0·7527	0·3801	1·94	0·37
0·04	4·0	0·7866	0·0827	0·0313	0·67 (0·71)				

Tables II.–VI. contain values of $4\overline{q'^2}$, $4\overline{\omega'^2}/3$, and of the separate terms in the formulæ (34) and (37) for R=300, 200, 100, 50, and 20 respectively, these being the values of R for which computations were carried out by Taylor and Green, whose values for $4\overline{q'^2}$ and $4\overline{\omega'^2}/3$ are shown in brackets.

The computations of Taylor and Green were pushed far beyond the limits of accuracy of their series in t' (in their tables as well as in their graphs); and in order that comparisons may be made of the numerical values obtained by the two methods results have been set out in Tables II.–VI., which in some cases are not given sufficiently accurately by the formulæ (34) and (37), if we judge by the values of the separate terms. The accuracy and range of reliability of the two methods are about the same

TABLE V. R=50.

T.	t'.	$e-6T$.	$-R^2F_1$.	$-R^4F_2$.	$4\overline{q'^2}$.	R^2G_1.	R^4G_2.	$4\overline{\omega'^2}/3$.	$3a^2\lambda^2/5$
0·005	0·25	0·9704	0·0001	0·0000	0·970	0·0060	0·0000	0·976	0·994
0·012	0·6	0·9305	0·0008	0·0000	0·930 (0·930)	0·0309	0·0007	0·962 (0·962)	0·967
0·02	1·0	0·8869	0·0033	0·0001	0·883 (0·884)	0·0755	0·0040	0·966 (0·965)	0·914
0·03	1·5	0·8353	0·0098	0·0006	0·825 (0·825)	0·1442	0·0143	0·994 (0·980)	0·830
0·032	1·6	0·8253	0·0116	0·0008	0·813 (0·813)	0·1588	0·0172	1·001 (0·982)	0·812
0·04	2·0	0·7866	0·0207	0·0020	0·764 (0·766)	0·2175	0·0311	1·035 (0·964)	0·738
0·048	2·4	0·7498	0·0325	0·0038	0·713 (0·726)	0·2742	0·0468	1·07	0·67
0·05	2·5	0·7408	0·0359	0·0044	0·700	0·2877	0·0508	1·08	0·65
0·075	3·75	0·6376	0·0901	0·0146	0·533	0·4235	0·0753	1·14	0·47
0·1	5·0	0·5488	0·1591	0·0222	0·367	0·4852	0·0057	1·04	0·35
0·15	7·5	0·4066	0·3003	−0·0342	0·140				

TABLE VI. R=20.

T.	t'	$e-6T$.	$-R^2F_1$.	$-R^4F_2$.	$4\overline{q'^2}$.	R^2G_1.	R^4G_2.	$4\overline{\omega'^2}/3$.	$3a^2\lambda^2/5$
0·01	0·2	0·9418	0·0001	0·0000	0·942	0·0035	0·0000	0·945 (0·945)	0·997
0·03	0·6	0·8353	0·0016	0·0000	0·834	0·0231	0·0004	0·859 (0·858)	0·971
0·05	1·0	0·7408	0·0057	0·0001	0·735	0·0460	0·0013	0·788 (0·781)	0·933
0·075	1·5	0·6376	0·0144	0·0004	0·623	0·0678	0·0019	0·707 (0·634)	0·881
0·1	2·0	0·5488	0·0255	0·0006	0·523	0·0776	0·0001	0·626	0·835
0·15	3·0	0·4066	0·0480	−0·0008	0·359	0·0676	−0·0110	0·463	0·775
0·2	4·0	0·3012	0·0642	−0·0061	0·243	0·0385	−0·0229	0·317	0·77
0·25	5·0	0·2231	0·0711	−0·0138	0·166	0·0091	−0·0267	0·205	0·81
0·3	6·0	0·1653	0·0704	−0·0212	0·116	−0·0125	−0·0217	0·131	0·89
0·35	7·0	0·1225	0·0646	−0·0263	0·084	−0·0250	−0·0121	0·085	0·99
0·4	8·0	0·0907	0·0562	−0·0285	0·063	−0·0302	−0·0024	0·058	
0·45	9·0	0·0672	0·0470	−0·0280	0·05	−0·0304	+0·0051	0·042	

at the highest Reynolds number considered (300); in both methods reliable results are obtained only for small values of the time at this Reynolds number. As the Reynolds number decreases the accuracy and range of reliability of the present formulæ increase, whereas those of the formulæ of Taylor and Green decrease. For $R \geqslant 50$, ω'^2 rises to a maximum value greater than its initial value; this increase illustrates the effect to be considered—namely, the increase of the mean square vorticity due to the stretching of the vortex-lines. It is interesting to observe that the method of Taylor and Green appears considerably

to underestimate both the maximum value of $\overline{\omega'^2}$ and the time at which it occurs. For R=50, according to Taylor and Green, $4\omega'^2/3$ never rises again to 1 after $t'=0$, and has a maximum value about 0·98 at $t'=1{\cdot}6$; according to our results it rises to a maximum value about 1·14 at $t'=3{\cdot}5$. The formulæ of Taylor and Green also underestimate the decay in $\overline{q'^2}$ (as they must if they underestimate $\overline{\omega'^2}$).

In his theory of isotropic turbulence G. I. Taylor * introduced a length λ, which he called " the average size of the smallest eddies, which are responsible for the dissipation of energy by viscosity." This length is such that

$$\frac{d}{dt}(\tfrac{1}{2}\overline{q^2})=-5\nu\overline{q^2}/\lambda^2, \quad \ldots\ldots\ldots \quad (50)$$

so from (43)

$$\lambda^2=5\overline{q^2}/\overline{\omega^2}. \quad \ldots\ldots\ldots \quad (51)$$

Values of λ^2 can be calculated from equation (51) for the motion here considered. In non-dimensional form we have the equation

$$a^2\lambda^2=5\overline{q'^2}/\overline{\omega'^2}.$$

The initial value of $a^2\lambda^2$ is 5/3, and in Tables II.–VI. values of $3a^2\lambda^2/5$ are shown in the last columns. These values have not been compared with the values computed by Taylor and Green, since they tabulate $a\lambda/\pi$, and the extra computation necessary for comparison did not appear worth while.

Although numerical computations of the average square pressure and pressure-gradient have not been carried out, we may point out formulæ which enable us to calculate the variation at the beginning of the motion of certain other quantities analogous to quantities occurring in Taylor's theory of isotropic turbulence, and related to the pressure or pressure gradient. The initial values of these quantities for the motion here considered have been calculated by Taylor †.

From equations (34) and (48) we see that the ratio of $\sqrt{\overline{p^2}}$ to $\frac{1}{2}\rho\overline{q^2}$, whose initial value is $3/2\sqrt{2}$, decreases at the beginning of the motion.

In his theory of isotropic turbulence Taylor ‡ introduces a length λ_η, which is connected with the Lagrangian representation of turbulent motion used to discuss diffusion experiments, and is such that

$$\overline{(\partial p/\partial y)^2}=2\rho^2\overline{(v^2)^2}/\lambda_\eta{}^2.$$

---

* Proc. Roy. Soc. A, cli. pp. 421–454 (1935).

† Proc. Camb. Phil. Soc. xxxii. pp. 380–384 (1936). For the initial values in the cases of motion more nearly isotropic than that represented by (1), see Green, Proc. Camb. Phil. Soc. xxxiv. pp. 534–539 (1938).

‡ Proc. Roy. Soc. A, cli. pp. 475–477 (1935).

For non-isotropic motion we may define λ_η by the equation

$$\lambda_\eta^2 = 2\rho^2 \frac{\Sigma(\overline{u^2})^2}{\Sigma(\overline{\partial p/\partial x})^2}, \quad \ldots\ldots\ldots \quad (52)$$

where the summations are over u, v, w and x, y, z. In non-dimensional form

$$a^2\lambda_\eta^2 = 2\frac{\Sigma(\overline{u'^2})^2}{\Sigma(\overline{\partial p'/\partial x'})^2}.$$

From equations (33) we see that

$$\Sigma(\overline{u'^2})^2 = \frac{1}{32}\left\{e^{-12T} - R^2\left(\frac{e^{-12T}}{192} - \frac{7e^{-18T}}{192} + \frac{3e^{-20T}}{64} - \frac{e^{-22T}}{64}\right) + \ldots\right\}$$

$$= \frac{1}{32}\left\{e^{-12T} - R^2\left(\frac{T^2}{16} - \frac{5}{6}T^3 + \ldots\right) + \ldots\right\}, \quad \ldots\ldots\ldots \quad (53)$$

and from equations (53) and (49) it follows that $a^2\lambda_\eta^2$, whose initial value is 4/5, increases at the beginning of the motion.

Taylor further assumes that in isotropic turbulence

$$\sqrt{(\overline{\partial p/\partial y})^2} = 3B\rho\sqrt{\{\overline{v^2}.(\overline{\partial v/\partial y})^2\}},$$

where B is constant, from which it follows that in isotropic turbulence λ/λ_η is constant. In the motion here considered, at the beginning of the motion λ decreases and λ_η increases, so λ/λ_η, whose initial value is $5/2\sqrt{3}$, decreases. Also for non-isotropic motion we may define B by the equation

$$B^2 = \frac{1}{9\rho^2}\frac{\Sigma(\overline{\partial p/\partial x})^2}{\Sigma\overline{u^2}(\overline{\partial u/\partial x})^2}. \quad \ldots\ldots\ldots \quad (54)$$

In non-dimensional form

$$B^2 = \frac{1}{9}\frac{\Sigma(\overline{\partial p'/\partial x'})^2}{\Sigma\overline{u'^2}(\overline{\partial u'/\partial x'})^2}, \quad \ldots\ldots\ldots \quad (55)$$

and from equation (33) it follows that

$$\Sigma\overline{u'^2}(\overline{\partial u'/\partial x'})^2 = \frac{1}{32}\left\{e^{-12T} - R^2\left(\frac{e^{-12T}}{192} - \frac{23}{384}e^{-18T} + \frac{3}{32}e^{-20T} - \frac{5}{128}e^{-22T}\right) + \ldots\right\}$$

$$= \frac{1}{32}\left\{e^{-12T} + R^2\left(\frac{T^2}{32} - \frac{25}{24}T^3 + \ldots\right) + \ldots\right\}. \quad \ldots\ldots \quad (56)$$

Hence B^2, whose inital value is 5/18, decreases at the beginning of the motion.

Finally, we may remark that for large values of T,

$$\left.\begin{aligned} u' &= e^{-3T}\left(1-\frac{R^2}{768}+\frac{1009R^4}{423886848}-\ldots\right)\cos x' \sin y' \sin z', \\ v' &= -e^{-3T}\left(1-\frac{R^2}{768}+\frac{1009R^4}{423886848}-\ldots\right)\sin x' \cos y' \sin z', \end{aligned}\right\} \quad . \quad (57)$$

approximately, and for values of R for which the series converges, it follows that λ, the ratio of $\sqrt{\overline{p^2}}$ to $\frac{1}{2}\rho\overline{q^2}$, λ_η, and B all tend to return to their initial values for large values of the time. Thus λ, $\sqrt{\overline{p^2}}/(\frac{1}{2}\rho\overline{q^2})$, λ/λ_η, B must all reach minimum values and increase again, and λ_η must reach a maximum value and then decrease.

X. *Finite Complexes in Crystals : a Classification and Review.*

By A. F. Wells, M.A., Ph.D.,
Crystallographic Laboratory, Cambridge *.

[Received June 23, 1939.]

Introduction.

In a crystal atoms are arranged in some regular manner in space as the result of the forces acting between the various atoms. The atoms may be of one or more kinds, and the interatomic forces may be of different types. Excluding the forces in metallic crystals, which will not be considered here, three main types of interatomic binding are recognized : ionic, homopolar, and van der Waals. The binding may be uniformly of one type throughout the crystal, as, for example, in rock-salt (ionic), or bonds of more than one kind may be present. In crystals of this latter type we may distinguish aggregates of atoms within which the binding is of one kind whereas that between one aggregate and the next is of a different nature. Such groups of atoms may be finite (molecules and complex ions) or they may extend throughout the crystal as infinite chains or layers. We may therefore recognize three main types :—

(1) Finite complexes.

(2) Infinite one-dimensional complexes (chains and bands).

(3) Infinite two-dimensional complexes (layers).

The problem to be dealt with here is the enumeration and classification of finite complexes which in general will contain atoms of more than one kind.

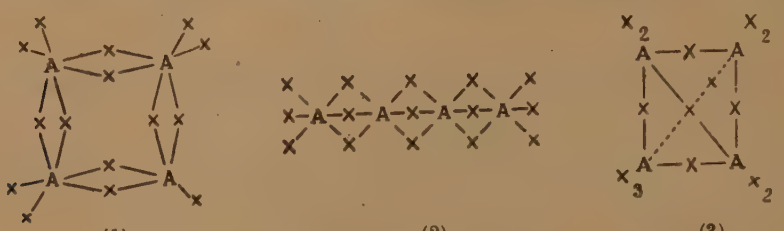
A finite complex may be characterized by the number, nature, and arrangement of its constituent atoms—in other words, by its chemical formula and point symmetry. We shall first consider the possibility of using one of these properties alone as a basis for classification.

(*a*) Any method based only on chemical formulæ has considerable

---

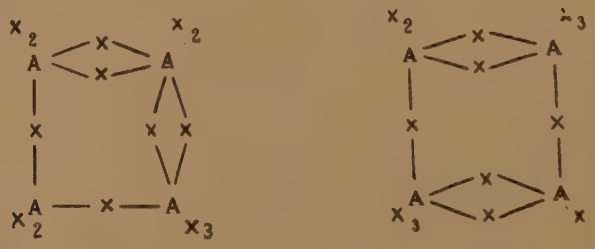
* Communicated by Professor J. D. Bernal, F.R.S.

disadvantages. Consider for example the hypothetical complex A_4X_{15}. This could exist in a very large number of forms, of which three are :



(1) (2) (3)

In the first place these complexes have nothing in common except their empirical formulæ, and it would appear that a more characteristic feature of (3), for example, is the arrangement of the four A atoms at the apices of a tetrahedron. Thus complexes of the same fundamental type as (3), but with different numbers of X atoms, should preferably be grouped with (3) rather than differentiated from (3) on account of their different formulæ A_4X_n. Secondly, it would be necessary in each class A_mX_n to devise some very elaborate system for distinguishing the various complexes one from another. It is not sufficient to state that of the fifteen X atoms nine, for example, are attached to one A atom and six to two A atoms, and that of the four A atoms two are attached to 6 X, one to 5 X, and the last to 4 X, for these conditions may be realized in a number of ways, *e. g.*,

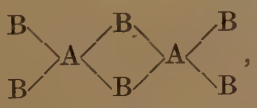


(4) (5)

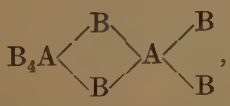
(*b*) Classification according to point symmetry is also undesirable for several reasons. The conception of maximum or " ideal " point symmetry would have to be introduced, since, although a group of atoms may in the free state exhibit its maximum possible point symmetry, this is often not the case in the crystal ; a small distortion of a molecule may reduce its point symmetry very considerably. Furthermore, the point symmetry of a complex is related to the symmetry of the lattice only if the complex occupies a special position in the lattice. In this case the coordinates of the points arising from the operation of the

point symmetry are listed in any space-group tables. When, however, the complex occupies a general position in the lattice its point symmetry and the lattice symmetry are unrelated, and for this reason it would be necessary to consider the less familiar non-crystallographic point groups in addition to the thirty-two crystallographic point groups. In the classification to be described below point symmetry will be treated as a secondary property and *not* as the basis of classification.

Before outlining the proposed scheme of classification it is interesting to examine briefly that of Niggli and Nowacki (1933). These authors attempted a classification of finite binary complexes A_nB_m based on a combination of chemical formulæ and point symmetry. They first defined a " finite binary regular point system " composed of a finite number of A and B atoms held together by links A–B as an arrangement in which all the atoms of one type (A or B) and all the distances of one type (A–A, B–B, or A–B) are equivalent and related by some point symmetry. (The condition regarding the equivalence of all the B–B distances was relaxed in certain cases.) In such complexes every A atom is necessarily joined to the same number of B atoms and *vice versa*. In many possible complexes A_nB_m, however, there are different numbers of B atoms attached to different A atoms, or conversely different B atoms may be joined to different numbers of A atoms. It was therefore necessary to extend the classification, and further complexes were derived obeying less stringent conditions. The equivalence of all the A atoms was still insisted on, but the only other requirement was that it must be possible in any complex to make all the links A–B of equal length. In this way they included complexes of the type



in which the A atoms are still equivalent but the B atoms are of two kinds. Further relaxing of the conditions would be necessary to include, for example,



but the classification was not carried further. Apart from the complexity of such a system, it is felt that the principle of first imposing restrictions and then relaxing them in order to extend a classification which is obviously too limited is to be avoided. The scheme developed here is comprehensive in that within the stated limits all complexes, known and unknown, are included.

It has been shown above that a system based on chemical formulæ soon becomes too complicated to be practical. This difficulty is surmounted by classifying complexes according to the number and arrangement of the atoms of one kind only. This atom A will, in general, be electropositive with respect to the atoms X, and will in general form more bonds than X. According to this scheme the above complexes (1)–(3) are referred to the simple forms

(1) (4) (5)

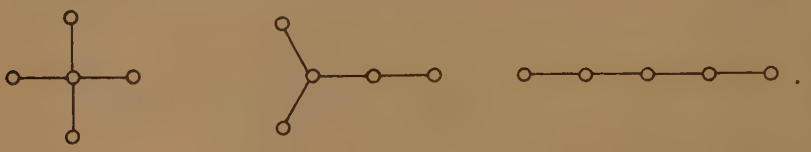
(2)

(3)

which retain the basic arrangement of the A atoms in the original complexes, regardless of the number and position of the X atoms. It might appear from this that the system takes account only of binary complexes A_mX_n, but it will be seen later that this is not the case.

A Classification of Finite Complexes.

The problem has been reduced to finding in which ways a specified number of points may be arranged in space to form a connected system. In general the minimum number of links required to connect together p points is $p-1$, the resulting systems containing any number from 2 to $p-1$ of singly-connected points. For example, for 5 points and 4 links :



The maximum number of points is $\frac{p}{2}\cdot(p-1)$, when each point is connected to every other point. If we denote by N_n the number of points connected to n others then in general

$$N_1+2N_2+3N_3 \ldots\ldots (p-1)\,N_{p-1}=2l \ldots\ldots \text{(i.)}$$

and $$N_1+N_2+N_3 \ldots\ldots N_{p-1}=p, \ldots\ldots \text{(ii.)}$$

where l=number of links and p=total number of points.

For the present purpose it is not necessary to consider systems involving such large numbers of links as $\frac{p}{2}\cdot(p-1)$, and various restrictions might be introduced. These are of two types :—

(1) We may restrict the total number of links to some function of p. For example, we might postulate convex polyhedra as the limiting cases,

since cross-connected polyhedra (*i. e.*, those with links between vertices of different faces) are of little interest here. The maximum permissible number of links would then be $3p-6$, corresponding to convex polyhedra of $2p-4$ triangular faces. Equations (i.) and (ii.) would then be solved for each system p_l (which denotes a system of p points and l links) for all possible values of $N_1, N_2, \ldots N_{p-1}$.

(2) Alternatively we may limit the number of links which may meet in a given point. This naturally also limits the total number of links, since if not more than n links may meet in a point then the maximum number of links is $\frac{np}{2}$, when every point is joined to n others. The difference from (1) is that we now consider only those solutions of (i.) and (ii.) which involve values of $N_1, N_2, \ldots N_n$.

Now in these complexes the links will ultimately represent chemical bonds, and we are therefore not interested in complexes in which large numbers of links meet in any of the points. In fact, for practical purposes, we may conveniently restrict ourselves to systems in which not more than four links meet in any point, though of course the classification could at any time be extended to include, for example, 6-connected points. Furthermore :

(*a*) Since it is always possible to add any number of singly-connected points to a finite system p_l by adding p points together with p links, such points will not be considered in the classification.

(*b*) Links starting and ending at one and the same atom without passing through other atoms will be entirely excluded.

(*c*) Multiple links are excluded at this stage and introduced later.

Equations (i.) and (ii.) now have the form

$$2N_2+3N_3+4N_4=2l, \quad \ldots\ldots \quad \text{(i. } a\text{)}$$

$$N_2+ N_3+ N_4= p. \quad \ldots\ldots \quad \text{(ii. } a\text{)}$$

The minimum number of links for p points is now p (polygon of p sides) and the maximum number $2p$. In the latter case every point is connected to four others. The problem has, therefore, been reduced to the evaluation of N_2, N_3, and N_4 for values of l from p to $2p$, and this has been carried out for p from 3 to 12 *. For a system consisting of a specified number of points and links (*e. g.*, 6_{10}) there will be a number of arrangements distinguished by different values of N_2, N_3, and N_4 satisfying

---

* See notes to Table I.

equations (i. a) and (ii. a). The generalized solutions for the system p are

N_2.	N_3.	N_4.
$2p-l$	—	$l-p$
$2p-l-1$	2	$l-p-1$
$2p-l-2$	4	$l-p-2$
	etc.,	

proceeding until either N_2 or N_4 becomes zero. These solutions are given in Table I., and for completeness the single point system 1_0 is included. We obtain in this way some 233 complexes characterized firstly by the number of points and links (p_l) and, secondly, by the numbers of points in which two, three, and four links meet (N_2, N_3, and N_4). These data do not, however, in general define a complex completely, since the environment of each point is not specified. Accordingly each of these complexes exists in one or more forms distinguished by having different *relative* arrangements of the 2-, 3-, and 4-connected points. This matter will be dealt with more fully as the different classes of complexes are considered in detail.

TABLE I.

p.	l.	N_2.	N_3.	N_4.
1	—	—	—	—
3	3	3		
4	4	4		
	5	2	2	
	6		4	
5	5	5		
	6	4	—	1
		3	2	
	7	3	—	2
		2	2	1
		1	4	
	8	1	2	2
			4	1
	9		2	3
	10			5
6	6	6		
	7	5	—	1
		4	2	
	8	4	—	2
		3	2	1
		2	4	
	9	3	—	3
		2	2	2
		1	4	1
			6	
	10	2	—	4
		1	2	3
			4	2
	11	1	—	5
			2	4
	12			6
7	7	7		
	8	6	—	1
		5	2	
	9	5	—	2
		4	2	1
		3	4	
	10	4	—	3
		3	2	2

TABLE I. (*cont.*).

p.	*l.*	N_2.	N_3.	N_4.
		2	4	1
		1	6	
	11	3	—	4
		2	2	3
		1	4	2
			6	1
	12	2	—	5
		1	2	4
			4	3
	13	1	—	6
			2	5
	14			7
8	8	8		
	9	7	—	1
		6	2	
	10	6	—	2
		5	2	1
		4	4	
	11	5	—	3
		4	2	2
		3	4	1
		2	6	
	12	4	—	4
		3	2	3
		2	4	2
		1	6	1
			8	
	13	3	—	5
		2	2	4
		1	4	3
			6	2
	14	2	—	6
		1	2	5
			4	4
	15	1	—	7
			2	6
	16			8
9	9	9		
	10	8	—	1
		7	2	
	11	7	—	2
		6	2	1
		5	4	
	12	6	—	3
		5	2	2
		4	4	1
		3	6	
	13	5	—	4
		4	2	3
		3	4	2
		2	6	1
		1	8	
	14	4	—	5
		3	2	4
		2	4	3
		1	6	2
			8	1
	15	3	—	6
		2	2	5
		1	4	4
			6	3
	16	2	—	7
		1	2	6
			4	5
	17	1	—	8
			2	7
	18			9
10	10	10		
	11	9	—	1
		8	2	
	12	8	—	2
		7	2	1
		6	4	
	13	7	—	3
		6	2	2
		5	4	1
		4	6	
	14	6	—	4
		5	2	3
		4	4	2
		3	6	1
		2	8	
	15	5	—	5
		4	2	4
		3	4	3
		2	6	2
		1	8	1
			10	
	16	4	—	6
		3	2	5
		2	4	4

TABLE I. (*cont.*).

p.	*l.*	N_2.	N_3.	N_4.
		1	6	3
			8	2
	17	3	—	7
		2	2	6
		1	4	5
			6	4
	18	2	—	8
		1	2	7
			4	6
	19	1	—	9
			2	8
	20			10
11	11	11		
	12	10	—	1
		9	2	
	13	9	—	2
		8	2	1
		7	4	
	14	8	—	3
		7	2	2
		6	4	1
		5	6	
	15	7	—	4
		6	2	3
		5	4	2
		4	6	1
		3	8	
	16	6	—	5
		5	2	4
		4	4	3
		3	6	2
		2	8	1
		1	10	
	17	5	—	6
		4	2	5
		3	4	4
		2	6	3
		1	8	2
			10	1
	18	4	—	7
		3	2	6
		2	4	5
		1	6	4
			8	3
	19	3	—	8
		2	2	7
		1	4	6
			6	5
	20	2	—	9
		1	2	8
			4	7
	21	1	—	10
			2	9
	22			11
12	12	12		
	13	11	—	1
		10	2	
	14	10	—	2
		9	2	1
		8	4	
	15	9	—	3
		8	2	2
		7	4	1
		6	6	
	16	8	—	4
		7	2	3
		6	4	2
		5	6	1
		4	8	
	17	7	—	5
		6	2	4
		5	4	3
		4	6	2
		3	8	1
		2	10	
	18	6	—	6
		5	2	5
		4	4	4
		3	6	3
		2	8	2
		1	10	1
			12	
	19	5	—	7
		4	2	6
		3	4	5
		2	6	4
		1	8	3
			10	2
	20	4	—	8
		3	2	7
		2	4	6
		1	6	5

TABLE I. (*cont.*).

p.	l.	N_2.	N_3.	N_4.	p.	l.	N_2.	N_3,	N_4.
			8	4			1	2	9
	21	3	—	9				4	8
		2	2	8		23	1	—	11
		1	4	7				2	10
			6	6		24			12
	22	2	—	10					

Notes to Table I.

(*a*) For any closed system of p points the maximum number of links is $\frac{p}{2} \cdot (p-1)$. Solutions are not possible, therefore, for the systems 3_4, 3_5, 3_6, 4_7, and 4_8.

(*b*) The following solutions cannot be realized and are omitted —

p.	l.	N_2.	N_3.	N_4.
4	5	3	—	1
4	6	2	—	2
5	8	2	—	3
5	9	1	—	4

(*c*) The complex 5_{10} is a special case where $l > 3p-6$, and is therefore realizable only as a cross-connected polyhedron.

(*d*) There is no convex 4-connected polyhedron 7_{14}.

It is not proposed to deal with all the complexes given in Table I., but rather to consider certain of the more important classes, and these may be described by the relationship between p and l.

I. $l=p$.—These are the simple polygons, consisting of p 2-connected points.

II. $l=3p/2$.—This class includes those convex polyhedra at each vertex of which three edges meet. Since the number of edges or links (l) must be an integer, the values of p must be even numbers, giving the complexes 4_6, 6_9, 8_{12}, 10_{15}, and 12_{18}. The number of faces (f) follows from Euler's rule, $p+f=l+2$. The complexes 4_6 and 6_9 each exist in one form only, the tetrahedron and trigonal prism respectively, but all higher polyhedra $p_{3p/2}$ exist in more than one form according to the number and nature of the polygons forming their surfaces. If f_3, f_4, f_5,

etc. denote the numbers of 3-, 4 , 5-, etc. sided faces forming the surface of the polyhedron, then all forms of the polyhedra satisfy the equations

$$3f_3+4f_4+5f_5, \text{ etc.} = 2l\ (=3p) \quad \text{(iii.)*}$$

and

$$p+f=l+2. \quad \text{(iv.)}$$

Finally, there may also be more than one arrangement of the same polygons (see fig. 1). The 3-connected polyhedra from 4_6 to 12_{18} are illustrated in fig. 1, adopting the convention of drawing the rest of the faces within one face of the polyhedron.

III. $3p/2<l<2p$.—Every system of this type (for which $N_2=0$) represents a polyhedron of a mixed type, in which some of the points are 3-connected and the rest 4-connected. The simplest of these mixed polyhedra are the tetragonal pyramid 5_8 and the trigonal bipyramid 5_9.

IV. $l=2p$.—These include the 4-connected polyhedra, 6_{12}, 8_{16}, 9_{18}, 10_{20}, 11_{22}, and 12_{24}. There is no polyhedron of this class for $p=7$. As in the case of the 3-connected polyhedra they may be further analysed (Table III.) according to the number and nature of the polygons forming their surfaces, and yet further by the relative arrangement of a given set

---

* The number of terms in equation (iii.) is strictly limited, since for a polyhedron with a given number of vertices (p) no face may have more than a certain number of corners. The size of this " maximum polygon " may be found as follows :—For a polyhedron of p points let the maximum polygon contain $p-q$ points. Then by representing the polyhedron drawn within the maximum polygon we have to add q points within this polygon ($p-q$). The q points must

p.	N_3.	N_3–N_4.	N_4.
4	3	—	—
5	—	4	—
6	4	4	3
7	—	5	(3)
8	5	6	4
9	—	6	4
10	6	7	5
11	—	8	5
12	7	8	6

form a connected system and must clearly possess the maximum number of free links. This number is $q+2$, which must equal $p-q$. Hence $p=2q+2$, and the maximum polygon is one of $(p+2)/2$ sides. For polyhedra of other types similar arguments apply, and the results are given in the accompanying table. The second, third, and fourth columns show the size of the " maximum polygon " for 3-, mixed 3- and 4-, and 4-connected polyhedra.

of polygons (faces). They are illustrated in fig. 2 up to 12_{24}, the first member being the octahedron 6_{12}.

TABLE II.

The Three-connected Polyhedra from 4_6 to 12_{18}.

p.		f_3.	f_4.	f_5.	f_6.	f_7.	Point symmetry.
4		4					$\bar{4}3m$
6		2	3				$3/mmm$
8	(1)	2	2	2			mm
	(2)		6				$m3m$
10	(1)	2	3	—	2		mm
	(2)	3	—	3	1		$3mm$
	(3)	2	2	2	1		2
	(4)	1	3	3			$3mm$
	(5)		5	2			$5/mmm$
12	(1)	2	4	—	—	2	mm
	(2)	3	1	2	1	1	1
	(3)	2	3	1	1	1	1
	(4)	2	2	3	—	1	m
	(5)	4	—	—	4		$43m$
	(6)	3	1	1	3		m
	(7 a)	2	2	2	2		mm
	(7 b)	2	2	2	2		1
	(8)	1	4	1	2		m
	(9)	—	6	—	2		$6/mmm$
	(10)	2	1	4	1		mm
	(11)	1	3	3	1		m
	(12)	2	—	6			$32m$
	(13)	—	4	4			m

I (a).—One other class of some importance arises when $l=p+1$ (complexes 6_7, 7_8, etc.). Application of equations (i. a) and (ii. a) gives the following solutions for p_{p+1} :—

$$N_2=p-1 \text{ and } N_4=1.$$

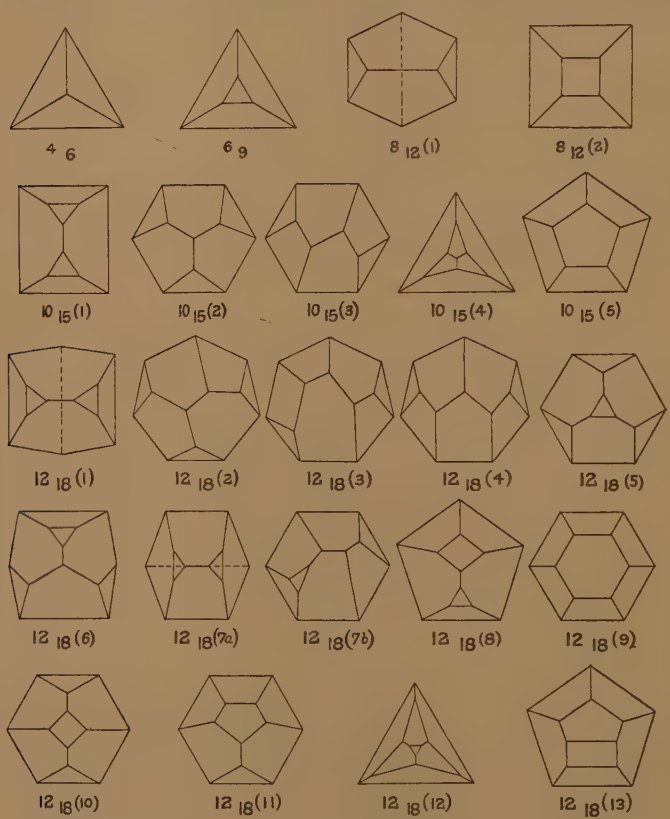
$$N_2=p-2 \text{ and } N_3=2.$$

The first solution represents two polygons with a common corner. The second gives rise to various forms, the simplest of which consists of two polygons with a common side. All the possible forms of these complexes from 4_5 to 7_8 are illustrated in fig. 3.

The maximum symmetry of each polyhedron of figs. 1 and 2 is given in Tables II. and III. respectively. One advantage of this method of drawing the polyhedra is that each diagram represents all the forms of a given convex polyhedron with point symmetries ranging from the maximum down to 1. For example, the most symmetrical form of 4_6 is the tetrahedron, with point symmetry $\bar{4}3m$, but the first diagram in fig. 1 also represents the following figures, the point symmetries of which are

Fig. 1.

The data relating to these polyhedra are given in Table II.

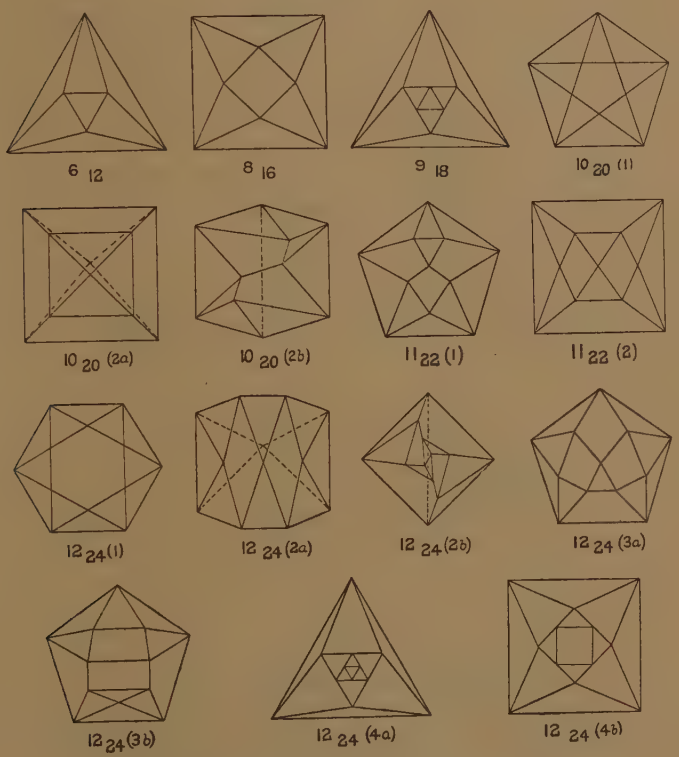


The 3-connected polyhedra from 4_6 to 12_{18}.

The polyhedra may be visualized by imagining the vertices within the edges bounding the diagram to be raised above the surface of the paper, so that the polyhedron is seen standing on one face. In order to show the symmetry it has been necessary in certain cases to depart from this convention, and edges which would be invisible to the observer are shown as dotted lines.

sub-groups of $\bar{4}3m$:—tetragonal bisphenoid, $\bar{4}2m$, trigonal pyramid, $3m$, and rhombic bisphenoid, 222. Similarly the trigonal trapezohedron, the rhombohedron, and the tetragonal and rhombic prisms are less symmetrical forms of the polyhedron 8_{12} (cube). The polyhedral complexes A_mX_n discussed below are obtained by placing X atoms on the edges and faces of polyhedra A_m, and the only forms derived are those which retain the original symmetry of the polyhedron. Thus by placing an X atom on

Fig. 2.



The 4-connected polyhedra from 6_{12} to 12_{24}.

(or above) each face of an octahedron A_8 we obtain the complex A_8X_8 with point symmetry $m3m$, but we could omit the X atoms from opposite faces giving a complex A_6X_6 with trigonal symmetry. The first member of this family of complexes is A_4X_4 (4 *b*, p. 115), and higher members may be derived from the polyhedra $2n_{4n}$ with axes of rotatory inversion (see Table III.).

Having outlined the scheme of classification it is now proposed to deal in more detail with the various classes of complexes, giving examples.

It is to be understood that the examples given are those whose constitution has been established by X-ray or electron diffraction unless a statement to the contrary is made.

TABLE III.

The Four-connected Polyhedra from 6_{12} to 12_{24}.

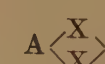
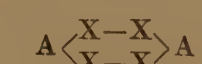
p.		f_3.	f_4.	f_5.	f_6.	Point symmetry.
6		8				$m3m$
8		8	2			$\bar{8}$
9		8	3			$3/mmm$
10	(1)	10	—	2		$\bar{5}$
	(2 *a*)	8	4			$4/mmm$
	(2 *b*)	8	4			2
11	(1)	9	3	1		m
	(2)	8	5			$2mm$
12	(1)	12	—	—	2	$\bar{1}\bar{2}$
	(2 *a*)	10	2	2		mm
	(2 *b*)	10	2	2		2
	(3 *a*)	9	4	1		m
	(3 *b*)	9	4	1		m
	(4 *a*)	8	6			$\bar{3}m$
	(4 *b*)	8	6			$m3m$

Derivatives of the Complex 1_0.

The complexes of this group are characterized by the absence of a closed system of A atoms. They may be further subdivided as below :—

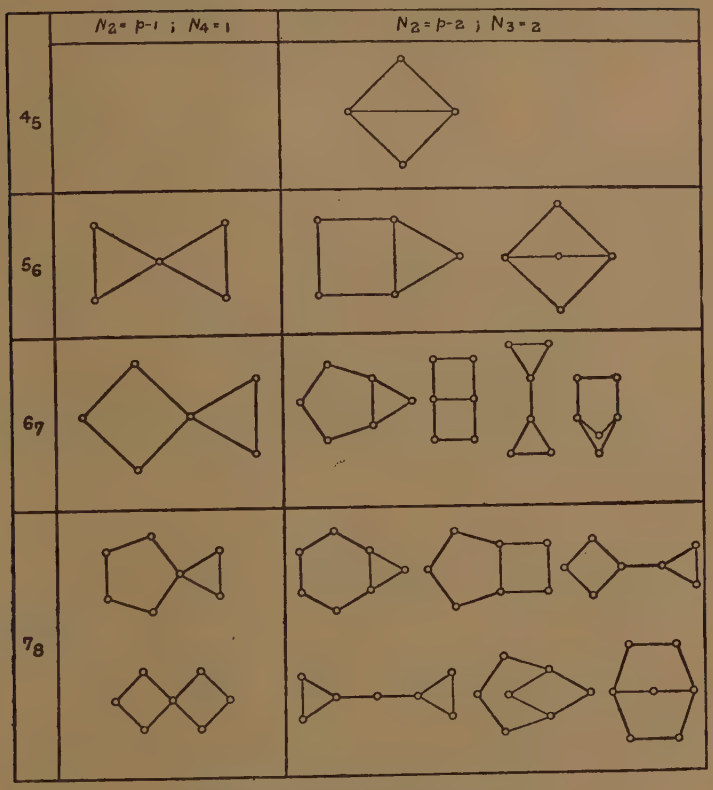
1_0 A

1_0 (2 *a*) A—A A—X—A A—X—X—A

(2 *b*) A=A   Higher classes similarly.

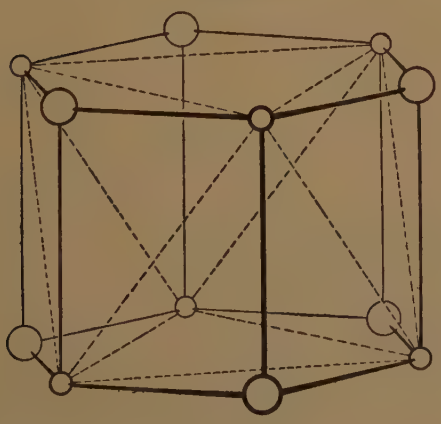
(2 *c*) A≡A

Fig. 3.



The complexes p_{p+1} from 4_5 to 7_8.

Fig. 4.



(2 *d*) A≡A

1_0 (3 *a*) A—A—A A—X—A—X—A A—X—X—A—X—X—A

(3 *b*) A—A=A A—X—A etc.

(3 *c*) A=A=A A A A

(3 *d*) A—A≡A

and similarly for higher classes.

It is to be understood that any number of singly-connected atoms may be attached to any A atom in the above table, the complex being referred always to the fundamental form involving A atoms only.

The Complex 1_0.

This class includes all complexes AY_n, and in the following table these are further classified according to the disposition of the bonds from A. For a list of the elements forming directed bonds of the various types reference may be made to Pauling and Huggins (1934). A definition of a complex ion is suggested and briefly discussed in the Annual Reports of the Chemical Society (Bernal and Crowfoot, 1933). Examples are given only of the first six groups. Those of the tetrahedral and octahedral complexes are too numerous to quote here, while examples of the other three possible types are not yet known in crystals. References are given only to work published since the completion of the fourth volume of the ' Strukturbericht.'

Type.	*Examples.*
AY	CO
AY_2 linear	$[Ag(NH_3)_2]^+$*, $[Ag(CN)_2]^-$**, $[Tl(CH_3)_2]^+$***, CO_2, O=C=S, $[ICl_2]^-$, $[HF_2]^-$
AY_2 non-linear	NO_2^-, $[ClO_2]^-$
AY_3 planar	CO_3^{2-}, NO_3^-
AY_3 pyramidal	SO_3^{2-}, ClO_3^-, BrO_3^-

*, in $Ag_2SO_4.4NH_3$; **, in $KAg(CN)_2$; ***, in $[Tl(CH_3)_2]I$. Unless otherwise stated complex ions have been studied in their K, Na, or NH_4 salts.

AY_4 planar	$AuCl_4^-$, $AuBr_4^-$, $CuCl_4^{2-}$, $CuCl_2(H_2O)_2$ $PdCl_4^{2-}$, $PtCl_4^{2-}$, ICl_4^-.
AY_4 tetrahedral	
AY_5	(See note below.)
AY_6 octahedral	
AY_6 trigonal prism	
AY_8 cubic	

Note added in proof.—In $Sb(CH_3)_3Cl_2$ the interatomic distances indicate a state intermediate between the covalent molecule and the ionic form $[Sb(CH_3)_3]Cl_2$. A similar trigonal bipyramidal configuration has been found for the molecules $MoCl_5$, PF_3Cl_2, PF_5, and PCl_5 in the vapour state. The only pentahalide yet examined in the crystalline state, PCl_5, has an ionic structure and may be formulated $[PCl_4][PCl_3]$. (Powell, Clark, and Wells, ' Nature,' Lond. cxlv. p. 149 (1940).)

The Complex 1_0 (2 *a*).

This class includes all diatomic molecules and ions and the simplest pyro- and per-ions, and bridged compounds of general formula $Y_m—A—X—A—Y_n$.

Type.	*Examples.*
A—A	Cl_2, Br_2, I_2, C_2^{2-} (in CaC_2), O_2^{2-} (in BaO_2)
YA—A	
YA—AY	
$Y_2A—A$	
$Y_2A—AY$	
$Y_2A—AY_2$	$O_2N—NO_2$
$Y_3A—A$	$O_3S—S^{2-}$
$Y_3A—AY$	
$Y_3A—AY_2$	$O_3S—SO_2^{2-}$
$Y_3A—AY_3$	$O_3S—SO_3^{2-}$, $H_3B—BH_3$

It is convenient to distinguish as pyro-ions those in which the atom X is shared between two A atoms (A—X—A and derivatives) as in the pyrophosphate ion $[O_3P—O—PO_3]^{4-}$. In per-ions X atoms attached to different A atoms are joined (A—X—X—A, etc.), an example being the persulphate ion $[O_3S—O—O—SO_3]^{2-}$. Throughout this paper the symbol X will be retained for an atom forming two bonds and Y for an atom forming only

one bond. On account of the large number of bridged compounds in which X forms two bonds it is convenient to tabulate some of the more important atoms and groups which can form (*a*) —X—, (*b*) —X→ or ←X→, and (*c*) Y— or Y→. In the following table (*b*) is distinguished from (*a*) because an atom or group of type (*b*) may act as Y, whereas those under (*a*) can do so only (1) by forming a multiple bond, as in O=C=O and C≡O, (2) by taking up an electron when forming an electron-pair bond as in SiO_4^{-4}, or (3) by receiving two electrons from a coordinate link, $S(\rightarrow O)_4$.

(*a*).	(*b*).	(*c*).
Type X.	Type X or Y.	Type Y.
$-CH_2-$	$-CN\rightarrow$	$-CH_3$
—NH—	$-\overset{\overset{O}{\|}}{C}-$ *	$\leftarrow NH_3$
—O—	$-NH_2\rightarrow$	
	$-OH\rightarrow$	
	$-Cl\rightarrow$	
	$\leftarrow H_2O\rightarrow$	
	$\leftarrow O_2\rightarrow$	

Apart from the pyro-ions already mentioned the following types of bridged compound are possible : Y—A—X—A—Y, $Y_2A-X-AY_2$ (*e.g.*, the hypothetical B_2O_5 ion, $\begin{matrix}O\\O\end{matrix}\!\!>\!B-O-B<\!\!\begin{matrix}O\\O\end{matrix}$), $Y_3A-X-AY_3$ (planar and tetrahedral), $Y_5A-X-AY_5$, etc. Examples of the last type include those binuclear cobaltammines in which one atom or group is shared between two cobalt atoms as in $[(NH_3)_5Co-NH_2-Co(NH_3)_5]Cl_5$. The structure of these compounds has not yet been studied by X-ray methods.

The Complex 1_0 (2 *b*).

In this class we have an important group of " bridged " compounds. The system $A\begin{matrix}X\\<\\X\end{matrix}\!\!>\!A$ is easily formed if the bond angles of A are approxi-

* The recent investigation (Powell and Ewens, 1939) of the structure of $Fe_2(CO)_9$ indicates that in this compound the carbon atoms of the shared CO groups form two normal covalent bonds, $-\overset{\overset{O}{\|}}{C}-$, and that the linking of the metal atoms is not of the type —C≡O— suggested by Sidgwick and Bailey (1933).

mately 90°, *i. e.*, for A atoms in 4-square or 6-octahedral coordination. As the angle X—A—X increases the A atoms are brought nearer together and there is less tendency for the formation of such compounds. Thus $Y.A\langle^{X}_{X}\rangle A.Y$ is not known. For 4-coordinated atoms there are three forms of $Y_2.A^1\langle^{X}_{X}\rangle A^2.Y_2$ according to the valency angles of A^1 and A^2:—

(*a*) A^1 tetrahedral — A^2 tetrahedral. The halides A_2Cl_6 (A=Al, Fe, etc.) exist in the vapour state as molecules of this type (Palmer and Elliott, 1938).

(*b*) A^1 tetrahedral—A^2 planar. No compound has yet been shown to have this configuration, though it has been suggested for the cuprous-cupric complex (Mellor and others, 1938)

$$\begin{matrix} Cl \\ \phi_2CH_3As \end{matrix}\!\!\searrow\!\!\nearrow Cu \swarrow\!\!\nwarrow \begin{matrix} Cl \\ Cl \end{matrix}\!\!\rangle Cu \swarrow\!\!\nwarrow \begin{matrix} AsCH_3\phi_2 \\ AsCH_3\phi_2 \end{matrix}.$$

(*c*) A^1 planar—A^2 planar. Tribromo-gold, Au_2Br_6, is presumably of this type and numerous binuclear bridged compounds such as

$$\begin{matrix} C_2H_5 \\ C_2H_5 \end{matrix}\!\!\rangle Au \swarrow\!\!\nwarrow \begin{matrix} Br \\ Br \end{matrix}\!\!\searrow\!\!\nearrow Au\langle \begin{matrix} C_2H_5 \\ C_2H_5 \end{matrix}$$

(Burawoy and others, 1937).

$$(CH_3)_3As \searrow \begin{matrix} Cl \\ Cl \end{matrix}\!\!\swarrow\!\!\nwarrow Pd \swarrow\!\!\nwarrow \begin{matrix} Cl \\ Cl \end{matrix}\!\!\searrow Pd \swarrow\!\!\nwarrow Cl \nwarrow As(CH_3)_3$$

(Wells, 1938).

The derivatives $Y_4\ A\langle^{X}_{X}\rangle A\ Y_4$, consisting of two octahedra sharing an edge, are possible for many elements A, and include the binuclear ammines of cobalt and chromium, and probably also certain palladous-palladic compounds. Examples are:

$$\left[(NH_3)_4Co\langle^{OH}_{OH}\rangle Co(NH_3)_4\right]Cl_4,$$

$$\left[(NH_3)_4Co\langle^{NH_2}_{NH_2}\rangle Co(NH_3)_4\right]Cl_4,$$

$$\begin{matrix} & Cl & & Cl & \\ NH_3\searrow & | & \swarrow Cl\rangle & | & NH_3 \\ NH_3\nearrow & Pd & \nwarrow Cl\rangle & Pd & \nwarrow NH_3. \\ & | & & | & \\ & Cl & & Cl & \end{matrix}$$

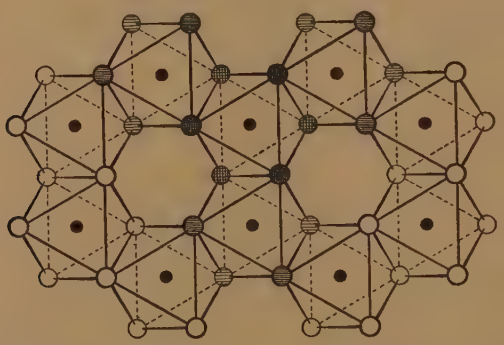
No compounds of this type have yet been studied by X-ray diffraction.

The simplest form in 1_0 (2 *c*), $A\langle^{X}_{X}\rangle A$, is not known, though it may formally be distinguished as the unit of the structure of corundum,

Al_2O_3*. The derivatives $Y_3A\begin{smallmatrix}X\\X\\X\end{smallmatrix}AY_3$ are important, and consist of two octahedra with one face in common. Examples are: the ions $Tl_2Cl_9^{3-}$ and $W_2Cl_9^{3-}$ in $Cs_3Tl_2Cl_9$ and $K_3W_2Cl_9$ respectively, cobaltammines of the type $[(NH_3)_3Co(OH)_3Co(NH_3)_3]Cl_3$, and iron enneacarbonyl, $Fe_2(CO)_9$.

The group 2 (*d*) would include complexes in which two cubes share a face, as in the hypothetical molybdenum compounds $Y_4Mo.X_4.MoY_4$, which are as yet unknown.

Fig. 5.



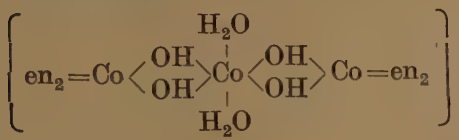
The complex $\left[Co\left\{\begin{smallmatrix}OH\\OH\end{smallmatrix}\right\rangle Co(NH_3)_4\right\}_3\Big]^{6+}$ and its relation to the layer lattice AX_3 of, *e. g.*, $Al(OH)_3$. The small black circles represent metal atoms in the plane of the paper, and the large circles X atoms above and below that plane forming octahedra around the metal atoms. In the cobalt complex a central octahedron $Co(OH)_6$ shares an edge with each of three other octahedra $Co(NH_3)_4(OH)_2$ (NH_3 shaded horizontally, OH horizontally and vertically).

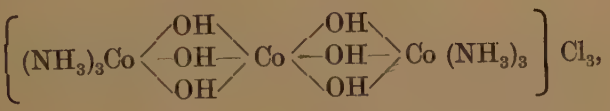
The next class includes all triatomic molecules and ions (apart from those in which the three A atoms form a ring) and all derivatives containing this nucleus. Examples of the complex A_3 are: N_3^-, S_3^{2-}, and I_3^- and of

---

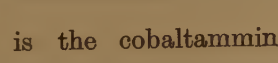
* (Niggli and Nowacki, 1933). This structure is, however, preferably regarded as a homopolar lattice of oxygen atoms in approximately hexagonal close-packing with aluminium atoms occupying two-thirds of the octahedral holes.

a derivative $Y_3A_3Y_3$ the trithionate ion $O_3S—S—SO_3^{2-}$. The trinuclear cobaltammines





are examples of derivatives of A—X_2—A—X_2—A and A—X_3—A—X_3—A which await examination.

Few examples are known of complexes A_n, where n is greater than 3, in which the A atoms do not form a closed system. One important example is the cobaltammine  , which is historically interesting as being the first purely inorganic compound to be resolved into its optical antimers. The probable structure of this cation is shown in fig. 5.

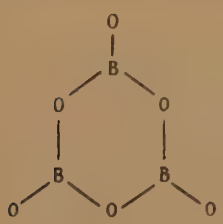
The Simple Ring Complexes p_p and their Derivatives.

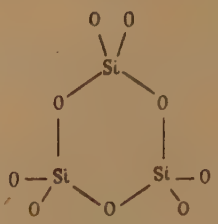
As in the case of the derivatives of the single-point complex, one or more X atoms may be placed on the links between the A atoms, and these links may be single or multiple. The following derivatives, for example, arise from the triangle 3_3:

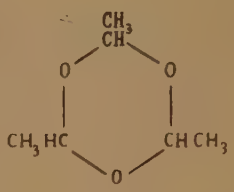
The Complex 3_0.

No triatomic molecules or ions of this type are known, but a number of derivatives are of interest. The actual nature of the ring (*i. e.*, whether planar or buckled) will depend on the bond angles of A and X. For example, the $B_3O_6^{3-}$ ion in $NaBO_2$ forms a plane ring (1) (Ssu-Mien Fang, 1938). The analogous form for tetrahedral atoms A is represented by the $Si_3O_9^{6-}$ ion in benitoite, $BaTiSi_3O_9$, and wollastonite

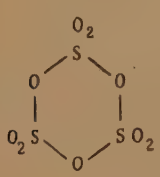
$CaSiO_3$ (2). In the vapour state paraldehyde forms the molecule (3). Electron diffraction studies of the vapour of $B_3N_3H_6$ indicate a benzene-

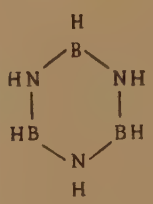

(1).

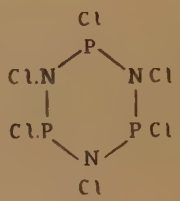

(2).


(3).

like ring (Stock and Wierl, 1931; Bauer, 1938), and 3-membered rings have been suggested for $(SO_3)_3$ and $(PNCl_2)_3$.

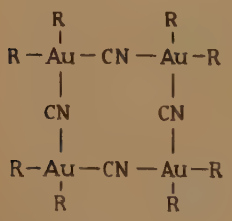


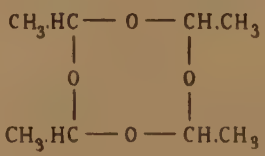




The Complex 4_4.

An example of this complex is the planar auric compound $(AuCN.R_2)_4$, where $R=C_3H_7$, and similar palladous compounds $(PdXY_2)_4$ could exist. Metaldehyde, $(CH_3CHO)_4$, forms a four-membered ring in the crystal, and pentaborane, B_5H_9, has been assigned a ring structure as the result of electron diffraction studies of the vapour.





The Complex 5_5.

No five-ring complexes are yet known. A ring of five tetrahedra $A_5X_5Y_{10}$ will be planar if the bond angles of X are 180°, since the internal angle of a regular pentagon is approximately equal to the tetrahedral angle.

The Complex 6_6.

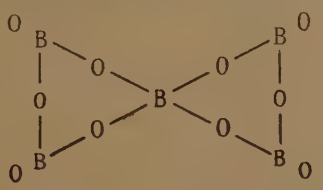
The simple ring C_6 forms the basis of aromatic organic chemistry. The ring Si_6O_{18} is found in beryl, $Be_3Al_2Si_6O_{18}$.

The Complex 8_8.

The only example yet known is the S_8 molecule of rhombic sulphur.

Complexes of the Class p_{p+1}.

This extensive class (see fig. 3 above) is as yet represented by only one example, the pentaborate ion, B_5O_{10} in $KH_2(H_3O)_2B_5O_{10}$ (Zachariasen, 1937). It is remarkable for showing the element boron



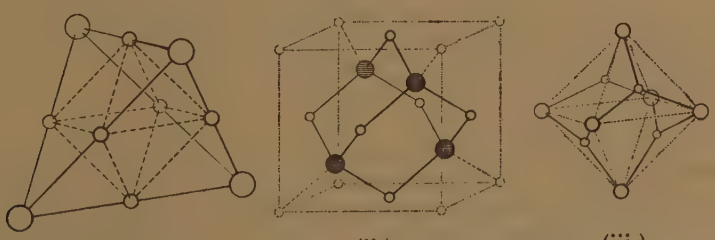
in both 3-planar and 4-tetrahedral coordination in the same complex ion.

The Polyhedral Complexes.

The Tetrahedral Complex 4_6.

The simple tetrahedral molecule A_4 is not known in any crystal, but it has been shown by electron diffraction that phosphorus and arsenic form such molecules P_4 and As_4 in the vapour state. Similar tetrahedral molecules may exist in crystalline yellow phosphorus and arsenic.

Fig. 6.



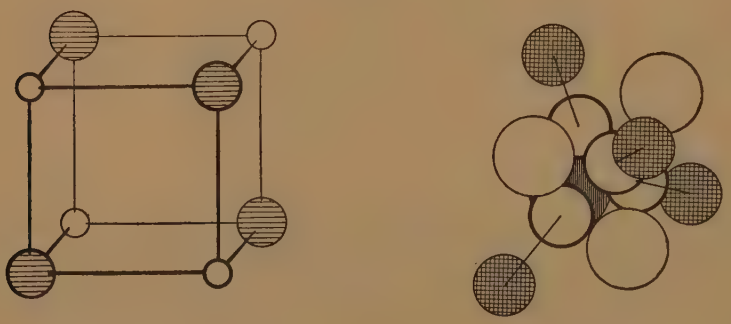
(i.) (ii.) (iii.)

4_6 (*a*).—If atoms X are placed on the edges of the tetrahedron we obtain a complex A_4X_6 in which the X atoms lie at the apices of an octahedron. The cubic forms of As_4O_6 and Sb_4O_6 contain molecules of this sort, and the molecules of hexamethylene tetrammine, $N_4(CH_2)_6$, have a similar

configuration. The form (i.) is drawn for pyramidal A bonds and co-linear X bonds, while (iii.) shows the configuration if the bonds from A are coplanar. The intermediate form (ii.) shows the relation of this complex to the structure of zinc blende, the edges of the unit cell being indicated by dotted lines. The oxides P_4O_6 and P_4O_{10} in the vapour state are further examples of this complex. Electron diffraction experiments show that the nucleus of the molecule of the latter is similar to that of P_4O_6, but there is an additional oxygen atom attached to each phosphorus (Hampson and Stosick, 1938). Finally the molecule of basic beryllium acetate, $Be_4(CH_3COO)_6O$, has the tetrahedral nucleus A_4X_6 with, in addition, an oxygen atom at the centre of the tetrahedron of beryllium atoms.

4_6 (*b*).—If atoms are placed on the faces of the tetrahedron then the complex has the formula A_4X_4. One form is shown in fig. 7, with cubic symmetry $43m$. No complex A_4X_4 of this simple type is known,

Fig. 7.



(*a*) (*b*)

(*a*) Idealized form of the complex A_4X_4.

(*b*) The complex $Cu_4(Et_3As)_4I_4$ consisting of tetrahedra of Cu atoms, of iodine (large open circles), and of arsenic (crossed shading). The twelve ethyl groups attached to the arsenic atoms are omitted.

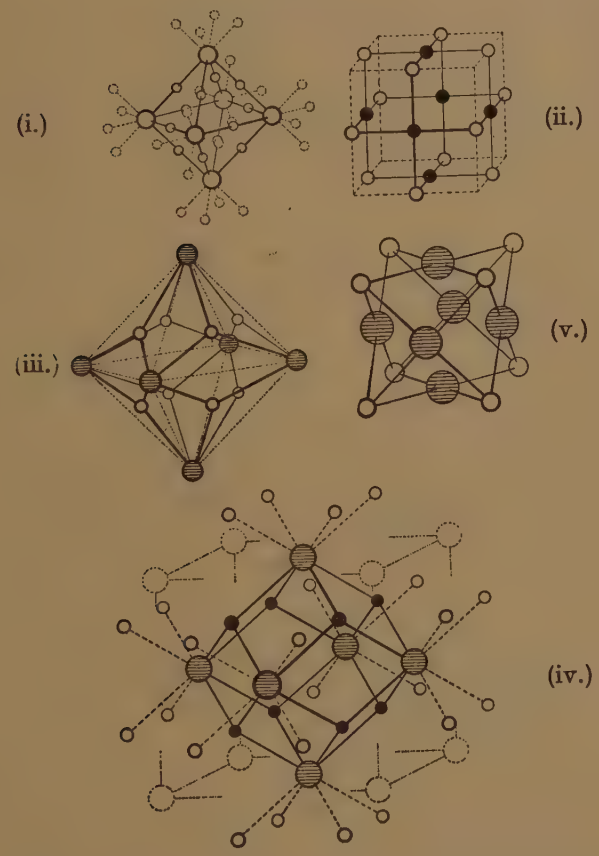
but the derivative $A_4X_4Y_4$ occurs as the fourfold macro-molecule $Cu_4(Et_3As)_4I_4$. The four triethylarsine groups are attached to the copper atoms of the complex, as shown below, and from the point of view of this classification are exactly analogous to the four outer oxygen atoms of P_4O_{10}.

The Octahedral Complex 6_{12}.

The simple octahedral complex A_6 is not known, nor as yet are any of its derivatives.

6_{12} (*a*).—The complex A_6X_{12} arises from placing X atoms along the edges of the octahedron, and may possess full cubic symmetry, $m3m$. As in the case of all complexes the configuration will depend on the valency angles of A and X. The form (i.) is not likely to occur as such, since it would require an element A forming four pyramidal bonds, but

Fig. 8.



it is possible for an element forming eight cubic bonds, the additional twenty-four Y atoms being indicated as dotted circles. Movement of the A atoms along the tetrad axes leads to the form (ii.), in which may be distinguished planar AX_4 groups, the X bond angles being 90°.

6_{12} (*b*).—Placing the X atoms on the faces of the octahedron gives the complex A_6X_8. Possible configurations range from (iii.) to (v.), the intermediate form (iv.) showing the relationship of this complex to the

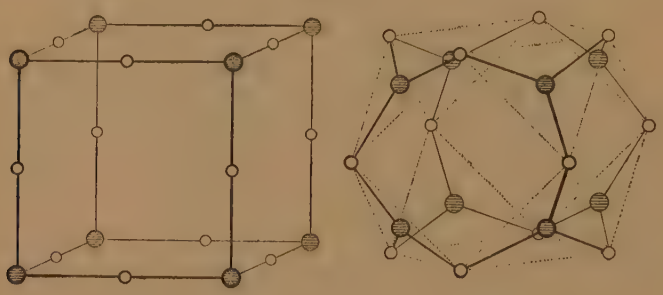
fluorite structure. The large circles in (iv.) outline the unit cell of the CaF_2 structure, in which they would be Ca atoms, and the diagram shows how an atom forming eight cubic bonds (*e. g.* Mo) could form such a complex $A_6X_8Y_{24}$.

The Cubic Complex 8_{12}.

In this group we have the simple cube A_8 and the derivatives A_8X_{12} and A_8X_6, the X atoms lying along the edges and on the faces of the cube respectively. Since the cube is the inverse of the octahedron the complex A_8X_6 is merely A_6X_8, (6_{12} *b*), with A and X interchanged.

8_{12} (*a*).—The complex A_8X_{12} may have full cubic symmetry, the eight A atoms lying on the triad axes. It could be formed by atoms A having 3 planar, 3 pyramidal, 4 tetrahedral, or 6 octahedral bonds,

Fig. 9.



(i.) (ii.)

giving complexes with the formulæ A_8X_{12}, A_8X_{12}, $A_8X_{12}Y_8$, and $A_8X_{12}Y_{24}$. For an atom such as boron the complex would adopt the cubo-octahedral configuration (ii.) which is derived from (i.) by movement of the A atoms along the triad axes.

The Complex 10_{12}.

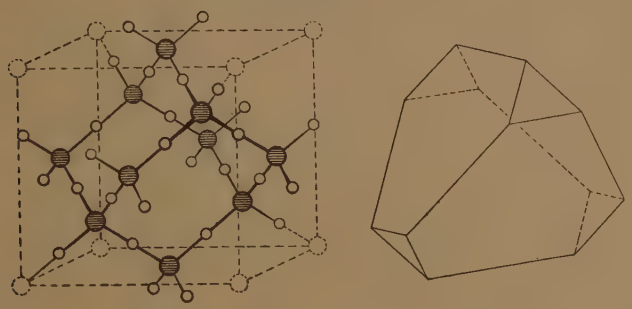
Although not a polyhedral complex ($N_2=6$, $N_3=4$) it is illustrated here to show its relation to the β-cristobalite structure which is mentioned later (fig. 10 (*a*)). As in previous cases the broken lines indicate the unit cell.

The Complex 12_{18} (5).

One of the forms of the polyhedron 12_{18} is interesting since it may have cubic symmetry, 43m. (fig. 10 (*b*)). For an element A forming four tetrahedral bonds the formula of the resulting complex would

be $A_{12}X_{18}Y_{12}$, and for octahedral bonds $A_{12}X_{18}Y_{36}$ (see the section on the heteropolyacids).

Fig. 10.



(*a*) (*b*)

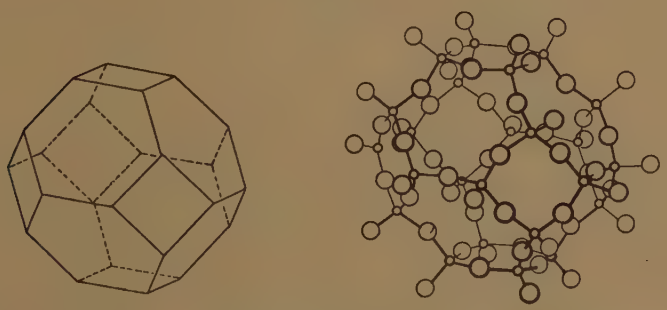
(*a*) The structure of β-cristobalite. The full lines and circles show the related complex $A_{10}X_{12}Y_{16}$.
(*b*) The polyhedron 12_{18} (5).

The Complex 12_{24} (4 *b*).

As will be shown below, this polyhedron may be considered the basis of the heteropolyacid ion $[PW_{12}O_{40}]^{3-}$.

Although complexes of more than 12 points are outside the scope of this paper, one higher polyhedron will be mentioned here. The complex shown here ($A_{24}X_{36}Y_{24}$) is derived from the polyhedron 24_{36}

Fig. 11.



(i.) (ii.)

($f_4=6$, $f_6=8$). This finite complex is not known, but it is related to the structure of ultramarine in the same way that the complex A_4X_6 is related to the structure of zinc blende or A_8X_8 to that of fluorspar.

In concluding this section of the paper it is worth while considering the possibility of the existence of these large basket-like complexes in crystals. In the above account of the polyhedral complexes the geometrical relationship of certain forms of these finite complexes to three-dimensional structures has been emphasized. Of the various finite complexes only the smallest (*e. g.*, A_4X_6, $A_4X_4Y_4$, $A_4X_6Y_4$) are known to exist as independent units in crystals. The large basket-like complexes ($A_{12}X_{18}$, $A_{24}X_{36}$) enclose a considerable volume of space unoccupied by atoms, and this is equally true whether the particular assemblage of atoms exists as a finite complex or whether it is merely to be distinguished as part of a three-dimensional lattice. Now although there are crystals in which the packing of the atoms leaves cavities or tunnels (*e. g.*, zeolites), in general atoms are packed in crystals in the most compact manner possible, as is seen from the large number of structures based on close-packing. Accordingly it is of interest to consider how these large complexes or the open three-dimensional frameworks corresponding to them enter into crystals, and we find the following possibilities :—

(1) *Framework Structures.*

(*a*) Interpenetration of two similar frameworks, one being displaced relative to the other. The relationship of the structure of cuprite, Cu_2O, to that of β-cristobalite, SiO_2, is that in cuprite there are two identical interpenetrating lattices of the cristobalite type, these lattices being displaced by the length of the side of the cubic cell relative to one another. The low density of β-cristobalite (2·27) compared with that of quartz (2·655) illustrates the open nature of the former structure.

(*b*) Filling of the holes in the structure by " foreign " material, as, for example, cations, S (Se, Te), and acid radicles in ultramarine.

(2) *Finite Complexes.*

Addition of atoms within the complex covalently linked to those of the framework of the complex. The simplest example is the structure of the molecule of basic beryllium acetate. A more complicated case is the relation of the ion $PW_{12}O_{40}$ to the complex 12_{24} (4 *b*) which is dealt with below.

This account of finite complexes concludes with a short discussion of the heteropolyacids. It has been the author's intention throughout this paper to show that a formal classification of complexes, whilst enabling us to group together related complexes and to suggest others as yet unknown, also emphasizes the three-dimensional nature of chemical molecules and the close connexion between classical stereochemistry and the stereochemistry of crystals.

The Constitution of the Heteropolyacids.

The general formulæ of the complex ions of the heteropolyacids may be written $(P_xW_yO_z)^{n-}$, it being understood that P may be replaced in this formula by As, Si, B, I, etc., and W by Mo, V, etc. This discussion will be confined to those in which $y/x \geqslant 6$, and these may be divided into three classes :

(*a*) $y/x=6$.

(*b*) $6<y/x<12$.

(*c*) $y/x=12$.

For many years these compounds were classified according to the Miolati-Rosenheim scheme, which was based on Werner's coordination theory. For the "saturated" acids, classes (*a*) and (*c*), typical constitutional formulæ were $[As(MoO_4)_6]^{7-}$ and $[P(W_2O_7)_6]^{7-}$. The intermediate compounds, however, presented more difficulty. For example, the salt with the empirical formula $K_6H_6As_2Mo_{18}O_{65}$ had to be written as a polynuclear compound

$$K_6H_6[_{(Mo_2O_7)_4}^{O}As-(Mo_2O_7)-As_{(Mo_2O_7)_4}^{O}]$$

in order to preserve the 6-coordination of the arsenic atoms. The first great advance in elucidating the constitution of these compounds was due to Pauling (1929). He accepted the formulation of the 6-ions as a central AsO_6 octahedron surrounded by, and sharing corners with, six WO_4 tetrahedra. It may be mentioned here that there is an alternative structure for these 6-ions which consists of a ring of six WO_6 octahedra each sharing two edges. The arsenic or other atom would then occupy the octahedral hole at the centre of the ring (compare the layer structures of $Mg(OH)_2$ and $Al(OH)_3$). It may be true, however, as Pauling suggested, that only corners and not edges of octahedra will be shared on account of the large charge on the W atoms.

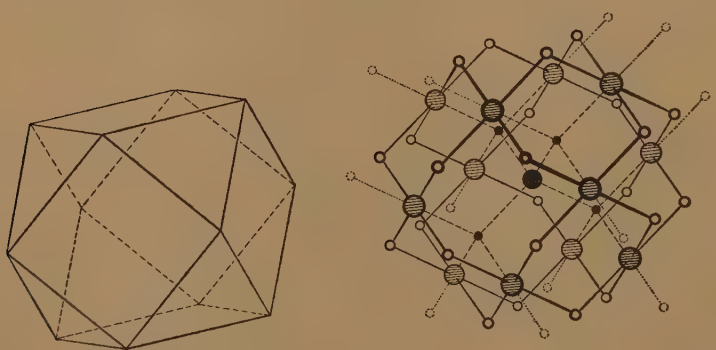
For the 12-acids Pauling suggested an elegant structure composed of twelve WO_6 octahedra each sharing a corner with each of three other octahedra. The structure may be derived from the 3-connected polyhedron 12_{18} (5) * by placing W atoms at the vertices and O atoms midway along the edges, giving a shell of composition $W_{12}O_{18}$. If the W atoms are now displaced outwards and the octahedra around them completed by attaching three OH groups to each then we obtain a cluster of twelve octahedra, $W_{12}O_{18}(OH)_{36}$, within which a PO_4 tetrahedron may be placed. The final formula was therefore $[PO_4W_{12}O_{18}(OH)_{36}]^{3-}$. It is now known, however, that although this structure was very near to the true one these complexes are actually based on the 4-connected polyhedron 12_{24}

* See fig. 10 (*b*).

(4 *b*) from which they may be derived in the following way. The most symmetrical form of this polyhedron is the cubo-octahedron. If we place O atoms on the edges of this figure we obtain the complex $W_{12}O_{24}$ possessing full cubic symmetry. These O atoms may now be moved outwards from the centre of the complex until each W atom is surrounded by four O atoms in a plane, the original symmetry of the complex being retained. The octahedra of oxygen atoms around the tungsten atoms may now be completed by the addition of sixteen oxygen atoms, four within the complex and twelve outside (dotted circles). The four atoms inside the shell form a tetrahedron, at the centre of which an atom of phosphorus is placed, the result being the complex ion $[PW_{12}O_{40}]^{3-}$.

" Unsaturated " acids, class (*b*), appear to exist in which the ratio of W to P atoms has the values 8, $8\frac{1}{2}$, 9, 10, $10\frac{1}{2}$, and 11, or doubling

Fig. 12.



The relation between the cubo-octahedron 12_{24} (4 *b*) and the 12-heteropolyacid ion $[PW_{12}O_{40}]^{3-}$.

The tungsten atoms, situated at the vertices of the cubo-octahedron, are shown as shaded circles. Small circles represent oxygen atoms. The atoms (PO_4) within the polyhedral shell are shown in black.

these values, P_2W_{16} to P_2W_{22}. Pauling suggested that these might arise from two or more 12-anions sharing certain of their octahedra, and he derived for the P_2W_{18} anion the formula $[(PO_4)_2W_{18}O_{30}(OH)_{48}]^{6-}$, but the formulæ would become very complicated for the other members of the series. Apart from this, all formulæ based on 3-connected polyhedra contain large numbers of OH groups which we now know are not present in these anions. It would appear, therefore, that these " unsaturated " acids are derived from 4-connected polyhedra, and are exactly analogous to the 12-anion $PW_{12}O_{40}$. We shall assume that the polyhedra concerned are those with 3- and 4-sided faces as in the 12-anion, since the oxygen atoms in the polyhedral shell are likely to be closely

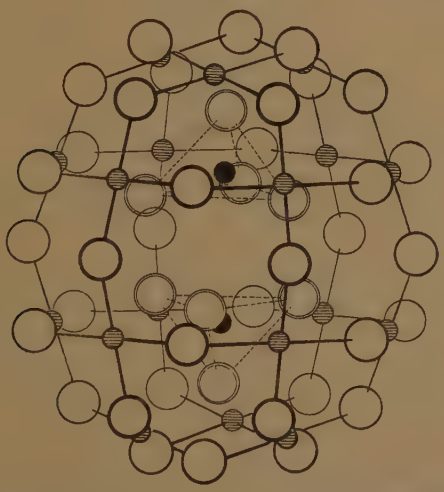
packed. Certain of the more symmetrical of these polyhedra are shown in Table IV. Each has eight triangular faces and p-6 quadrilateral faces, p being the number of vertices. As in the derivation of $PW_{12}O_{40}$ from the polyhedron 12_{24} W atoms are placed at the vertices, O atoms on the edges, and one O atom outside the shell for each W atom. For p_{2p} this

TABLE IV.

4-connected Polyhedra.

Number of vertices (p) and symmetry.			No. of edges (l=2p).	f_3.	f_4.
Trigonal.	Cubic.	Tetragonal.			
	12		24	8	6
15			30	8	9
		16	32	8	10
18			36	8	12
		20	40	8	14
21			42	8	15
	24		48	8	18

Fig. 13.



The nucleus of the ion $[P_2W_{18}O_{62}]^{6-}$, the 18 oxygen atoms outside the polyhedral shell being omitted. The phosphorus atoms within the PO_4 tetrahedra are shown as black circles.

gives W_pO_{3p}. Within the polyhedron we now have to place two PO_4 groups, giving the final formula $P_2W_pO_{3p+8}$. These complexes carry a charge of —6. Examples would be $H_6[P_2W_{18}O_{62}]$, $H_6[P_2W_{20}O_{68}]$.

A suggested structure for the trigonal $[P_2W_{18}O_{62}]$ anion is illustrated in fig. 13.

Our general picture of the heteropolyacids is therefore as follows :— After the 4- and 6-anions, PW_4O_{16} and PW_6O_{24}, based presumably on tetrahedral PO_4 and octahedral PO_6 groups respectively, there is a gap until we reach a polyhedral group large enough to contain one PO_4 tetrahedron at $PW_{12}O_{40}$. There is then a further gap until there is room for two PO_4 tetrahedra, apparently at $P_2W_{16}O_{56}$. The higher members, up to $P_2W_{22}O_{74}$, will then be based on 4-connected polyhedra as indicated for $P_2W_{18}O_{62}$. (The less symmetrical polyhedra, *e. g.*, 17_{34} and 22_{44}, are omitted from Table IV., but are of the same type.)

Summary.

In crystals throughout which the interatomic binding is not uniformly of one type we may distinguish aggregates of atoms within which the binding is of one kind whereas that between one aggregate and the next is of a different nature. Such groups of atoms may be finite (molecules and complex ions) or they may extend throughout the crystal as infinite chains or layers. The first part of the present paper deals with the classification of such finite complexes. This is followed by a comprehensive survey of finite complexes, and the paper concludes with a short discussion of the constitution of the heteropolyacids.

Acknowledgments.

The author is greatly indebted to Professor J. D. Bernal, F.R.S., for his generous advice and encouragement during the preparation of this paper. The work was carried out during the tenure of a Senior Research Award of the Department of Scientific and Industrial Research.

References.

Bauer, J. Amer. Chem. Soc. lx. p. 524 (1938).
Bernal and Crowfoot, Ann. Reps. Chem. Soc. p. 407 (1933).
Burawoy, Gibson, Hampson, and Powell, J. Chem. Soc. p. 1690 (1937).
Hampson and Stosick, J. Amer. Chem. Soc. lx. p. 1814 (1938).
Mellor, Burrows, and Morris, ' Nature,' Lond., cxli. p. 414 (1938).
Mooney, *Z. Kristallogr.* xcviii. p. 377 (1938).
Niggli and Nowacki, *Z. Kristallogr.* lxxxvi. p. 65 (1933).
Palmer and Elliott, J. Amer. Chem. Soc. lx. p. 1852 (1938).
Pauling, J. Amer. Chem. Soc. li. p. 2868 (1929).
Pauling and Huggins, *Z. Kristallogr.* lxxxvii. p. 228 (1934).
Powell and Ewens, J. Chem. Soc. p. 286 (1939).
Sidgwick and Bailey, Proc. Roy. Soc. A, cxliv. p. 521 (1933).
Ssu-Mien Fang, *Z. Kristallogr.* xcix. p. 1 (1938).
Stock and Wierl, *Z. anorg. allgem. Chem.* cciii. p. 228 (1931).
Wells, Proc. Roy. Soc. A, clxvii. p. 169 (1938).
Zachariasen, *Z. Kristallogr.* xcviii. p. 266 (1937).

XI. *Lorentz Transformation and Hamilton's Quaternions.*

By Otto F. Fischer ‡.

[Received February 14, 1939.]

1. *Introduction.*

In a recent paper, "Hamilton's Quaternions and Minkowski's Potentials," § I suggest a paving of the way for a wider use of Hamilton's quaternions in the treatment of modern physics. Hypercomplex numbers in general are acquiring increasing interest. Dirac's q-numbers, van der Waerden's spinors, Klein-Kaluza-Schouten-van Dantzig's projectors are examples hereof.

In the present paper I take a second step, treating the Lorentz Transformation. Some time further on, of course, I must reach the limit, when quantities of a higher order than quaternions are needed. Then, however, I may carry on still a while by the resource of the manner of writing, which I call an " axion," combining the essential features of the principal values and the corresponding transformation (" eigenstates ").

2. *Quaternions and Quaternation.*

As in the above-mentioned paper, I indicate quaternions by italics,

$$O=\mathbf{O}+\sigma=\mathrm{O}_1\mathbf{i}_1+\mathrm{O}_2\mathbf{i}_2+\mathrm{O}_3\mathbf{i}_3+\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

simply defining them as consisting of a scalar (" c-number ") and a hereto added three-dimensional vector (" q-number "), referred to orthogonal axes indicated by the unit vectors $\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$, and only distinguishing itself from the usual Gibbs's vector by the sign being altered by scalar multiplication as by imaginary numbers. Writing the complete quaternion multiplication as follows,

$$O\times F=O\times F+O.F=\mathbf{O}\times\mathbf{F}-\mathbf{O}.\mathbf{F}+\sigma.f+\sigma.\mathbf{F}+\mathbf{O}.f, \quad . \quad . \quad (2)$$

with Gibbs's signs . and $\times$ for resp. scalar and vector multiplication, we are able accordingly to make use also of Gibbs's further terms and conceptions when we like. For instance, writing the four-dimensional radius vector

$$L=\mathbf{r}+l=x\mathbf{i}_1+y\mathbf{i}_2+z\mathbf{i}_3+ict, \quad l=ict \quad . \quad . \quad . \quad . \quad . \quad (3)$$

---

‡ Communicated by the Author.
§ Phil. Mag. xxvii. p. 375 (1939).

we introduce the corresponding " Nabla operator "

$$\diamondsuit = \mathbf{i}_1 \frac{\partial}{\partial x} + \mathbf{i}_2 \frac{\partial}{\partial y} + \mathbf{i}_3 \frac{\partial}{\partial z} + \frac{\partial}{\partial l} = \Delta + \frac{\partial}{\partial l} = \nabla - \frac{i}{c}\frac{\partial}{\partial t} \quad . \quad . \quad (4)$$

and " Laplace operator "

$$\square = \widetilde{\diamondsuit} \times \diamondsuit = \diamondsuit \times \widetilde{\diamondsuit} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Delta + \frac{\partial^2}{\partial l^2}, \quad (5)$$

defining the " adjungate quaternion " :

$$\tilde{O} = \sigma - \mathbf{O} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the " norm "

$$N(O) = \tilde{O} \times O = O \times \tilde{O} = O_1^2 + O_2^2 + O_3^2 + \sigma^2.$$

If the coordinates are complex we also may introduce the corresponding " conjugate complex " quantities. In order to reduce the number of signatures I shall, however, endeavour to escape further such signs by writing the complex quaternions in full in cases of doubt. The assumption of real coordinates under ordinary circumstances then gives us the advantage of dealing with " the only hypercomplex number, which follows all the usual rules with the exception of the commutative " (Frobenius).

A special sign $*$ and the special term " Quaternation " I shall allot to the important and characteristic rigid transformation (rotation):

$$\mathbf{O}' = \gamma * \mathbf{O} = \gamma \times \mathbf{O} \times \tilde{\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

effectuated by the unit quaternion

$$\gamma = \gamma_1 \mathbf{i}_1 + \gamma_2 \mathbf{i}_2 + \gamma_3 \mathbf{i}_3 + \gamma_4 = \boldsymbol{\gamma} + \gamma_4 = \cos\theta + \mathbf{i}_\theta \sin\theta = e^{i_\theta \theta}, \quad . \quad . \quad (8)$$

$$\cos\theta = \frac{\gamma_4}{R_\gamma}, \quad \sin\theta = \frac{r_\gamma}{R_\gamma}, \quad \tilde{\gamma} = \cos\theta - \mathbf{i}_\theta \sin\theta, \quad \mathbf{i}_\theta^2 = -1, \quad . \quad (9)$$

$$R_\gamma = |\gamma| = \sqrt{N(\gamma)} = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2} = +1 \quad r_\gamma = |\boldsymbol{\gamma}| = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} \quad . \quad . \quad . \quad (10)$$

on the ordinary three-dimensional vector $\mathbf{O}$.

I may for future convenience indicate the details of the quaternation here by the multiplication table (or corresponding matrix):

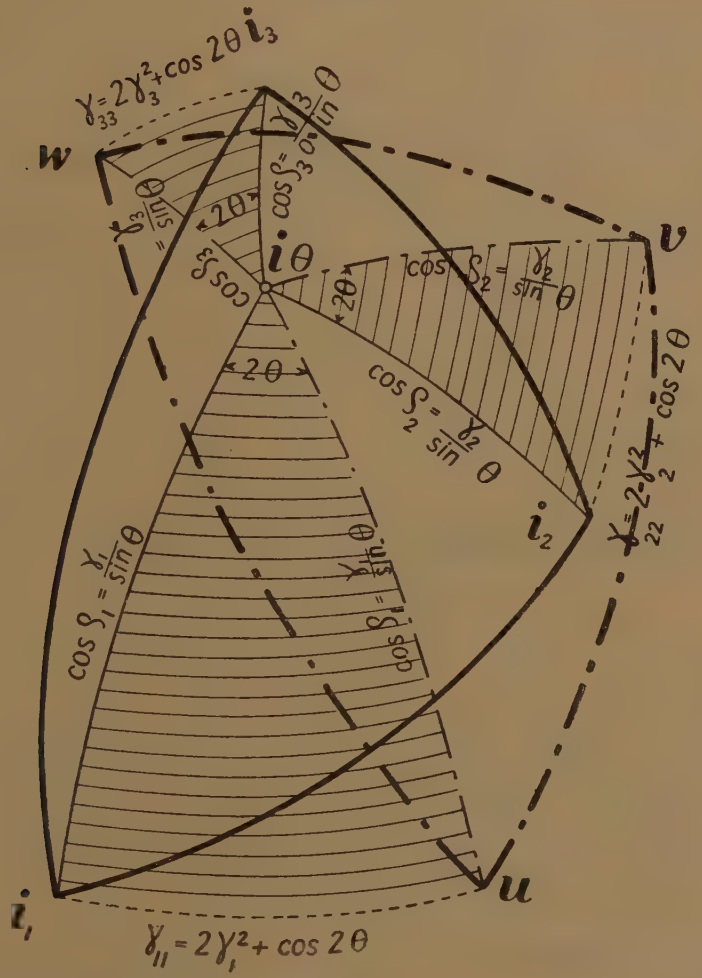
	O_1.	O_2.	O_3.
O_1'	γ_{11}	γ_{12}	γ_{13}
O_2'	γ_{21}	γ_{22}	γ_{23}
O_3'	γ_{31}	γ_{32}	γ_{33}

$$= \gamma * \begin{pmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} \end{pmatrix} = \tilde{\gamma} *, \quad (11)$$

where

$$\left.\begin{array}{lll} \gamma_{11}=\gamma_4^2+\gamma_1^2-\gamma_2^2-\gamma_3^2, & \gamma_{12}=2(\gamma_1\gamma_2-\gamma_3\gamma_4), & \gamma_{13}=2(\gamma_3\gamma_1+\gamma_2\gamma_4), \\ \gamma_{21}=2(\gamma_1\gamma_2+\gamma_3\gamma_4), & \gamma_{22}=\gamma_4^2+\gamma_2^2-\gamma_1^2-\gamma_3^2, & \gamma_{23}=2(\gamma_2\gamma_3-\gamma_1\gamma_4), \\ \gamma_{31}=2(\gamma_3\gamma_1-\gamma_2\gamma_4), & \gamma_{32}=2(\gamma_2\gamma_3+\gamma_1\gamma_4), & \gamma_{33}=\gamma_4^2+\gamma_3^2-\gamma_1^2-\gamma_2^2, \end{array}\right\} \quad . \quad . \quad . \quad (12)$$

Fig. 1.



with the conditions of orthogonality

$$\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22}+\gamma_{13}\gamma_{23}=0, \quad \gamma_{11}\gamma_{12}+\gamma_{21}\gamma_{22}+\gamma_{31}\gamma_{32}=0 \ldots \text{cycl.} \quad (13)$$

normalization

$$\gamma_{11}^2+\gamma_{12}^2+\gamma_{13}^2=1, \quad \gamma_{11}^2+\gamma_{21}^2+\gamma_{31}^2=1 \ldots \text{cycl.} \quad . \quad . \quad (14)$$

and substitution:

$$\gamma_1=\frac{\gamma_{32}-\gamma_{23}}{4\gamma_4},\quad \gamma_2=\frac{\gamma_{13}-\gamma_{31}}{4\gamma_4},\quad \gamma_3=\frac{\gamma_{21}-\gamma_{12}}{4\gamma_4},$$

$$8\gamma_4^2=\frac{(\gamma_{13}-\gamma_{31})^2-(\gamma_{21}-\gamma_{12})^2}{\gamma_{22}-\gamma_{33}}=\frac{(\gamma_{21}-\gamma_{12})^2-(\gamma_{32}-\gamma_{23})^2}{\gamma_{33}-\gamma_{11}}$$

$$=\frac{(\gamma_{32}-\gamma_{23})^2-(\gamma_{13}-\gamma_{31})^2}{\gamma_{11}-\gamma_{22}}. \quad . \quad (15)$$

The contragredient transformation

$$\mathbf{u}=\tilde{\gamma}*\mathbf{i}_1,\quad \mathbf{v}=\tilde{\gamma}*\mathbf{i}_2,\quad \mathbf{w}=\tilde{\gamma}*\mathbf{i}_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

of the axes $\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$, considered as a rotation through the angle 2θ around the rotation axis $\mathbf{i}_\theta$ into the new directions uvw, is in fig. 1 indicated by the spherical triangles described by the intersection points of the two systems of coordinate axes with a sphere of unit radius, the centre of which coincides with their common origin. The spherical "radii vectores" are

$$\cos\rho_1=\frac{\gamma_1}{\sin\theta},\quad \cos\rho_2=\frac{\gamma_2}{\sin\theta},\quad \cos\rho_3=\frac{\gamma_3}{\sin\theta}, \quad . \quad . \quad . \quad (17)$$

and the spherical "roads" travelled by the intersection points:

$$\gamma_{11}=2\gamma_1^2+\cos 2\theta,\quad \gamma_{22}=2\gamma_2^2+\cos 2\theta,\quad \gamma_{33}=2\gamma_3^2+\cos 2\theta. \quad . \quad (18)$$

3. *The "Longitudinal" Lorentz Transformation.*

If this quaternation is applied to a complete quaternion

$$O'=\gamma*O=\gamma*\mathbf{O}+\gamma*\sigma=\mathbf{O}'+\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

according to the multiplication rule (2) nothing happens to the scalar part:

$$\gamma*\sigma=\sigma=\sigma'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

According to Cayley it is necessary, in order to obtain the most general rigid transformation in four-dimensional space, that one of the "rotational quaternions," for instance γ, is generalized. I shall, however, not immediately go to the length of complete generalization, but only take a first step and substitute the unit quaternion

$$\theta'=\cos\theta'+\mathbf{i}_\theta\sin\theta',\quad \theta'-\theta=\Delta_\theta \quad . \quad . \quad . \quad . \quad . \quad (21)$$

of the same "rotation axis" i_θ. The transformation then becomes

$$O'=\theta'\times O\times\tilde{\gamma}=\mathfrak{L}O, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where I put

$$\theta'=A\gamma-B,\quad A=\frac{\sin\theta'}{\sin\theta},\quad B=\frac{\sin\Delta_\theta}{\sin\theta}, \quad . \quad . \quad . \quad . \quad (23)$$

resulting in

$$O' = \mathrm{A}\gamma * O - \mathrm{B}O \times\!\!\!\cdot\, \tilde{\gamma}. \quad \ldots\ldots \quad (24)$$

Making use of the matrices

$$\gamma * = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \ldots\ldots \quad (25)$$

$$\times\!\!\!\cdot\, \tilde{\gamma} = \begin{bmatrix} \gamma_4 & -\gamma_3 & \gamma_2 & -\gamma_1 \\ \gamma_3 & \gamma_4 & -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 & \gamma_4 & -\gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}, \quad \ldots\ldots \quad (26)$$

we finally obtain

$$\mathscr{L} = \begin{bmatrix} \mathrm{A}\gamma_{11} - \mathrm{B}\gamma_4 & \mathrm{A}\gamma_{12} + \mathrm{B}\gamma_3 & \mathrm{A}\gamma_{13} - \mathrm{B}\gamma_2 & \mathrm{B}\gamma_1 \\ \mathrm{A}\gamma_{21} - \mathrm{B}\gamma_3 & \mathrm{A}\gamma_{22} - \mathrm{B}\gamma_4 & \mathrm{A}\gamma_{23} + \mathrm{B}\gamma_1 & \mathrm{B}\gamma_2 \\ \mathrm{A}\gamma_{31} + \mathrm{B}\gamma_2 & \mathrm{A}\gamma_{32} - \mathrm{B}\gamma_1 & \mathrm{A}\gamma_{33} - \mathrm{B}\gamma_4 & \mathrm{B}\gamma_3 \\ -\mathrm{B}\gamma_1 & -\mathrm{B}\gamma_2 & -\mathrm{B}\gamma_3 & \mathrm{A} - \mathrm{B}\gamma_4 \end{bmatrix} \quad (\text{F. I.})$$

representing what I shall call the *longitudinal Lorentz Transformation* for reasons given in a following paragraph.

As the three-dimensional quaternation has already been sufficiently considered, I shall eliminate this and simplify the expression by making

$$\theta = \frac{\pi}{2} \sin\theta = 1, \quad \cos\theta = 0, \quad \mathrm{A} = \sin\theta', \quad \mathrm{B} = \sin \Delta_\theta, \quad \ldots \quad (27)$$

and also choose the i_1 axis for rotation axis, coinciding with

$$\gamma = -\tilde{\gamma} = \mathbf{i}_0 = \mathbf{i}_1, \quad \gamma_1 = 1, \quad \gamma_2 = \gamma_3 = \gamma_4 = 0, \quad \ldots\ldots \quad (28)$$

resulting in

$$\mathscr{L}_1 = \begin{bmatrix} \mathrm{A} & 0 & 0 & \mathrm{B} \\ 0 & -\mathrm{A} & \mathrm{B} & 0 \\ 0 & -\mathrm{B} & -\mathrm{A} & 0 \\ -\mathrm{B} & 0 & 0 & \mathrm{A} \end{bmatrix} \quad \ldots\ldots \quad (\text{F. I.}a)$$

If the usual shape

$$x' = \frac{x - q_t}{\sqrt{1 - \frac{q^2}{c^2}}} \qquad t' = \frac{t - \frac{q}{c^2} x}{\sqrt{1 - \frac{q^2}{c^2}}} \qquad l' = ict' \quad \ldots \quad (29)$$

of the Lorentz Transformation is wanted, I must make

$$A=\cos\Delta_\theta=\frac{1}{\sqrt{1-\beta^2}},\quad B=\sin\Delta_\theta=\frac{i\beta}{\sqrt{1-\beta^2}},\quad \beta=\frac{q}{c}. \quad . \quad (30)$$

If also the coordinates y and z are to remain unaltered, I must add a "relativity twist σ," determined by

$$y''=-y'\cos\sigma+z'\sin\sigma,\quad z''=-z'\cos\sigma-y'\sin\sigma,\quad \sigma=\Delta_\theta \quad . \quad (31)$$

and effecting the additional quaternation,

$$\mathfrak{L}_2=g*=\begin{bmatrix}1&0&0&0\\0&-A&-B&0\\0&B&-A&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}g_{11}&g_{12}&g_{13}&0\\g_{21}&g_{22}&g_{23}&0\\g_{31}&g_{32}&g_{33}&0\\0&0&0&1\end{bmatrix} \quad . \quad . \quad (F.\,II.)$$

by means of the rotational quaternion,

$$g=\frac{B}{\sqrt{2(A+1)}}+i_1\sqrt{\frac{A+1}{2}}=\sqrt{\frac{1-A}{2}}+i_1\sqrt{\frac{1+A}{2}}. \quad (F.\,II.\,a)$$

Then the final expression for the "longitudinal" Lorentz Transformation in quaternions becomes

$$\begin{aligned}O''&=\mathfrak{L}_0O=\mathfrak{L}_2\mathfrak{L}_1O\\&=g*O'=g\times\theta'\times O\times\tilde{\gamma}\times\tilde{g}=\end{bmatrix}\begin{bmatrix}A&0&0&B\\0&1&0&0\\0&0&1&0\\-B&0&0&A\end{bmatrix}O. \quad . \quad (F.\,III.)\end{aligned}$$

Again I may consider this expression from a somewhat different point of view. With reference to (23) I split up the expression into two parts :

$$\mathfrak{L}_0O=Ag\times\gamma\times O\times\tilde{\gamma}\times\tilde{g}-Bg\times O\times\tilde{\gamma}\times\tilde{g}, \quad . \quad . \quad . \quad (32)$$

and substitute a single quaternation for the double one, which is contained in the first member :

$$\lambda*O=g\times\gamma\times O\times\tilde{\gamma}\times\tilde{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\lambda=g\times\gamma=\sqrt{\frac{1-A}{2}}i_1-\sqrt{\frac{1+A}{2}}, \quad . \quad . \quad . \quad . \quad (34)$$

after which I also alter the second member

$$g\times O\times\tilde{\gamma}\times\tilde{g}=\tilde{\gamma}\times\lambda*O, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

with the final result

$$\mathfrak{L}_0O=(A-B\tilde{\gamma}\times)\lambda*O, \quad . \quad . \quad . \quad . \quad . \quad (F.\,IV.)$$

where

$$\lambda* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & -B & A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\tilde{y} \times O \times \tilde{\gamma} \times \tilde{g} = \tilde{\gamma} \times \lambda * O = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -B & A & 0 \\ 0 & -A & -B & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} O. \quad . \quad (37)$$

4. *Six-dimensional Vectors.*

In Minkowski's representation of four-dimensional space much use is made of six-dimensional vectors. Even these I propose to substitute by quaternions.

In the following formulas (compare (4) (2)) I introduce special terms for the scalar and vector part as follows:

$$\diamondsuit \times O = \text{Curl}\, O - \text{Div}\, O, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\text{Curl}\, O = \text{curl}\, \mathbf{O} + \nabla\sigma - \frac{i}{c}\dot{\mathbf{O}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\text{Div}\, O = \nabla \mathbf{O} + \frac{i}{c}\dot{\sigma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

leading, for instance, to the expressions

$$\tilde{\text{C}}\text{url}\, \phi_M = \text{curl}\, \mathbf{A}_M + i\nabla\phi_M + \frac{i}{c}\dot{\mathbf{A}}_M = \mathbf{H} - i\mathbf{E}, \quad . \quad . \quad . \quad (41)$$

$$\tilde{\text{D}}\text{iv}\, \phi_M = \nabla \mathbf{A}_M + \frac{i}{c}\dot{\phi}_M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$\frac{1}{c} P_F = \diamondsuit \times \tilde{\text{C}}\text{url}\, \phi_M = \diamondsuit \times \tilde{\text{D}}\text{iv}\, \phi_M = \frac{1}{c}(\mathbf{I} + i\rho c), \quad . \quad . \quad (43)$$

where $\mathbf{H}$ and $\mathbf{E}$ indicate respectively the magnetic and the electric forces, $\mathbf{P}_F$ is the four-dimensional current and ϕ_M is a four-dimensional potential indicated by

$$\phi_M = -(\mathbf{A}_M + i\phi_M), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\mathbf{H} = \text{curl}\, \mathbf{A}_M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$-\mathbf{E} = \frac{1}{c}\mathbf{A}_M + \nabla\phi_M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

In general I produce a " six-dimensional vector " of the type (41) according to the following principle.

In three-dimensional vector-algebra we distinguish between ordinary " polar " vectors or " monovectors," representing line elements of certain longitudinal directions, and " axial " vectors or " bivectors," representing normals of plane elements of certain rotational directions.

I shall attempt to introduce a similar distinction into the quaternion algebra by using the term " bimultiquaternions "' (resp. " trimulti-", " quadrumulti- ") when desiring to indicate the origin as a complete quaternion product of two (or more) " constituents " (the term " bi-quaternion " having already been disposed of for other purpose). For convenience with regard to signs I shall hereby specify that one of the factors, say the prefactor, shall enter in its adjungate shape, in order to increase the similarity to the conditions of ordinary scalar multiplication of vectors (compare (7)).

The general shape then becomes

$$C=\tilde{O}\times F=-\mathbf{O}\times\mathbf{F}+\mathbf{O}.\mathbf{F}+\sigma.f+\sigma.\mathbf{F}-\mathbf{O}.f=\mathrm{V}C+\mathrm{S}C. \quad . \quad (47)$$

The scalar part of this product I shall take as determining the " angle " α between the constituents as follows :

$$\mathrm{S}C=(OF)=|O|.|F|\cos\alpha=\mathbf{O}.\mathbf{F}+\sigma.f=\mathrm{O}_1\mathrm{F}_1+\mathrm{O}_2\mathrm{F}_2+\mathrm{O}_3\mathrm{F}_3+\sigma f. \quad (48)$$

If the constituents are " perpendicular " to each other (hereby meaning that $\cos\alpha=0$) leading to the disappearance of this scalar part, I shall call C a *transversal quaternion*. If, further, the scalar part of one constituent and the vector part of the other are imaginary :

$$O=\mathbf{O}+i\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$F=i\mathbf{F}+f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$C\dagger=-i\mathbf{O}\times\mathbf{F}-(\sigma.\mathbf{F}+f.\mathbf{O})=-(\mathbf{C}^\circ+i\mathbf{C}^*). \quad . \quad . \quad . \quad (51)$$

I shall call $C\dagger$ a *hexavector*, as it corresponds to Minkowski's six-dimensional vector under such circumstances (compare (41)).

Then I shall make use of the following signatures :—

$$\mathrm{C}_{yz}=\mathrm{C}^*_{xl}=\sigma\mathrm{F}_1+f\mathrm{O}_1=\mathrm{C}^\circ_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\mathrm{C}_{zx}=\mathrm{C}^*_{yl}=\sigma\mathrm{F}_2+f\mathrm{O}_2=\mathrm{C}^\circ_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$\mathrm{C}_{xy}=\mathrm{C}^*_{zl}=\sigma\mathrm{F}_3+f\mathrm{O}_3=\mathrm{C}^\circ_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\mathrm{C}_{xl}=\mathrm{C}^*_{yz}=i(\mathrm{O}_2\mathrm{F}_3-\mathrm{O}_3\mathrm{F}_2)=i\mathrm{C}^*_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\mathrm{C}_{yl}=\mathrm{C}^*_{zx}=i(\mathrm{O}_3\mathrm{F}_1-\mathrm{O}_1\mathrm{F}_3)-i\mathrm{C}^*_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$\mathrm{C}_{zl}=\mathrm{C}^*_{xy}=i(\mathrm{O}_1\mathrm{F}_2-\mathrm{O}_2\mathrm{F}_1)=i\mathrm{C}^*_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

in connexion with

$$\mathbf{C}^\circ=\mathrm{C}_{yz}\mathbf{i}_1+\mathrm{C}_{zx}\mathbf{i}_2+\mathrm{C}_{xy}\mathbf{i}_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

$$i\mathbf{C}^*=\mathrm{C}_{xl}\mathbf{i}_1+\mathrm{C}_{yl}\mathbf{i}_2+\mathrm{C}_{zl}\mathbf{i}_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where I shall call C* the "dual" quaternion in conformity with the terms used in v. Laues's 'Die Relativitätstheorie.'

If the constituents of the general bimultiquaternion degenerate into ordinary three-dimensional vectors ($\sigma=f=0$) I shall call C a *planar quaternion*, as in that case it actually is "perpendicular" to each constituent and accordingly may be considered as a "normal" to the hyperplane, which they represent. Otherwise, however, the scalar product between the bimultiquaternion and one of its constituents, O, cannot be counted on to disappear, as it leaves the residuum : $f\mathrm{N}(O)$.

5. *The Transversal Lorentz Transformation.*

Since the conceptions of transversal quaternions and hexavectors have been defined in the previous paragraph it becomes of interest to investigate their transformation, when both constituents individually undergo the ordinary "longitudinal Lorentz Transformation" (F. III.). The resulting transformation of C I shall call the *transversal Lorentz transformation* in connexion with the above representation.

I shall consider the general bimultiquaternion (47) in this respect. Then the transformation of the hexavector, the transversal quaternion, and the planar quaternion follow automatically as special cases.

In order to conform with the usual shape of the result I shall apply the reverse (contragredient) transformation (F. III.) to the constituents :

$$O=(\mathrm{AO_1'}-\mathrm{B}\sigma')\mathrm{i}_1+\mathrm{O_2'i_2}+\mathrm{O_3'i_3}+\mathrm{BO_1'}+\mathrm{A}\sigma' \quad . \quad . \quad . \quad (60)$$

$$F=(\mathrm{AF_1'}-\mathrm{B}f')\mathrm{i}_1+\mathrm{F_2'i_2}+\mathrm{F_3'i_3}+\mathrm{BF_1'}+\mathrm{A}f', \quad . \quad . \quad . \quad (61)$$

which, with reference to (2), gives us

$$\begin{aligned}C=&(\mathrm{AC_1'}-\mathrm{BC_4'})\mathrm{i}_1+\mathrm{C_2'i_2}+\mathrm{C_3'i_3}+\mathrm{BC_1'}+\mathrm{AC_4'}\\ =&\mathrm{i}_1\{(\mathrm{BO_1'}+\mathrm{A}\sigma')(\mathrm{AF_1'}-\mathrm{B}f')-(\mathrm{BF_1'}+\mathrm{A}f')(\mathrm{AO_1'}-\mathrm{B}\sigma')\\ &\qquad -\mathrm{O_2'F_3'}+\mathrm{O_3'F_2'}\}\\ &+\mathrm{i}_2\{(\mathrm{BO_1'}+\mathrm{A}\sigma')\mathrm{F_2'}-(\mathrm{BF_1'}+\mathrm{A}f')\mathrm{O_2'}-\mathrm{O_3'}(\mathrm{AF_1'}-\mathrm{B}f')\\ &\qquad +(\mathrm{AO_1'}-\mathrm{B}\sigma')\mathrm{F_3'}\}\\ &+\mathrm{i}_3\{(\mathrm{BO_1'}+\mathrm{A}\sigma')\mathrm{F_3'}-(\mathrm{BF_1'}+\mathrm{A}f')\mathrm{O_3'}-(\mathrm{AO_1'}-\mathrm{B}\sigma')\mathrm{F_2'}\\ &\quad +\mathrm{O_2'}(\mathrm{AF_1'}-\mathrm{B}f')\}+(\mathrm{BO_1'}+\mathrm{A}\sigma')(\mathrm{BF_1'}+\mathrm{A}f')\\ &\quad +(\mathrm{AO_1'}-\mathrm{B}\sigma')(\mathrm{AF_1'}-\mathrm{B}f')+\mathrm{O_2'F_2'}+\mathrm{O_3'F_3'},\quad . \quad . \quad . \quad (62)\end{aligned}$$

resulting in

$$\begin{aligned}\mathrm{C_1}&=\mathrm{C_1'}\\ \mathrm{C_2}&=\mathrm{AC_2'}-\mathrm{BC_3'}\\ \mathrm{C_3}&=\mathrm{BC_2'}+\mathrm{AC_3'}\\ \mathrm{C_4}&=\mathrm{C_4'}\end{aligned} \qquad \mathscr{L}_t=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \mathrm{A} & \mathrm{B} & 0\\ 0 & -\mathrm{B} & \mathrm{A} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad . \quad . \quad \text{(F. V.)}$$

as the familiar description (compare Laue) of what I have called the transversal Lorentz transformation.

Comparing with (36) we find the interesting fact that

$$\mathscr{L}_t=\lambda *, \quad \ldots\ldots\ldots \quad \text{(F. V. } a\text{)}$$

$$\mathscr{L}_0=(A-B\tilde{\gamma}\times)\mathscr{L}_t, \quad \ldots\ldots \quad \text{(F. V. } b\text{)}$$

which seems to be a simple substitute for the usual 36-dimensional matrix.

As an application the formula may be used on the transformation of (41). For that purpose I shall generalize this formula, introducing

$$\frac{1}{c}\psi=H-iE-\left(e_F+\frac{i}{c}\phi_F\right)=\tilde{\diamondsuit}\times\phi_M=H-iE \quad \ldots \quad (63)$$

in order to treat a more general bimultiquaternion instead of the specialized hexavector. Writing

$$\psi=\psi_1'i_1'+(A\psi_2'-B\psi_3')i_2+(B\psi_2'+A\psi_3')i_3'+\psi_4', \quad \ldots \quad (64)$$

I then arrive at

$$E_1=E_1', \quad \ldots\ldots\ldots \quad (64)$$

$$E_2=A(E_2'+\beta H_3'), \quad \ldots\ldots\ldots \quad (65)$$

$$E_3=A(E_3'-\beta H_2'), \quad \ldots\ldots\ldots \quad (66)$$

$$\phi_F=\phi_F', \quad \ldots\ldots\ldots \quad (67)$$

$$H_1=H_1', \quad \ldots\ldots\ldots \quad (68)$$

$$H_2=A(H_2'-\beta E_3'), \quad \ldots\ldots\ldots \quad (69)$$

$$H_3=A(H_3'+\beta E_2'), \quad \ldots\ldots\ldots \quad (70)$$

$$e_F=e_F', \quad \ldots\ldots\ldots \quad (71)$$

of which the six upper equations are the well-known results while the two last refer to dielectrics.

6. *Some Radius Vector Properties.*

In connexion with the developments of the last paragraphs and their physical and geometrical meaning with regard to the theory of relativity, it may be useful to consider some properties of the radius vector.

The expression (3) is known to be a first invariant, I_1, against Lorentz transformation as well as its " size " :

$$I_2: \; L=\sqrt{x^2+y^2+z^2-c^2t^2}.$$

Hereto correspond the invariants : $I_3: \diamondsuit$ and $I_4: \square$ (compare (4) and (5)). The " relativity twist " δ (31) further discloses

$$I_5: \; \rho_\theta=\rho_0\sqrt{y+^2z^2} \quad \text{and} \quad I_6: \; \rho_l=\sqrt{x^2-c^2t^2}.$$

All these invariants may be regarded as belonging to the "space category."

The three-dimensional part r of the radius vector is not invariant. Its transformation may be written :

$$\mathbf{r}'=\rho_\theta+\mathbf{i}_1(x\cos\sigma+ict\sin\sigma). \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

The norm is

$$r'^2=\rho_\theta^2+\frac{(x-ct\beta)^2}{1-\beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

We may differentiate the latter with regard to β in order to find the extremes. One solution is $q=c$ corresponding to an infinite r'. Another invariant value equals I_5 corresponding to

$$x\cos\sigma+ict\sin\sigma=0 \quad . \quad . \quad , \quad . \quad . \quad . \quad . \quad (74)$$

and the special value

$$\frac{x}{t}=q^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

A third extreme value is obtained when attempting to make

$$r'^2=\rho_\theta^2+x'^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

writing

$$\rho_\theta=ix'=ix\cos\sigma_\tau-ct\,\mathrm{ein}\,\sigma_\tau. \quad . \quad . \quad . \quad . \quad . \quad (77)$$

In this special case the transformed time t' acquires the extreme value determined by

$$ict'=\mathrm{L}'=\mathrm{L}=\sqrt{x^2+y^2+z^2-c^2t^2} \quad . \quad . \quad . \quad . \quad . \quad (78)$$

(compare I_2) corresponding to what I shall define as the *principal time*

$$\mathrm{I}_7: \ \tau=t_\tau'=-\frac{i}{c}\sqrt{x^2+y^2+z^2-c^2t^2}=-\sqrt{t^2-\frac{r^2}{c^2}}, \quad \mathrm{L}=ic\tau. \quad . \quad (79)$$

As the inverse formulas of (29) are (29 *a*)

$$x=\frac{x'+qt'}{\sqrt{1-\beta^2}}, \quad ct=\frac{ct'+\beta x'}{\sqrt{1-\beta^2}};$$

this in connexion with (77) gives

$$ct_\tau'=ct\sqrt{1-\beta_\tau^2}-\beta_\tau x_\tau'=ct\sqrt{1-\beta_\tau^2}+i\beta_\tau\rho_\theta=c\tau. \quad . \quad . \quad (80)$$

But only if $\beta_\tau\rho_\theta$ is a constant in four-dimensional space I find the usual formula (compare Laue) :

$$d\tau=\sqrt{1-\beta_\tau^2}\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

for which reason I usually shall adopt the definition (79).

Instead, however, I shall introduce a new invariant, which I shall call the *relativity starting time* :

$$\mathrm{I}_8: \ \tau^*=\sqrt{\tau^2+\frac{\rho_\theta^2}{c^2}}=i\frac{\rho_l}{c}$$

combining I_7 and I_5. Introducing

$$\rho_\theta^2 = c^2(\tau^{*2} - \tau^2) = y^2 + z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

(which follows from the definition) into

$$c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

(compare (79)) I find

$$c^2\tau^{*2} = c^2t^2 - x^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

which shows that $x=0$ by $t=\tau^*$. If this is introduced in the transformations (29 *a*), (80) I obtain unconditionally

$$\tau^* = \sqrt{1-\beta_{\tau^*}^2}\, t, \quad d\tau^* = \sqrt{1-\beta_{\tau^*}^2}\, dt \quad . \quad . \quad . \quad . \quad . \quad (85)$$

for $d\tau^*$, supposing $\beta\tau^*$ to be constant.

The corresponding relativity twist may be introduced as follows :—

$$t = \tau^* \cos \sigma_{\tau^*}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$x = -ic\tau^* \sin \sigma_{\tau^*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

If the radius vector quaternion is represented in its " Euler shape "

$$\left.\begin{aligned} L &= r\mathbf{i}_r + ict = \mathrm{L}(\cos\epsilon + \mathbf{i}_r \sin\epsilon) = \mathrm{L}e^{\epsilon}, \\ \mathbf{r} &= r\mathbf{i}_r = \boldsymbol{\rho}_\theta + \mathbf{x} = y\mathbf{i}_2 + z\mathbf{i}_3 + x\mathbf{i}_1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (88)$$

indicating $\mathbf{i}_r$ as the unit direction vector corresponding to $\mathbf{r}$ the angle ϵ may be called its *quaternion angle*. It is so far from being invariant, that the vector $\boldsymbol{\epsilon} = \epsilon\mathbf{i}_r$ seems on the contrary to impersonate the characteristic features of the variation. Writing

$$\epsilon^* = \frac{\pi}{2} - \epsilon = i\eta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

I find, for instance,

$$\cos\epsilon = \sin\epsilon^* = i \sinh\eta = \frac{i}{2}(e^{\eta} - e^{-\eta}) = \frac{ict}{\mathrm{L}} = \frac{t}{\tau}, \quad . \quad . \quad . \quad (90)$$

$$\sin\epsilon = \cos\epsilon^* = \cosh\eta = \tfrac{1}{2}(e^{\eta} + e^{-\eta}) = \frac{r}{\mathrm{L}}, \quad . \quad . \quad . \quad . \quad (91)$$

$$e^{\eta} - e^{-\eta} = \frac{2ct}{\mathrm{L}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

$$e^{\eta} + e^{-\eta} = \frac{2r}{\mathrm{L}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

$$e^{\eta} = \frac{ct + r}{\mathrm{L}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

$$e^{-\eta} = \frac{r - ct}{\mathrm{L}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and, finally,

$$e^{\epsilon\eta}=\frac{r+ct}{r-ct}, \quad \ldots\ldots\ldots \quad (96)$$

$$\epsilon^*=\frac{i}{2}ln\frac{r+ct}{r-ct}, \quad \ldots\ldots\ldots \quad (97)$$

in combination with the relations (98)

$$\tau\cos\epsilon=\tau^*\cos\sigma_{\tau^*}=t, \quad r=\mathrm{L}\sin\epsilon, \quad \beta^2_{\tau^*}=1-\left(\frac{\tau^*}{\tau\cos\epsilon}\right)^2.$$

In analogy with the three-dimensional case I shall imagine a four-dimensional radius vector L to be able to represent

(*a*) A point in "universe."

(*b*) A four-dimensional line.

(*c*) A hyperplane determined by its "normal" (compare the "planar quaternion" of paragraph (4)).

In the latter case the angle α determined by

$$\mathrm{S}(L\times L')=(LL')=\mathrm{LL}'\cos\alpha=xx'+yy'+zz'-c^2tt' \quad \ldots \quad (99)$$

(compare (48)) shall be taken to define the angle between two hyperplanes. Or again, the one factor, L', for instance, may represent a line or "axis," while the other L represents a hyperplane by its normal. In that case α shall be interpreted as the angle between the axis and the normal of the plane, while

$$\alpha^*=\frac{\pi}{2}-\alpha \quad \ldots\ldots\ldots \quad (100)$$

shall represent the angle between the axis and the plane itself. The equation

$$\alpha=\text{a constant} \quad \text{resp.} \quad \alpha^*=\text{a constant} \quad \ldots\ldots \quad (101)$$

shall represent a "hypercone" produced by a tangent hyperplane as described revolving about the axis. α^* then represents half the "apex angle." The hypercone accordingly also may be produced by revolving a line L^* forming this angle with the axis L'.

Now I shall specialize this by making

$$L'=ict, \quad x'=y'=z'=0, \quad t'=t, \quad \ldots\ldots \quad (102)$$

or, in other words, make the axis L' coincide with the time axis. Then (99) turns into

$$ict\,\mathrm{L}\cos\alpha=-c^2t^2, \quad \ldots\ldots\ldots \quad (103)$$

$$\cos\alpha=\frac{ict}{\mathrm{L}}=\cos\epsilon. \quad \ldots\ldots\ldots \quad (104)$$

Thus the " quaternion angle " ϵ is identified with the " angle in four-dimensional space," which the quaternion in question forms with the time axis.

The hypercone formed by the hyperplane, which the quaternion indicates, when it revolves around the time axis, has the apex angle $2\epsilon^*$.

This hypercone becomes identical with the well-known hypercone of Minkowski's space (resp. the pseudoeuclidean geometry) when $\epsilon = \epsilon_M$

$$tg\,\epsilon_M = i, \quad \ldots\ldots\ldots \quad (105)$$

making

$$\sin\epsilon_M = i\cos\epsilon_M, \quad \ldots\ldots\ldots \quad (106)$$

$$r = -ct, \quad \ldots\ldots\ldots \quad (107)$$

$$L = \sqrt{r^2 - c^2t^2} = 0, \quad \ldots\ldots\ldots \quad (108)$$

$$\tau = 0, \quad \ldots\ldots\ldots \quad (109)$$

$$\tau^* = \frac{\rho_\theta}{c}. \quad \ldots\ldots\ldots \quad (110)$$

The space " inside " this hypercone (on the same side as the time axis) is " time-fashioned," the time coordinate is larger than the space coordinate r and $\epsilon_M > \epsilon$. The space outside is " space-fashioned " and $\epsilon_M < \epsilon$.

In Minkowski's pseudoeuclidean geometry the time axis, however, is real, and he works with real apex angles, whereas I obtain imaginary values as long as I desist from using imaginary quaternion angles (which does not seem to offer any advantage).

It is often claimed in the theory of relativity, that Minkowski's hypercone corresponds to

$$\beta = 1, \quad q = c. \quad \ldots\ldots\ldots \quad (111)$$

This is obvious in the ordinary two-dimensional representation of the " isotrope " asymptotes. If I, however, consider the consequences of (111) I arrive at a somewhat different general point of view. According to (98) the condition follows that

$$\frac{\tau^*}{\tau\cos\epsilon} = 0, \quad \frac{\tau^*}{L\cos\epsilon} = 0, \quad \ldots\ldots \quad (112)$$

satisfied alternatively by

$$\tau^* = 0, \quad \ldots\ldots\ldots \quad (113)$$

$$\tau = \infty, \quad \ldots\ldots\ldots \quad (114)$$

$$L = \infty, \quad \ldots\ldots\ldots \quad (115)$$

$$\cos\epsilon = \infty, \quad \ldots\ldots\ldots \quad (116)$$

In the case

$$L = \rho_\theta. \quad \ldots\ldots\ldots \quad (117)$$

I obtain (113), and consequently also (112).

7. *The Most General Rigid Transformation.*

The various Lorentz Transformations here considered are all special cases of the most general rigid transformation obtained by substituting an arbitrary unit quaternion β for the specially chosen θ' (compare (21)) (which possessed the same rotation axis as γ). Accordingly I write, in general,

$$O'=\beta\times O\times\tilde{\gamma}=\mathcal{G}O, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

$$\beta=\beta_1 i_1+\beta_2 i_2+\beta_3 i_3+\beta_4=\cos b+i_\beta \sin b=e^b, \quad |\beta|=1, \quad . \quad (119)$$

where the affinor $\mathcal{G}$ is determined by the matrix

$$\mathcal{G}=\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & e_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

$$=\begin{pmatrix} \beta_4\gamma_4+\beta_1\gamma_1-\beta_2\gamma_2-\beta_3\gamma_3, & \beta_1\gamma_2+\beta_2\gamma_1-(\beta_3\gamma_4+\beta_4\gamma_3), \\ \quad \beta_1\gamma_3+\beta_3\gamma_1+\beta_2\gamma_4+\beta_4\gamma_2, & \beta_3\gamma_2-\beta_2\gamma_3+\beta_1\gamma_4-\beta_4\gamma_1, \\ \beta_1\gamma_2+\beta_2\gamma_1+\beta_3\gamma_4+\beta_4\gamma_3, & \beta_4\gamma_4+\beta_2\gamma_2-\beta_1\gamma_1-\beta_3\gamma_3, \\ \quad \beta_2\gamma_3+\beta_3\gamma_2-(\beta_1\gamma_1+\beta_4\gamma_1), & \beta_1\gamma_3-\beta_3\gamma_1+\beta_2\gamma_4-\beta_4\gamma_2, \\ \beta_1\gamma_3+\beta_3\gamma_1-(\beta_2\gamma_4+\beta_4\gamma_2), & \beta_2\gamma_3+\beta_3\gamma_2+\beta_1\gamma_4+\beta_4\gamma_1, \\ \quad \beta_4\gamma_4+\beta_3\gamma_3-\beta_1\gamma_1-\beta_2\gamma_2, & \beta_2\gamma_1-\beta_1\gamma_2+\beta_3\gamma_4-\beta_4\gamma_3, \\ \beta_3\gamma_2-\beta_2\gamma_3-(\beta_1\gamma_4-\beta_4\gamma_1), & \beta_1\gamma_3-\beta_3\gamma_1-(\beta_2\gamma_4-\beta_4\gamma_2), \\ \quad \beta_2\gamma_1-\beta_1\gamma_2-(\beta_3\gamma_4-\beta_4\gamma_3), & \beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\beta_4\gamma_4. \end{pmatrix} \quad (120)$$

It is easy to check that the " longitudinal " Lorentz transformation $\mathcal{L}$ (F. I.) is obtained from (120) by making

$$\mathfrak{i}_b=\mathfrak{i}_\theta, \quad b=\theta', \quad \beta=\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$\cos\theta'=\beta_4, \quad \cos\theta=\gamma_4, \quad \sin\theta'=\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}, \quad \sin\theta=\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}, \quad . \quad . \quad . \quad (122)$$

$$i_\theta=\frac{\beta_1 i_1+\beta_2 i_2+\beta_3 i_3}{\sin\theta'}=\frac{\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3}{\sin\theta}, \quad . \quad . \quad . \quad (123)$$

$$\beta_1\sin\theta=\gamma_1\sin\theta', \quad \beta_2\sin\theta=\gamma_2\sin\theta', \quad \beta_3\sin\theta=\gamma_3\sin\theta', \quad . \quad (124)$$

$$\beta_1=A\gamma_1, \quad \beta_2=A\gamma_2, \quad \beta_3=A\gamma_3, \quad \beta_4=A\gamma_4-B. \quad . \quad . \quad . \quad (125)$$

If, on the other hand, the rotation axes $\mathfrak{i}_b$ and $\mathfrak{i}_\theta$ are perpendicular to each other we may speak of the *Hypertransversal transformation* $\mathcal{H}$ determined by

$$\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

$$\mathfrak{i}_H=\mathfrak{i}_b\times\mathfrak{i}_\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

$$\mathfrak{i}_H=\tilde{H}*\mathfrak{i}_1=H_{11}\mathfrak{i}_1+H_{12}\mathfrak{i}_2+H_{13}\mathfrak{i}_3, \quad . \quad . \quad . \quad . \quad . \quad (128)$$

$$\mathbf{i}_b = \tilde{H} * \mathbf{i}_2 = H_{21}\mathbf{i}_1 + H_{22}\mathbf{i}_2 + H_{23}\mathbf{i}_3, \quad . \quad . \quad . \quad . \quad . \quad (129)$$

$$\mathbf{i}_\theta = \tilde{H} * \mathbf{i}_3 = H_{31}\mathbf{i}_1 + H_{32}\mathbf{i}_2 + H_{33}\mathbf{i}_3, \quad . \quad . \quad . \quad . \quad . \quad (130)$$

$$\beta_1 = H_{21}\sin b, \quad \beta_2 = H_{22}\sin b, \quad \beta_3 = H_{23}\sin b, \quad \beta_4 = \cos b, \quad . \quad (131)$$

$$\gamma_1 = H_{31}\sin\theta, \quad \gamma_2 = H_{32}\sin\theta, \quad \gamma_3 = H_{33}\sin\theta, \quad \gamma_4 = \cos\theta, \quad . \quad (132)$$

$$\beta_2\gamma_3 - \beta_3\gamma_2 = H_{11}\sin b\sin\theta, \quad \beta_3\gamma_1 - \beta_1\gamma_3 = H_{12}\sin b\sin\theta,$$

$$\beta_1\gamma_2 - \beta_2\gamma_1 = H_{13}\sin b\sin\theta, \quad . \quad (133)$$

which may be transformed by

$$\mathbf{i}_H = \mathbf{i}_1, \quad i_b = i_2, \quad i_\theta = i_3, \quad H_4 = 1, \quad H_1 = H_2 = H_3 = 0. \quad . \quad (134)$$

$$\beta_1 = 0, \quad \beta_2 = \sin b, \quad \beta_3 = 0, \quad \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = \sin\theta \quad . \quad . \quad (135)$$

into

$$\mathscr{H}_1 = \begin{bmatrix} \cos b\cos\theta, & -\cos b\sin\theta, & \sin b\cos\theta, & -\sin b\sin\theta, \\ \cos b\sin\theta, & \cos b\cos\theta, & \sin b\sin\theta, & \sin b\cos\theta, \\ -\sin b\cos\theta, & \sin b\sin\theta, & \cos b\cos\theta, & -\cos b\sin\theta, \\ -\sin b\sin\theta, & -\sin b\cos\theta, & \cos b\sin\theta, & \cos b\cos\theta, \end{bmatrix} \quad \text{(F. VI.)}$$

or, further, by

$$\cos b\cos\theta = R, \quad \cos b\sin\theta = S,$$

$$\sin b\cos\theta = U, \quad \sin b\sin\theta = V.$$

$$R^2 + S^2 = \cos^2 b, \quad U^2 + V^2 = \sin^2 b, \quad R^2 + U^2 = \cos^2\theta, \quad S^2 + V^2 = \sin^2\theta$$

$$. \quad . \quad . \quad (136)$$

into

$$\mathscr{H}_1 = \begin{bmatrix} R & -S & U & -V \\ S & R & V & U \\ -U & V & R & -S \\ -V & -U & S & R \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad \text{(F. VII.)}$$

If, finally, I make

$$b = \theta = \tfrac{1}{2}\Theta, \quad C = \cos\Theta, \quad D = \sin\Theta, \quad . \quad . \quad . \quad . \quad (137)$$

I obtain

$$\mathscr{H}_0 = \begin{bmatrix} C & -D & D & C \\ D & C & -C & D \\ -D & -C & C & -D \\ C & -D & D & C \end{bmatrix} \quad . \quad . \quad . \quad \text{(F. VIII.)}$$

Stockholm.
9th February, 1939.

XII. *The Sine Relation and Minimum Ellipticity in Foucault's Pendulums with Copper Suspensions.*

By B. Dasannacharya, M.A., Dr. Phil., F.Inst.P., Professor of Physics, Benares Hindu University, Benares, India, and Balram Singh Gautom, M.Sc. (Benares) *.

[Received June 21, 1939.]

1. *Introduction.*

It was shown by Dasannacharya and Hejmadi † that it becomes increasingly difficult to get correct values for the rotation of the earth, or, assuming the earth's rotation as known, for the rotation of the plane of oscillation of a Foucault's pendulum as the length of the pendulum is made smaller and smaller. While they could get excellent values with a length of about 250 cm., it was stated by them that it was exceedingly difficult to get correct values with a length of 95 cm., and that at that short length the pendulum does not continue to give correct values for any considerable length of time.

The answer to this was given by Dasannacharya ‡. He showed that the development of the minor axis, which is the chief cause of the difficulty, can be brought under the following relation :

$$\gamma_t = \gamma_t^0 \sin 2\phi. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Here γ_t is the ratio of the minor to the major axis of the elliptical orbit of the bob of the pendulum at a time equal to t after starting the oscillations of the pendulum. γ_t^0 is the maximum value for γ_t as a function of ϕ, the initial direction of oscillation, the zero of ϕ, being taken as the direction for which $\gamma_t = 0$.

We can, however, make the equation more general by introducing a constant ϕ_0 in it, so that we get

$$\gamma_t = \gamma_t^0 \sin 2(\phi - \phi_0). \quad \ldots\ldots\ldots\ldots \quad (2)$$

Then we can define ϕ as any arbitrary direction and ϕ_0 as the particular direction ϕ for which γ_t is zero.

---

* Communicated by the Authors.

† B. Dasannacharya and D. Hejmadi, Phil. Mag. (7) xxiii. pp. 65–88 (1937).

‡ B. Dasannacharya, Phil. Mag. (7) xxv. pp. 601–622 (1938).

This equation was established for a length of pendulum equal to about 250 cm., using a pear-shaped bob of brass weighing 377 gm. Braided strings of silk of diameter about 0·4 mm. as well as 1·0 mm. were used. It was shown that the sine relation was strictly applicable. Secondly, it was found that γ_t^0, *i. e.*, the maximum value of γ_t for a particular setting of the string, could be diminished to a very great extent by so fixing the string at the top support that no asymmetry develops in that part of the string held inside of the clutch of the support. The smallest value of γ_t^0 there obtained was γ_{30}^0 equal to 0·03, *i. e.*, in a time t equal to 30 minutes.

The present work has been undertaken to see if the sine relation holds good for metallic suspensions of copper wire, using various lengths, and, secondly, to investigate the lower limit $(\gamma_t^0)_{\text{min.}}^l$ to which γ_t^0 can be pushed for a length l of the pendulum.

We have been able to show the validity of the sine relation for lengths from 250 cm. down to 15 cm. Secondly, it is found that $(\gamma_t^0)_{\text{min.}}^l$ is inversely proportional to l^2 when the angular amplitude is kept constant while going from one length to the other.

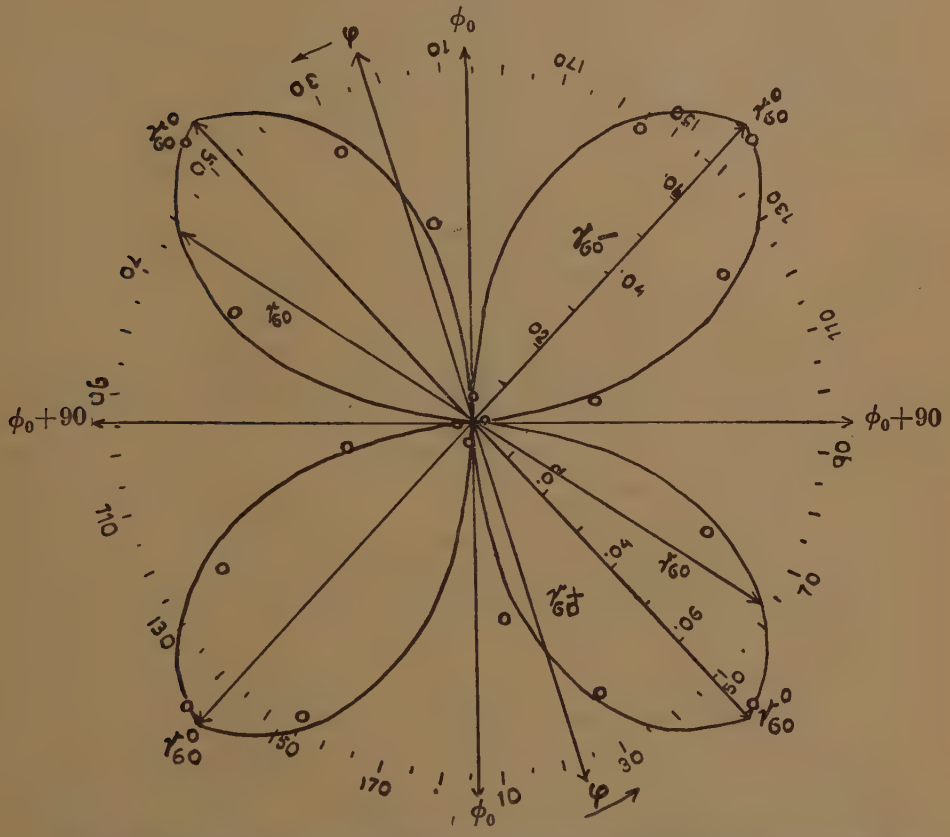
2. *Experimental Arrangement.*

The same pear-shaped brass bob weighing 377 gm., used by Dasannacharya (*loc. cit.*), has been used throughout, and the diameter of the copper wire used was about 0·3 mm., and in all the experiments it was taken out of the same lot. For experiments with pendulums of length about 250 cm. we have used the same apparatus used by Dasannacharya and Hejmadi (*loc. cit.*) and by Dasannacharya. For lengths smaller than this a separate smaller arrangement was used, also made of wood but of a more solid construction than the previous one. A very effective screening against breeze was obtained by using a glass wind-screen arrangement, with an adjustable sliding window to enable the loop used for drawing the bob aside to be tied to a support outside. As soon as the pendulum is started the loop is taken out and the window let down, leaving only a thin slit through which, by means of a stiff wire, the rider on the circular scale used for measuring ϕ, the direction of oscillation of the pendulum, could be rotated to coincide with the direction of vibration at any time. A 100 candle-power lamp was used. The plane of the filament of the lamp is made to coincide with the plane formed by the initial direction of oscillation and the vertical, whereby the shadow of the pin-point attached to the lower tip of the bob may be sharp owing to the absence of the penumbra. The other changes effected will be described in their proper places, particularly those referring to the double support for the copper wire at the top and the arrangement for determining the tightness or otherwise of the fixing of the copper wire to the top support.

3. *Pendulum of length 250 cm.*

To verify the sine relation it is advantageous if γ_t^0 is not too small, as otherwise observations have to be extended for long periods of time. In fig. 1 the different values of γ_{60} are plotted as distances from the centre point of the figure, at angles ϕ the initial directions of oscillation. The continuous line gives the values of the function sin $2(\phi-\phi_0)$ for the best

Fig. 1.



$\gamma_{t0}=\gamma^\circ_{60}\sin 2(\phi-\phi_0)$ in full line against experimental values of γ_{60} for a pendulum of copper wire suspension and of effective length 250 cm. For ϕ_0 and ϕ_0+90 γ_{60} is zero, and is maximum for $\phi=48°$ and $142°$.

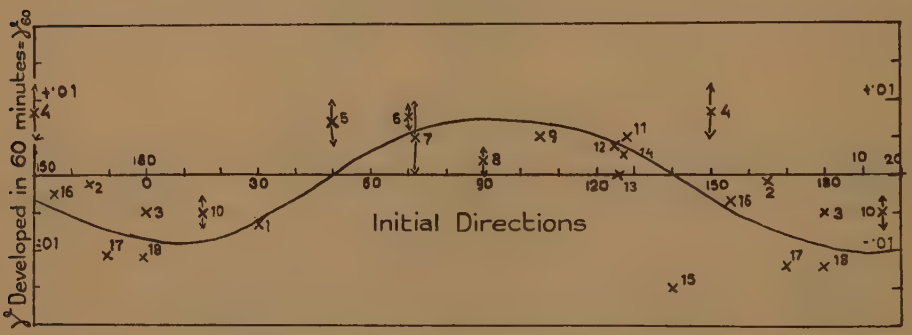
fit with experimental points, namely, with γ^0_{60} equal to 0·08 for directions 142° and 48°. The directions for ϕ_0 will be 45° away from these directions. The fit between the points and the curve are as close as can be expected.

Now we shall see to what extent γ_t^0 can be reduced by carefully fixing

the wire to the top support so as to produce as little asymmetry as possible. Fig. 2 shows the success obtained.

Here ϕ is plotted as abscissa and γ_{60} as ordinate. The order in which the successive measurements are made are shown against each point. Where repetitions are made for the same directions the points are shown at slightly different values of ϕ, so as to avoid overlapping of points. The minor axis developed is exceedingly small, being measured in fractions of a millimetre, although the amplitude to start with was as great as 11 cm. Therefore γ_t is subject to considerable errors of observation. The arrow marks show wherever indicated the limits of accuracy of the observed point. As we shall see later the value for γ_t^0 for this setting is much less than is to be expected (see section 4), and must have been obtained, perhaps, by accidental balancings of some disturbing factors.

Fig. 2.



Variations of γ_{60} with different initial directions of oscillations for a pendulum of copper wire suspension, length 250 cm., diameter about 0.3 mm., specially fixed to reduce γ°_{60} to minimum. Numbers on points give the serial order of measurement.

4. *A Method for the Determination of the Minimum Value for* γ_t^0.

It has been shown by Dasannacharya (*loc. cit.*) that fixing of the string too tightly at the top clutch produces in the string within the chuck a corrugated structure, a cross-section of which at right angles to the length of the string would resemble a cross. Such a fixing gives rise to a pronounced value for γ_t^0. If, however, the fixing is to be free from strain of this type it has to be very light, but then there is the danger of the string with the bob slipping from the top support altogether.

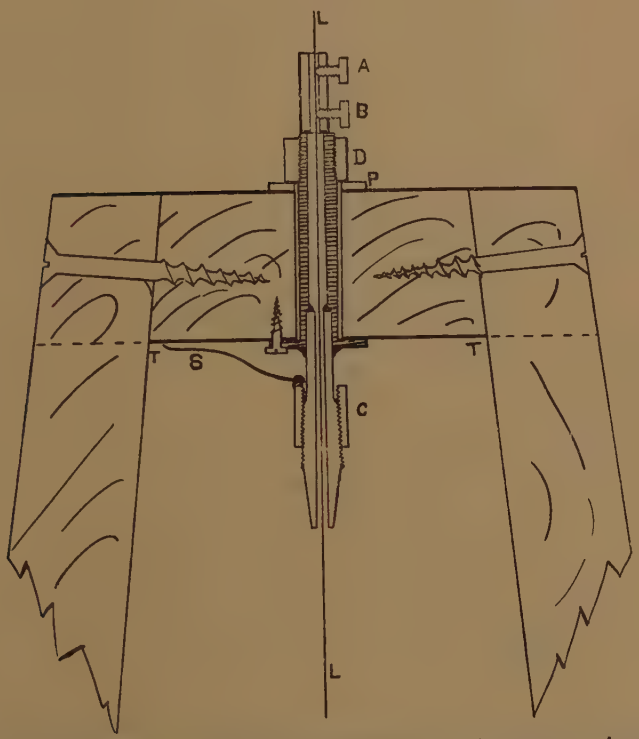
In the present investigation we have obviated this difficulty by introducing a system of double supports (see (fig. 3).

The string with the bob hanging from its lower end is supported by the binding screws A and B at the top. Only the limitation of the freedom

of movement of the string is brought about by the lower support, namely, the four-jawed chuck, which formerly used to carry the whole weight of the string and bob, and naturally could not be opened too widely. The position of the chuck could be read off on a circular scale by attaching a pointer S to the nut of the chuck.

To determine γ_t^0 for any particular fixing of the copper wire we have only to determine γ_t for two different directions ϕ_1 and ϕ_2, and calculate, using the formula 2. In this way γ_t^0 could be determined as a function

Fig. 3.



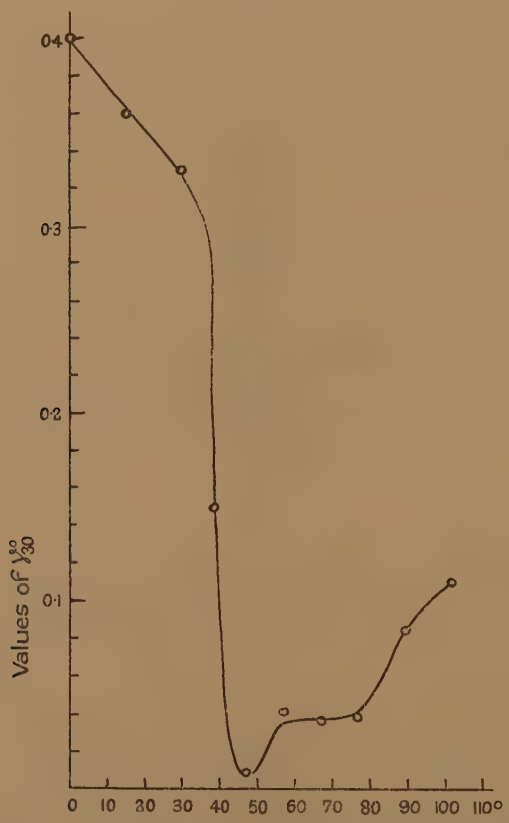
Double support system. AB are binding screws ; LL, copper wire suspension ; S, pointer moving on a circular scale TT gives the twist on the chuck C ; P is a loose circular plate, and nut D tightens the support.

of the fixing as measured by the readings of the pointer S, which we shall denote as twist of the chuck.

In fig. 4 we have the relations shown for a length of the pendulum equal to 241 cm. For small values of the twist the values of γ_t^0 for t equal to 30 minutes are very high, of the order of about 0·35. At a twist of about 38°, corresponding to a position where the string, namely, the

copper wire, is just held, it drops suddenly. At 47° it is seen to be 0·009, the lowest value. It then rises to 0·04, and then, as we tighten the chuck still further, it goes on rising to 0·11. We shall therefore be justified in concluding that 0·009 is the smallest value possible for γ^0_{30} for a length of the string equal to 241 cm., and that the optimum twist for the smallest value, namely $(\gamma^0_{30})^{241}_{\text{min.}}$ is 47°.

Fig. 4.



γ^0_{30} as a function of twist on the chuck for a pendulum of length 241 cm. Curve shows the best twist to be 47°, and $(\gamma^0_{30})^{l}_{\text{min.}}$ as 0·009 for the length 241 cm.

5. *The Verification of the Sine Relation and the Determination of* $(\gamma^0_{30})^{l}_{\text{min.}}$ *for different but smaller Values of* l.

We fix the twist at the optimum value according to the method given in the last section, but in the smaller apparatus. The γ^0_l thus obtained

will also be the smallest for that particular length of the pendulum. We start with the greatest length possible in the smaller apparatus, and, keeping the top fixing untouched, we proceed to verify the sine relation in the manner shown in section 3 for the longer length. Then we cut a certain length of the copper wire from the lower end and verify the sine relation for the pendulum of the diminished length. Thus we proceed further for smaller lengths. Down to a length of 15 cm. the sine

Fig. 5.

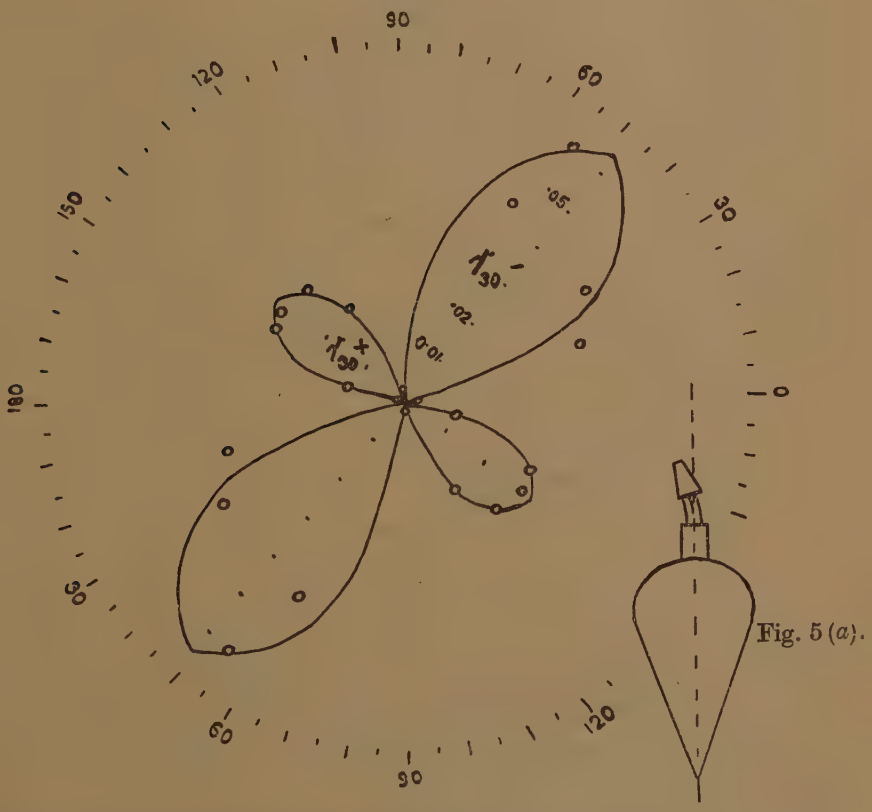


Fig. 5 (*a*).

The values of γ_t^0 when the bob is unsymmetrical (fig. 5 (*a*)) are not the same in adjacent quadrants. Pendulum length 132 cm.

relation is fulfilled. In two directions at right angles to each other γ_t assumes the value zero. For two adjacent quadrants separated by these zero directions γ_t has different signs, if in one it is positive, in the other it is negative ; but the γ_t^0, *i. e.*, the absolute value of the maximum in the adjacent quadrants, is the same, and the directions where these

values of the maximum occur are at right angles to each other and midway between the two zero directions.

A very interesting experiment may be referred to here, where γ^0_{30} *has not got* the same value, in absolute magnitude, in the adjacent quadrants. It is shown in fig. 5. The length of the pendulum was 132 cm., suspension wire of copper as usual in the present series, but the upper part of the bob, due to an accidental drop, got bent away from the axis, as shown in fig. 5(*a*) at the bottom of fig. 5. When, however, the part was made straight the usual type of the sine relation with γ^0_{30} equal in the adjacent quadrants was obtained.

The difficulty of measuring γ_t is enhanced, due to the fact that γ varies rapidly with time as the length of the pendulum is diminished so that γ_t will not change linearly with time, as θ rapidly sweeps out to values which have nothing to do with the small amount of the Foucault rotation. This is due to what has been named by Dasannacharya as " Drift " (*loc. cit.*).

The following table shows the values for γ^0_t for a value of t equal to 30 minutes. The values are also plotted in fig. 6.

It can be seen that γ^0_t increases very rapidly as the length l of the pendulum is diminished. The third column shows that it is not inversely

Pendulum length l (cm.).	$(\gamma^0_t)^l_{\text{min.}}$ t=30 minutes.	$(\gamma^0_t)^l_{\text{min.}} \cdot l$ t=30 minutes.	$(\gamma^0_t)^l_{\text{min.}} \cdot l^2$ t=30 minutes.
241	0·009	2·17	522
132	0·035	4·6	612
77	0·16	12·4	946
33	0·36	11·2	392
15	2·0	30·0	450

proportional to l, the fourth shows that *it is inversely proportional to the square of the length*, within the limits of the experimental accuracy.

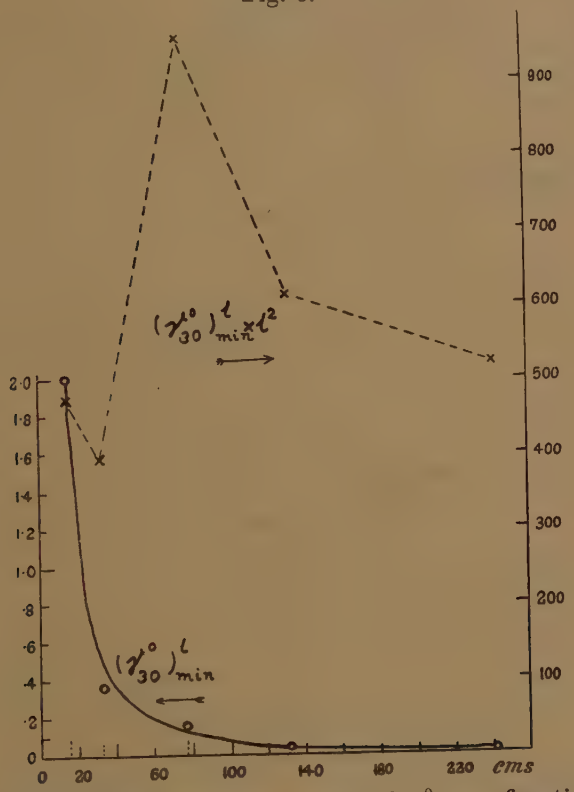
In the above experiments whatever the length of the pendulum used the initial amplitude P was such that P/l was maintained equal to 1/20—that is, the initial angular displacement was kept independent of the length. Under this condition $(\gamma^0_t)^l_{\text{min.}}$ was found to be inversely proportional to the square of the length. So it follows that for unit or constant linear displacement, $(\gamma^0_t)^l_{\text{min.}}$ is inversely proportional to the cube of the length. We know that in the case of a beam fixed at one end and free at the other the force required to be applied per unit linear displacement of the free end is given by

$$W = 3qM/l^3.$$

Here q is Young's modulus for the material of the suspension of the wire, M the moment of inertia of a cross-section about a diameter of the

same. The pendulum may be considered as a beam fixed at one end and getting a certain displacement at the free end. From the above considerations it follows that $(\gamma_t^0)^l_{\text{min.}}$ for unit angular displacement, to correspond to the working conditions of the present series of experiments, should be proportional to $Wl=3qM/l^2$. If the relation is true we can correlate the performance of a pendulum with the Young's modulus of the material of the wire used for the suspension and the dimensions of the wire. These considerations will be subject-matter of a separate investigation.

Fig. 6.



Length of pendulum. The minimum value of γ^0_{30} as function of length; and dotted curve shows that it is approximately inversely proportional to the square of the length.

6. *Summary and Conclusion.*

The present investigation deals with the verification of the relation derived by Dasannacharya * that

$$\gamma_t=\gamma_t^0 \sin 2(\phi-\phi_0)$$

* B. Dasannacharya, Phil. Mag. (7) xxv. pp. 601–622 (1938).

for the ellipticity γ or the ratio of the minor to the major axis developed in a time t as a function of the direction ϕ. Dasannacharya verified the relation for braided silk strings of about 250 cm. in length. The present investigation has verified the relation using suspensions of copper wire of about 0·3 mm. diameter for effective lengths of the pendulum equal to 250, 132, 77, 33, and 15 cm.

It is further found that γ_t^0 cannot be reduced to zero. Its value varies with the length of the string. Keeping the initial angular amplitude given to the bob of the pendulum as 1/20 and constant as we go from one length to the others, it is found that $(\gamma_t^0)_{\text{min.}}^l$ varies inversely as the square of the length of the pendulum. It has been pointed out that this suggests that one of the causes for $(\gamma_t^0)_{\text{min.}}^l$ is the Young's modulus of the material of the wire used for suspension.

Department of Physics,
Benares Hindu University, India.

XIII. *An Analogy between the Theories of Potential and Vibrations.*

By Harold Jeffreys *.

[Received August 21, 1939.]

Let $V(\xi)$ be a given real function for $-\infty<\xi<\infty$; we want a function that will satisfy Laplace's equation in two dimensions for $y>0$, and will reduce to $V(x)$ when $y\to 0$. Put $z=x+iy$, and consider the integral

$$f(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty} V(\xi)\frac{d\xi}{\xi-z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty} V(\xi)\frac{y\,d\xi}{(\xi-x)^2+y^2}-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{V(\xi)(\xi-x)\,d\xi}{(\xi-x)^2+y^2} \quad . \quad . \quad (2)$$

$$=R+iI, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

say, on separating the real and imaginary parts. If $\int V(\xi)\,d\xi/\xi$ converges, $f(z)$ represents a function regular when y is positive, and R and I are both solutions of Laplace's equation. If we put

$$\xi=x+\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$R=\frac{1}{\pi}\int_{-\infty}^{\infty} V(x+\eta)\frac{y\,d\eta}{\eta^2+y^2};\quad I=-\frac{1}{\pi}\int_{0}^{\infty}\{V(x+\eta)-V(x-\eta)\}\frac{\eta\,d\eta}{\eta^2+y^2}. \quad (5)$$

Now put

$$\eta=y\tan\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then

$$R=\frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} V(x+\eta)\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Consider the behaviour of R when $y\to 0$. Suppose first that $V(\xi)$ is continuous at $\xi=x$; then we can choose a range such that

$$|V(x+\eta)-V(x)|<\delta, \quad -\alpha\leqslant\eta\leqslant\alpha, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where δ is arbitrarily small, and

$$R=\frac{1}{\pi}\int_{-\tan^{-1}\alpha/y}^{\tan^{-1}\alpha/y} V(x+\eta)\,d\theta+\frac{1}{\pi}\int_{\tan^{-1}\alpha/y}^{\frac{1}{2}\pi}\{V(x+\eta)+V(x-\eta)\}d\theta. \quad . \quad (9)$$

By (8) the first term is

$$\frac{1}{\pi}2\tan^{-1}\frac{\alpha}{y}\{V(x)+\lambda\delta\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

---

* Communicated by the Author.

where $|\lambda| \leqslant 1$; and when $y \to 0$ this tends to $V(x)+O(\delta)$. The second integral tends to 0 if $V(\xi)$ is bounded, as it must be, because (1) converges. Hence the limit R differs from $V(x)$ by an arbitrarily small amount; and therefore

$$\lim_{y \to 0} R = V(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

at any point where $V(x)$ is continuous. A simple modification of the argument shows that if $V(\xi)$ is discontinuous at $\xi = x$ the limit is

$$\tfrac{1}{2}\{V(x-0)+V(x+0)\}.$$

But in a potential problem V will always be continuous, and in other problems, where there may be points of discontinuity, the integral of V over any range of x will still be given correctly.

For I we have

$$I = -\frac{1}{\pi}\int_0^\alpha \{V(x+\eta)-V(x-\eta)\}\frac{\eta\, d\eta}{\eta^2+y^2} - \frac{1}{\pi}\int_\alpha^\infty \{V(x+\eta)-V(x-\eta)\}\frac{\eta\, d\eta}{\eta^2+\eta^2}, \quad . \quad . \quad . \quad (12)$$

and if α is such that $V(x+\eta)-V(x-\eta)$ retains the same sign for $0<\eta<\alpha$, the first integral is less numerically than that obtained by putting $y=0$. Hence if the latter integral converges we have

$$\lim_{y \to 0} I = -\frac{1}{\pi}\int_0^\infty \{V(x+\eta)-V(x-\eta)\}\frac{d\eta}{\eta}. \quad . \quad . \quad . \quad . \quad (13)$$

It will converge wherever $V(x)$ is differentiable and diverge wherever $V(x)$ is discontinuous. If $V(x)$ is continuous but not differentiable, as, for instance, if

$$V(x+\eta)-V(x-\eta) = 1/\log|\eta|$$

the integral may diverge. Such cases, however, do not seem physically important.

The condition that $\int V(\xi)\, d\xi/\xi$ converges can be weakened for some of these results. For the existence of R, the fact that it satisfies Laplace's equation, and the establishment of its behaviour as $y \to 0$, it is sufficient that $\int V(\xi)\, d\xi/\xi^2$ should converge. We might, indeed, have $V(\xi)$ equal to a constant A; then we have immediately $R = A$ for all x, y. If then V tends to the same limit A as $\xi \to \pm\infty$, we can define

$$f(z) = A + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{V(\xi)-A}{\xi-z}\, d\xi = R + iI, \quad . \quad . \quad . \quad . \quad (14)$$

provided that $\int\{V(\xi)-A\}\, d\xi/\xi$ converges. The form (13) remains valid. Again, if $V(\xi)=0$ for $\xi>0$, and $=A$ for $\xi<0$, we get

$$R = \frac{A}{\pi}\tan^{-1}\frac{y}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which satisfies the conditions; but the function I is now $(A/\pi)\log r$ and is not regular at $y=\infty$. The form (14) seems to be the most generally important one in potential problems, since it is in practice the differences between the potentials or the charge functions at different places that concern us, and a mere additive constant is trivial. We can then take

$$V(x, y)=R=A+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\{V(\xi)-A\}y\,d\xi}{(\xi-x)^2+y^2}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$U(x, y)=I=-\frac{1}{\pi}\int_{0}^{\infty}\frac{\{V(x+\eta)-V(x-\eta)\}\eta\,d\eta}{\eta^2+y^2}, \quad . \quad . \quad . \quad (17)$$

$$U(x, 0)=-\frac{1}{\pi}\int_{0}^{\infty}\{V(x+\eta)-V(x-\eta)\}\frac{d\eta}{\eta}, \quad . \quad . \quad . \quad . \quad (18)$$

provided that (1) V tends to the same finite limit A as $\xi\to\pm\infty$ (2) $\int\{V(\xi)-A\}d\xi/\xi$ converges; (3), for (18), that the integral converges at $\eta=0$.

If U $(x, 0)$ satisfies the conditions imposed on $V(\xi)$, we could equally define

$$g(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}U(\xi, 0)\frac{d\xi}{\xi-z}=R'+iI', \quad . \quad . \quad . \quad . \quad (19)$$

and R′ will be a potential function regular for $y>0$ and reducing to $U(x, 0)$ when $y\to 0$. Hence $R'=I$; but the conjugate functions to R′ and I are I′ and $-R$, and therefore also $I'=-R$. Hence

$$g(z)=I-iR=-if(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$f(z)+ig(z)=2f(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\{V(\xi, 0)+iU(\xi, 0)\}\frac{d\xi}{\xi-z}, \quad . \quad . \quad (21)$$

or

$$f(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi-z}d\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

a well-known result, which follows at once from Cauchy's theorem with suitable conditions on the behaviour of $f(z)$ for large argument. It follows also that

$$V(x, 0)-A=\frac{1}{\pi}\int_{0}^{\infty}\{U(x+\eta)-U(x-\eta)\}\frac{d\eta}{\eta}. \quad . \quad . \quad . \quad (23)$$

I originally obtained (16) (17) (18) by a limiting process, starting from Green's solution for the potential at a point internal or external to a sphere, given the potential over the surface. When the radius of the sphere tends to infinity the solution for a distribution of potential over a plane follows; and then, taking this to be a function of one coordinate only, the solution of the two-dimensional problem is found *. The

* M. N. R. A. S. Geophys. Suppl. iv. p. 312 (1938).

conjugate function U can then be written down at once from the integral obtained for V. This procedure is so obvious that I should have expected it to have been carried out immediately after the publication of Green's paper, but after considerable search and inquiry I have not succeeded in finding it in print. The results can also be obtained by a method similar to Green's, but applied directly to the two-dimensional problem. The form (22) is well known, but appears to require that both the real and the complex parts of $f(z)$ are known over $y=0$; whereas either of them separately will determine the other when $f(z)$ is restricted to be bounded at infinity and regular for $y>0$.

The nearest analogue that I know is the work of Titchmarsh on the conjugate or allied Fourier integral *. If

$$V(\xi)=\int_0^\infty \{a(\kappa)\cos\kappa x+b(\kappa)\sin\kappa x\}d\kappa,$$

then the potential regular for $y>0$ will be

$$V(x, y)=\int_0^\infty \{a(\kappa)\cos\kappa x+b(\kappa)\sin\kappa x\}e^{-\kappa y}d\kappa,$$

and the conjugate function will be

$$U(x, y)=\int_0^\infty \{-a(\kappa)\sin\kappa x+b(\kappa)\cos\kappa x\}e^{-\kappa y}dy.$$

The function obtained by putting $y=0$ in this is evaluated directly by Titchmarsh, and the result (18) is found. The additional variable y, which appears in the potential problem, does not occur in this treatment. Titchmarsh obtains (2) incidentally (5.3.5 and 5.3.6 on p. 123), so that the whole of these results follow from his ; but it seems to me that (2) is the most important result, and for physical purposes needs more prominence, while the expression as a Fourier integral is secondary. M. E. Grimshaw has dealt with the conjugate integral in difficult cases, using Césaro summation and Denjoy integration ; but in the potential problem the exponential factors introduce the more drastic Euler summation naturally, and the integration presents less difficulty.

In the two problems where this arose the question was, given V over the surface, to find $\partial V/\partial y$, the normal gradient. Since

$$\partial V/\partial y=-\partial U/\partial x,$$

the problem reduces to that of finding the conjugate function, the values of which over $y=0$ contain all the information required. The first problem was, given the variation of gravitational potential over the earth's surface from measures of the deflexion of the plumb-line, to find

* ' Theory of Fourier Integrals,' chapter v. (1937).

the variation of gravity; the second was, given the disturbance of temperature at sea-level due to the fact that the outer surface is at air temperature, whatever the inequalities of height, to find the disturbance of the vertical temperature gradient in the crust, as measured in boreholes. The mathematical complications for exceptional cases in the above treatment have physical analogues. In the potential problem V would always be differentiable if the surface is plane; but with an irregular surface the reduction to a plane by assuming a first approximation to the vertical gradient introduces a second-order error, which may be large locally if there is, for instance, a sharp ridge on the surface. In the electrical and thermal problems the potential is given over the actual surface, and that at external points is not the continuation of any potential function that can exist between the actual surface and the mean surface. Hence the most that can be hoped for is to get results valid except in the neighbourhood of such special points. The analysis will give the mean normal gradient in ranges including them, but not the distribution in detail.

A physically quite different problem involves precisely the same analysis. The sensitivity of many types of recording apparatus, particularly seismographs, is tested by applying harmonic disturbances of known periods and recording the magnification and phase shift. If the disturbance is $\cos \gamma t$, the response is $\mu \cos (\gamma t - \epsilon)$, where μ and ϵ are real. This may be expressed by saying that the response to a disturbance $\exp (i\gamma t)$ is $\mu \exp i(\gamma t - \epsilon)$, where μ is an even and ϵ an odd function of γ. In most cases μ tends to 0 when γ is small. When γ is large μ usually tends to a finite limit with mechanical recorders and to 0 with electrical ones. According to the general theory of vibrations we can generalize to complex values of γ by regarding $\mu e^{-i\epsilon}$ as a function of γ. The system is always stable and damped; the characteristic values of γ have therefore positive imaginary parts. Pairs of conjugate functions will be the real and imaginary parts of $\mu e^{-i\epsilon}$ or of $\log \mu - i\epsilon$, and if one of these is known, subject to the conditions of convergence, it will be possible to find the other by the above theorem, $\mu e^{-i\epsilon}$ being regarded as a function of γ regular when γ has a negative imaginary part. The second pair, however, fail for electrical recorders because $\log \mu$ tends to $-\infty$ at $\gamma = \infty$, and in all cases are inconvenient for the same reason at $\gamma = 0$. The first pair, $\mu \cos \epsilon$ and $-\mu \sin \epsilon$, behave satisfactorily, but are subject to the difficulty that if one of them is determinable from observation the other will be determinable too, so that the information will apparently be either insufficient or redundant.

Nevertheless, even if μ and ϵ are both measured the observed values will not be quite consistent, and each of $\mu \cos \epsilon$ and $\mu \sin \epsilon$ may be used to compute a set of values of the other. The mean of the original

and computed values for each pair will give a more consistent pair, and if desired the process may be repeated till it converges. The formulæ will be, if $\mu \to 0$ for $\gamma \to \infty$,

$$\pi\mu \sin\epsilon = -\int_0^\infty \{(\mu\cos\epsilon)_{\gamma+\eta} - (\mu\cos\epsilon)_{\gamma-\eta}\}\frac{d\eta}{\eta}, \quad . \quad . \quad . \quad (24)$$

$$\pi\mu \cos\epsilon = \int_0^\infty \{(\mu\sin\epsilon)_{\gamma+\eta} - (\mu\sin\epsilon)_{\gamma-\eta}\}\frac{d\eta}{\eta}, \quad . \quad . \quad . \quad (25)$$

where μ and ϵ on the left refer to coefficient γ and on the right to coefficients $\gamma \pm \eta$. If $\mu \to \mu_0$, $\epsilon \to 0$ for short periods (γ large), $\mu\cos\epsilon$ in (25) must be replaced by $\mu\cos\epsilon - \mu_0$. The signs are the same as in (18) and (23), the change of sign due to the region where $\mu e^{-i\epsilon}$ is regular being that of $\mathrm{I}(\gamma)$ negative being compensated by the fact that the imaginary part of $\mu e^{-i\epsilon}$ is $-\mu\sin\epsilon$.

If μ is recorded without ϵ, a first approximation may be sought by first adopting a convenient odd function of γ, continuous at $\gamma = 0$, and deriving trial values of $\mu\cos\epsilon$ and $\mu\sin\epsilon$. Each of these leads to an estimate of the other by (24) and (25), and the ratio gives an approximation to $\tan\epsilon$. Adopting this the process may be repeated to give a further approximation, and ϵ will be found when the approximations repeat. Conversely, if ϵ is known without μ, we may begin with a convenient form of μ and compute $\mu\cos\epsilon$ and $\mu\sin\epsilon$ from the resulting form of the other. These together yield a first approximation to μ, and the process can be repeated. It will be noticed that in this case only the ratios of the values of μ for different values of γ can be determined.

The response to an impulsive disturbance $\mathrm{H}(t)$, equal to 0 for $t<0$ and to 1 for $t>0$, can be obtained as follows. For a damped stable system, if the response to a disturbance $\exp(\lambda t)$ is $f(\lambda)\exp(\lambda t)$, the response to the disturbance suggested is

$$\mathrm{X} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(\lambda)e^{\lambda t}\frac{d\lambda}{\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where c is any positive quantity. $f(\lambda)/\lambda$ is regular on the imaginary axis except possibly at $\lambda = 0$, where it will still be regular if μ tends to 0 for long periods. For positive c, since $f(\lambda)$ is regular and tends to μ_0 for large λ,

$$2\pi i\{f(\lambda) - \mu\} = -\int_{-i\infty}^{i\infty} \{f(z) - \mu_0\}\frac{dz}{z-\lambda}, \quad . \quad . \quad . \quad . \quad (27)$$

the contour being a large semicircle, which contributes 0, and the imaginary axis taken negatively. The term μ_0 contributes $\mu_0\mathrm{H}/(t)$ to X.

The second term gives

$$\mathrm{X}_2 = \frac{1}{4\pi^2}\int_{c-i\infty}^{c-i\infty}\int_{-i\infty}^{i\infty} \{f(z) - \mu_0\}e^{\lambda t}\frac{d\lambda\, dz}{\lambda(z-\lambda)}. \quad . \quad . \quad . \quad . \quad (28)$$

Integration with respect to λ gives 0 for $t<0$. For $t>0$ it gives

$$\begin{aligned} X_2 &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \{f(z)-\mu_0\}(e^{zt}-1)\frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\mu e^{-i\epsilon}-\mu_0)(e^{i\gamma t}-1)\frac{d\gamma}{\gamma} \\ &= \frac{1}{\pi}\int_0^{\infty} \mu\{\sin(\gamma t-\epsilon)+\sin\epsilon\}\frac{d\gamma}{\gamma}, \quad . \quad . \quad . \quad . \quad . \quad (29) \end{aligned}$$

so that

$$X=\mu_0 H(t)+\frac{1}{\pi}\int_0^{\infty} \{(\mu\cos\epsilon-\mu_0)\sin\gamma t+\mu\sin\epsilon(1-\cos\gamma t)\}\frac{d\gamma}{\gamma}. \quad (30)$$

We notice that X increases suddenly by μ_0 when t increases through 0. This form is therefore convenient for t small. But we may also proceed directly from (26), for if $f(\lambda)/\lambda$ is regular at $\lambda=0$ and $f(\lambda)$ finite at $\lambda=\pm i\infty$ we can reduce the path to the imaginary axis ; then

$$X=\frac{1}{\pi}\int_0^{\infty} \bar{\mu}\sin(\gamma t-\epsilon)\frac{d\gamma}{\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

a simpler form, which, however, does not put in evidence the discontinuity at $t=0$ for mechanical recorders. It should be the more convenient when the magnification $\to 0$ for short periods.

XIV. *Some Integral Representations of the Associated Legendre Functions.*

By A. Erdélyi *.

[Received December 14, 1939.]

§ 1. In a recent paper [4] Professor MacRobert obtained, by "translating" two corresponding integral representations of modified Bessel functions of the third kind, two different integral representations of the associated Legendre functions of first kind, one of which was already known [[3] p. 262 (121)], the other one seeming to be new. In § 2 I wish to point out that a more general integral representation, containing both of MacRobert's formulæ as special cases, can be obtained by the same method, using Meijer's integral representation of Bessel functions of third kind [[5], (19)] instead of the integral representations used by MacRobert. In § 3 I show that this more general integral representation can be directly obtained from the known special cases by fractional integration by parts. In the last section I propose to show that all the integral representations of associated Legendre functions dealt with in this note are particular cases of an integral formula [[2], (21)] for the hypergeometric function.

§ 2. MacRobert obtained the integral representations

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Gamma(m+n)\Gamma(m-n)(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)=\Gamma(m)\int_0^\infty\frac{\cosh nt}{(z+\cosh t)^m}dt \qquad (1)$$

$$[\mathrm{R}(z)>-1,\quad \mathrm{R}(m\pm n)>0]$$

[*cf.* also [3], p. 262 (121)] and

$$2^{n-\frac{1}{2}}\Gamma(m-n)\Gamma(n+\tfrac{1}{2})(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)=\int_0^\infty\frac{\sinh^{2n}t}{(z+\cosh t)^{m+n}}dt. \qquad (2)$$

$$[\mathrm{R}(z)>-1,\quad \mathrm{R}(n+\tfrac{1}{2})>0,\quad \mathrm{R}(m-n)>0]$$

by replacing in

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}}\Gamma(m+n)\Gamma(m-n)(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)=\int_0^\infty e^{-\lambda z}\mathrm{K}_n(\lambda)\lambda^{m-1}d\lambda \qquad (3)$$

[[6], p. 388 (7)] Basset's function by the integral representations

$$\mathrm{K}_n(\lambda)=\int_0^\infty e^{-\lambda\cosh t}\cosh nt\,dt\quad[\mathrm{R}(\lambda)>0] \quad . \quad . \quad . \quad . \qquad (4)$$

---

* Communicated by the Author.

[(6), p. 181 (5)] and

$$K_n(\lambda)=\frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\left(\frac{\lambda}{2}\right)^n\int_0^\infty e^{-\lambda\cosh t}\sinh^{2n}t\,dt \quad . \quad . \quad . \quad . \quad (5)$$
$$[R(\lambda)>0, \quad R(n+\tfrac{1}{2})>0.]$$

[(6), p. 172 (5)] respectively.

A more general representation of $P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(z)$ can be obtained by using instead of (4) or (5) the integral representation

$$K_n(\lambda)=\frac{\sqrt{\pi}}{\Gamma(\alpha)}(2\lambda)^{\alpha-\frac{1}{2}}\int_0^\infty e^{-\lambda\cosh t}\sinh^{2\alpha-1}\frac{t}{2}$$
$$\times F\left(\frac{1}{2}-n,\ \frac{1}{2}+n\ ;\ \alpha\ ;\ -\sinh^2\frac{t}{2}\right)\cosh\frac{t}{2}\,dt \quad . \quad (6)$$
$$[R(\lambda)>0, \quad R(\alpha)>0],$$

which is derived from Meijer's integral representation of Hankel's function [(5) (19)] by means of the formula [(6), p. 78 (8)]

$$K_n(\lambda)=\tfrac{1}{2}\pi i e^{\frac{1}{2}n\pi i}H_n^{(1)}(i\lambda),$$

and replacing t in Meijer's formula by $\lambda(\cosh t-1)$.

Inserting $K_n(\lambda)$ from (6) into (3) and changing the order of integrations is easily justified ; the integration with respect to λ is then readily performed, and we find the integral representation

$$\Gamma(m+n)\Gamma(m-n)\Gamma(\alpha)(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(z)$$
$$=2^\alpha\Gamma(m+\alpha-\tfrac{1}{2})\int_0^\infty(z+\cosh t)^{\frac{1}{2}-m-\alpha}\sinh^{2\alpha-1}\frac{t}{2}$$
$$\times F\left(\frac{1}{2}-n,\ \frac{1}{2}+n\ ;\ \alpha\ ;\ -\sinh^2\frac{t}{2}\right)\cosh\frac{t}{2}\,dt \, . \quad . \quad (7)$$

valid whenever the integral is absolutely convergent, *i. e.*, whenever

$$|\arg(z+1)|<\pi, \quad z\neq-1\ ; \quad R(\alpha)>0, \quad R(m\pm n)>0.$$

Meijer (*l. c.* pp. 40 and 49) pointed out that (4) and (5) are particular cases of (6) corresponding to $\alpha=\frac{1}{2}$ and $\alpha=\frac{1}{2}+n$ respectively. By the same argument (1) and (2) are particular cases of (7), corresponding to values $\alpha=\frac{1}{2}$ and $\alpha=\frac{1}{2}+n$ respectively.

§ 3. In two former notes (1), (2) I pointed out that fractional integration by parts is often useful in transforming hypergeometric integrals. For instance, (4), (5), and (6) can be transformed into each other by means of fractional partial integration. (Actually I performed this transformation on a generalization of these integrals representing Whittaker's function $W_{k,m}(z)$ in my paper, Proc. Benares Math. Soc. New Series, i. pp. 39–53 (1939).) Hence it seems probable that (1), (2), and (7) also can be transformed into each other by means of fractional integration by

parts. Although it is easy to perform this transformation quite rigorously by means of the theorems proved by Young and Love [(7), esp. Theorem C], for the sake of brevity I will restrict myself to a purely formal sketch of this transformation. I use the notation of (1), (6), (7), (8), putting $a=0$, $b=\infty$. I shall prove that (1), which is a well-known integral representation of associated Legendre functions, can be transformed into (7). All the other transformations follow similar lines.

We start with [see, *e. g.*, (5), (42)]

$$\cosh nt=\cosh\frac{t}{2}\,\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\ \frac{1}{2};\ \frac{1-\cosh t}{2}\right)$$

and

$$\Gamma(m)(z+\cosh t)^{-m}=\Gamma(m+\alpha-\tfrac{1}{2})\frac{d^{\frac{1}{2}-\alpha}(z+\cosh t)^{\frac{1}{2}-m-\alpha}}{d(-\cosh t)^{\frac{1}{2}-\alpha}},$$

according to the definition (1), (8). Using this,

$$\begin{aligned}
&\Gamma(\alpha)\Gamma(m)(\tfrac{1}{2}\pi)^{\frac{1}{2}}\int_0^\infty\frac{\cosh nt}{(z+\cosh t)^m}\,dt\\
&\quad=(2\pi)^{-\frac{1}{2}}\Gamma(\alpha)\Gamma(m+\alpha-\tfrac{1}{2})\int_{t=0}^\infty\left(\frac{\cosh t-1}{2}\right)^{-\frac{1}{2}}\\
&\qquad\times\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\ \frac{1}{2};\ \frac{1-\cosh t}{2}\right)\frac{d^{\frac{1}{2}-\alpha}(z+\cosh t)^{\frac{1}{2}-m-\alpha}}{d(-\cosh t)^{\frac{1}{2}-\alpha}}\,d(\cosh t)\\
&\quad=(2\pi)^{-\frac{1}{2}}\Gamma(\alpha)\Gamma(m+\alpha-\tfrac{1}{2})\int_{t=0}^\infty(z+\cosh t)^{\frac{1}{2}-m-\alpha}\\
&\qquad\times\frac{d^{\frac{1}{2}-\alpha}}{d(\cosh t)^{\frac{1}{2}-\alpha}}\left\{\left(\frac{\cosh t-1}{2}\right)^{-\frac{1}{2}}\right.\\
&\qquad\left.\times\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\ \frac{1}{2};\ \frac{1-\cosh t}{2}\right)\right\}d(\cosh t)\qquad(8)
\end{aligned}$$

according to the rule of fractional integration by parts.

Now, by (2), (10),

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\Gamma(\frac{1}{2})}\frac{d^{\frac{1}{2}-\alpha}}{d(\cosh t)^{\frac{1}{2}-\alpha}}\left\{\left(\frac{\cosh t-1}{2}\right)^{\frac{1}{2}}.\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\ \frac{1}{2};\ \frac{1-\cosh t}{2}\right)\right\}\\
&\qquad=2^{\alpha-\frac{1}{2}}\left(\frac{\cosh t-1}{2}\right)^{\alpha-1}.\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\,\alpha\,;\ \frac{1-\cosh t}{2}\right),
\end{aligned}$$

and hence (8) becomes

$$\begin{aligned}
&2^{\alpha-1}\Gamma(m+\alpha-\tfrac{1}{2})\int_{t=0}^\infty(z+\cosh t)^{\frac{1}{2}-m-\alpha}\left(\frac{\cosh t-1}{2}\right)^{\alpha-1}\\
&\qquad\times\mathrm{F}\left(\frac{1}{2}-n,\ \frac{1}{2}+n\,;\,\alpha\,;\ \frac{1-\cosh t}{2}\right)d(\cosh t),
\end{aligned}$$

which is identical with the right-hand side of (7). Putting $\alpha=\frac{1}{2}+n$, *i. e.*, using derivatives of order $-n$, we get from (1), instead of (7), (2). Similarly the other transformations can be effected.

§ 4. I conclude by pointing out that even the more general integral representation (7) is but a particular case of a still more general formula.

According to [3], p. 190 (13),

$$\Gamma(m+n)\Gamma(m-n)(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)$$

$$=\frac{\Gamma(m+n)\Gamma(m-n)}{2^{m-\frac{1}{2}}\Gamma(m+\frac{1}{2})}\,\mathrm{F}\left(m+n,\, m-n\,;\, m+\frac{1}{2};\, \frac{1-z}{2}\right),$$

and hence, due to [2], (21),

$$\Gamma(m+n)\Gamma(m-n)(z^2-1)^{\frac{1}{4}(\frac{1}{2}-m)}\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)$$

$$=\frac{2^{\frac{1}{2}-m}\Gamma(m+n)\Gamma(m-n)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(m-\lambda+\mu-\nu+\frac{1}{2})}\int_0^1 x^{\lambda+\nu-m-\frac{3}{2}}(1-x)^{m-\lambda+\mu-\nu-\frac{1}{2}}$$

$$\times\mathrm{F}(m-\lambda+\tfrac{1}{2},\, m-\nu+\tfrac{1}{2},\, m-\lambda+\mu-\nu+\tfrac{1}{2}\,;\, 1-x)$$

$$\times{}_3\mathrm{F}_2\left(m-n,\, m+n,\, \mu\,;\, \lambda\,;\, \nu;\, \frac{1-z}{2}x\right)dx, \quad \ldots \quad (9)$$

$$[|\arg(z+1)|<\pi,\ z\neq-1\,;\ \mathrm{R}(\lambda)>0,\ \mathrm{R}(\nu)>0,\ \mathrm{R}(m-\lambda+\mu-\nu+\tfrac{1}{2})>0|]$$

This is a fairly general integral representation of $\mathrm{P}^{\frac{1}{2}-m}_{n-\frac{1}{2}}(z)$. (7) is a particular case of (9). To obtain (7) put in (9):

$$\lambda=m-n,\quad \mu=\alpha+m-\tfrac{1}{2},\quad \nu=m+n,\quad x=\left(\cosh\frac{t}{2}\right)^{-2}.$$

References.

(1) A. Erdélyi, Quart. J. Math. Oxford x. pp. 129–34 (1939).
(2) A. Erdélyi, *Ibid.* pp. 176–89.
(3) E. W. Hobson, ' Spherical and Ellipsoidal Harmonics ' (1931).
(4) T. M. MacRobert, Phil. Mag. (7) xxvii. pp. 703–5 (1939).
(5) C. S. Meijer, Nieuw. Arch. Wiskde, xviii. pp. 35–57 (1934).
(6) G. N. Watson, ' Bessel Functions ' (1922).
(7) L. C. Young and E. R. Love, Proc. London Math. Soc. (2) xliv. pp. 1–34 (1938).

The University, Edinburgh.

XV. *Notices respecting New Books.*

The Variate Difference Method. By Gerhard Tintner. [Pp. xiii+175.] (Principia Press, Inc., Bloomington, Indiana, 1940. Price $2.50.)

This well-bound pamphlet is published jointly by the Cowles Commission for Research in Economics and the Department of Economics and Sociology, Iowa State College. It is intended rather for the economist than for the mathematical statistician, and is a very able advocacy of the variate difference method of analysing statistical series which may contain random fluctuations as well as a secular trend. The method is not new, and has been the subject of criticism by some writers, though others as eminent have upheld the use of the method. In the present volume the objections are mentioned, but are rapidly brushed aside, and the greater part of the space is devoted to an exposition of the technique of the method. This is well set out, and is illustrated by computations on actual observed statistics ; improvements in the method, due to the author himself, are included, together with sufficient general statistical theory to enable anyone to calculate and appreciate the significance of the results, including those of standard errors and correlation coefficients. There are some useful tables to facilitate the calculations. J. H. A.

Elasticité et Photoélasticimétrie. By H. Le Boiteux and R. Boussard. [Pp. iv+361.] (Paris : Hermann et Cie, 1940. Price 180 frs.)

The aim of this book is to present the subject of photoelasticity to the engineer, and the aim is carried out in an admirably logical manner by setting out first the mathematical theory of elasticity and then that of light. The former occupies four chapters (kinematics, stresses, Hooke's law, and two-dimensional problems) and the latter two, of which one develops the theory of optics, including polarization, from Maxwell's equations and the other is devoted to optics in anisotropic media. Having thus laid the foundations, the main subject of the book is reached, and occupies eight chapters, including one on the design of apparatus for photoelastic measurements and another on the construction of the necessary models of the structures to be investigated. There are ten plates in the book, more than half of them coloured, and these show in the clearest and most convincing manner how the stress contours are marked out by the method. There is also a short bibliography of books and papers on the subject. J. H. A.

Sound. By Dr. E. G. Richardson. [Pp. v+339.] (London : Edward Arnold & Co., 1940. Price 16*s.* net.)

Dr. Richardson's excellent treatise now appears in a third edition. Certain chapters—those on sound reproduction, impedance and supersonics—have been revised and extended, but in the main this edition retains the characteristics and scope of its predecessors. Those who know Dr. Richardson's contributions to the Progress Reports of the Physical Society will expect to find in his book some notice of the latest advances in the science of acoustics. This is the case ; and the third edition will take the place of the earlier editions as a thoroughly up-to-date treatise on Sound well suited to the needs of students in university classes.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

CONTENTS OF No. 200.

Page

XVI. Some Notes on End-corrections. By L. C. Tyte, B.Sc., Ph.D., F.Inst.P., Projectile Development Establishment 173

XVII. Solution of the Equation $f'(x)=f(1/x)$. By Ludwik Silberstein, Ph.D. 185

XVIII. The Fermi-Dirac Functions. By J. L. B. Cooper, D.Phil..... 187

XIX. Partial Derivatives of Derivatives. By W. S. Kimball, Ph.D., Robert College, Istanbul, Turkey 190

XX. Some Acoustical Source-Observer Problems. By S. W. Redfearn 223

XXI. Measurements of the Noise from Aerofoils and Streamline Wires. By W. F. Hilton, Ph.D., A.R.C.Sc., B.Sc., D.I.C., A.F.R.Ae.S. (of the National Physical Laboratory) 237

XXII. Some Measurements of High Frequency Permeability. By Willis Jackson, D.Sc., Electrotechnics Department, Manchester University 247

XXIII. Castigliano's Principle of Minimum Strain-Energy, and the Conditions of Compatibility for Strain. By R. V. Southwell 252

XXIV. Notices respecting New Books :—

Profs. J. E. Mack and M. J. Martin's The Photographic Process 259

Mr. H. T. Pledge's Science since 1500. A short History of Mathematics, Physics, Chemistry, Biology 259

Prof. O. K. Rice's Electronic Structure and Chemical Binding.. 260

Messrs. H. S. Allen and R. S. Maxwell's A Text-book of Heat.—Part II. 260

*** It is requested that all Communications for this Work may be addressed, post-paid, to the care of Messrs. Taylor and Francis, Ltd., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

XVI. *Some Notes on End-corrections.*

By L. C. Tyte, B.Sc., Ph.D., F.Inst.P.,
Projectile Development Establishment *.

[Received July 22, 1939.]

Abstract.

It is shown that the experimental results of Bate and Wilson on the end-corrections of cylindrical pipes surmounted by conical flanges are in agreement with the values calculated from an expression derived earlier by the writer for a pipe with an infinite conical flange.

The effect of plane flanges placed against one or two sides of a square organ-pipe is then discussed from considerations of electrical conductance, and the ratio of the calculated corrections is in agreement with experimental values previously obtained.

Finally, for the end-correction due to an orifice in a diaphragm an empirical expression has been derived relating the reciprocal of the end-correction to the area of the orifice.

---

§ 1. *Conical Flanges on Cylindrical Pipes.*

In a recent paper Bate and Wilson [1] have shown that, if a cylindrical pipe is surmounted by a flange in the form of a cone for semi-apical angles of less than 10°, the system behaves as a conical pipe ; further, the end-corrections for such conical pipes have been determined. For semi-apical angles of 10° and above, however, the system behaves as a flanged cylindrical pipe. From the least distances of the nodes in the cylindrical pipe from the junction of the cylindrical pipe and conical flange given in the paper, and the quarter wave-length of the frequency concerned (292 c.p.s.), the distances of the antinodes outside the cylindrical pipe can be obtained, giving the open-end-corrections.

In an earlier paper [2], dealing with the open-end-correction of a square organ-pipe, the present author has calculated the end-correction for a conically flanged pipe on the electrical conduction analogy ; (i.) the case of an infinite cone, and (ii.) the case of a truncated cone taken as a weighted mean of two truncated cones treated as made up of plane disks and conical shells respectively, having been discussed.

---

* Communicated by the Author.

The expressions derived were :—

Case (i.) for an infinite cone :

$$E=\sqrt{\pi}\,\bar{r}\sqrt{\frac{1}{\omega}-\frac{1}{4\pi}},$$

where E was the length of the cylindrical pipe of radius r equivalent to the flanged open end, and ω the solid angle of the cone.

Case (ii.) for a truncated cone :

$$E=\bar{r}\tan A\left\{1-\frac{\bar{r}}{\bar{R}}\right\}\left\{\frac{1}{\tan^2 A}\right\}^{L/L+U}\left\{\frac{1}{2\log_e \sec A}\right\}^{U/L+U},$$

where E and $\bar{r}$ had the same significance, $\bar{R}$ was the radius of the outer edge of the conical flange of semi-apical (or generating) angle A, and L and U were the weights ascribed to the lower and upper limits respectively, being given by the ratio

$$L : U=(\bar{R}^2+\bar{r}^2) : (\bar{R}^2-\bar{r}^2)\operatorname{cosec}^2 A.$$

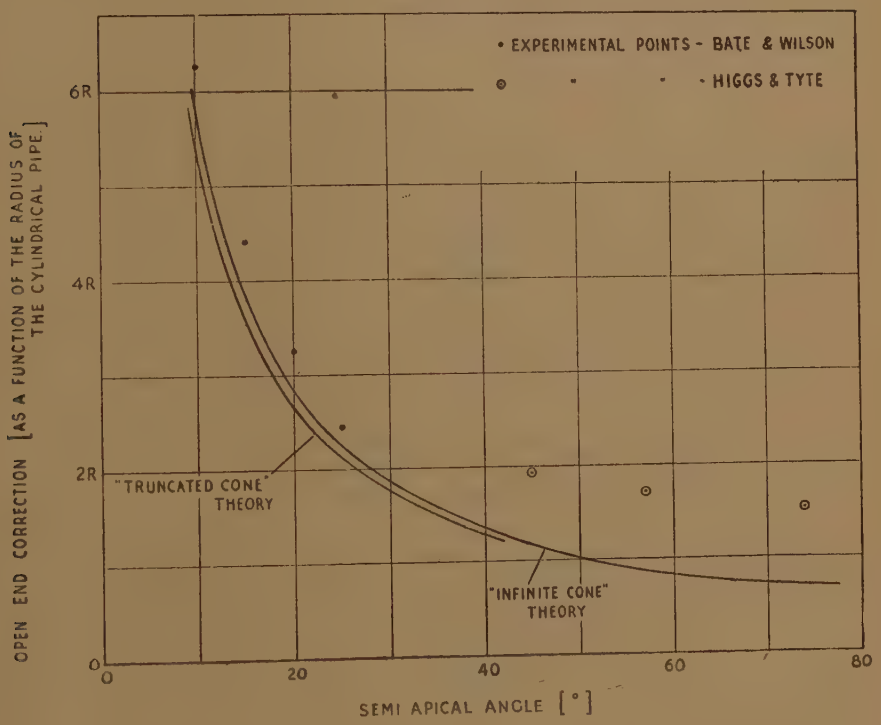
The corrections for flanges of the proper dimensions have been calculated over the relevant range of semi-apical angles, and are shown graphically in fig. 1, together with the experimental values derived from Bate and Wilson. The agreement between these experimental and the theoretical values is exceedingly good, and justifies the extension of the electrical analogy.

The " infinite cone " theory values lie rather nearer the experimental figures, and this curve has been extended to cover the angles used by the writer for investigating flange systems on square organ-pipes. The experimental values obtained have also been plotted, the square pipe values being converted to corrections for a circular pipe of the same cross-sectional area. These experimental points are well above the theoretical curve, and two possible explanations of this discrepancy can be suggested. It may be that the principle of converting corrections for square organ-pipes to those for circular pipes of the same area is unsound; however, as the experiments of Bate [3] have shown that a square orifice has practically the same conductance as a circular orifice of equal area, this explanation does not seem to be very probable. The flanges used by the writer were constructed of stout wood and covered with green baize, whereas the experiments of Bate and Wilson [1] were performed with sheet-metal flanges, and it may well be that the difference between a fairly absorbent surface and a perfectly reflecting one materially affects the open-end-correction, the theoretical value agreeing with the case involving no lateral loss. It would be of interest to check this point experimentally.

§ 2. *Plane Flanges on the Side of Square Organ-pipes.*

It is well known that the resistance to electric current flow from a charged disk of radius r attached to the surface of a semi-infinite medium of unit conductivity gives the value $4r$ for the conductivity (see ref. 4), and if the disk is embedded in an infinite medium the resistance is halved, the conductivity becoming $8r$. Further, if a half-disk be arranged with the diameter in the surface of the semi-infinite medium and normal to it the conductivity is also $4r$.

Fig. 1.



Bate [5] has attacked the problem of the conductance of the open-end of a cylindrical pipe by treating it as the electrical problem of a disk embedded in an infinite medium but with one face insulated. Consideration is directed to such a disk, divided into two equal parts by an infinite plane insulating lamina normal to its plane and by a similar plane in its own plane, thus dividing the infinite material into four equal portions each containing a semicircular conductor. It is then shown that the presence of an infinite plane in the plane of the disk has the effect of impeding the flow to one quadrant instead of two and reducing

the conductivity to $1/\sqrt{2}$ of its unimpeded value, and hence, as the conductivity into each quarter infinite space is $2r$, the conductivity of this half-disk into a semi-infinite space is $2r\sqrt{2}$.

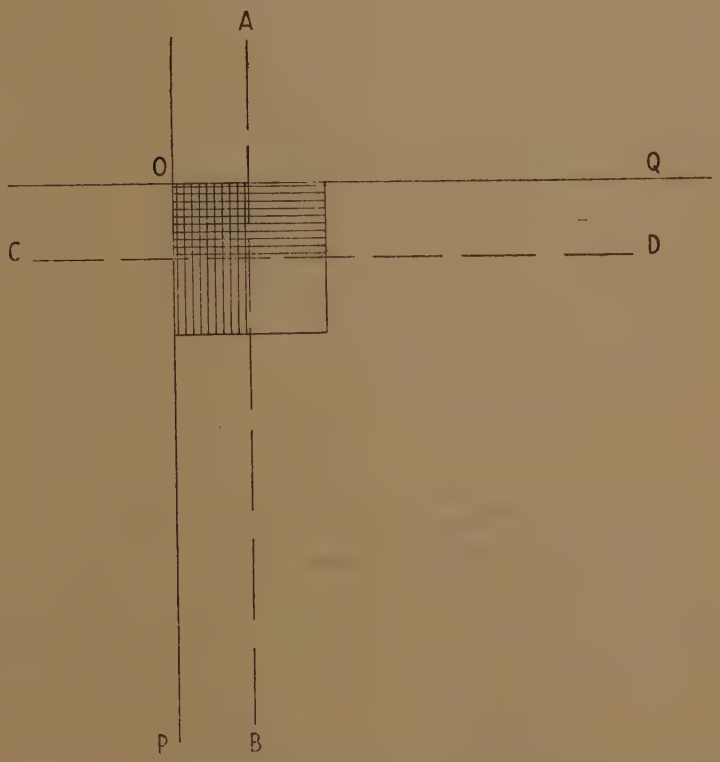
Let us now consider the problem of a square organ-pipe with an infinite plane flange placed against one side, as shown in fig. 2. The analogous

Fig. 2.

electrical problem will be that of a square plate, insulated on one side, with an edge in the surface of a semi-infinite medium and normal to it. Consider an infinite insulating lamina normal to the equivalent plate and parallel to the infinite flange (shown as a dotted line AB in the figure). Thus the upper surface of the square plate will be divided into two rectangles, which can be treated as semicircular disks of equal area. The equivalent half-disk to the right of the dotted line has a conductance into

semi-infinite space of $2R\sqrt{2}$, where R is the radius of the equivalent circle. The equivalent half-disk to the left of the dotted line can be treated as a half-disk with its diameter in the surface (defined by the flange) of the semi-infinite medium and normal to it. The conductance of this half-disk insulated on one side is 2R. Hence, assuming that there is no interference between the lines of flow of the left- and right-hand

Fig. 3.



equivalent disks, the conductivity of the two sections in parallel will be given by

$$2R+2R\sqrt{2}=2R(1+\sqrt{2})$$
$$=4R\frac{(1+\sqrt{2})}{2}.$$

This assumption, of course, is not strictly true, neither is the assumption that the left-hand disk is in a quarter-infinite medium, but, as shown below, they can be accepted as offering a theoretical solution close to the experimental facts.

We shall now press the argument a stage further, and consider a square organ-pipe with infinite plane flanges placed against two neighbouring sides, as shown in fig. 3. Consider in the electrical analogue two infinite insulating laminæ normal to the plate and parallel to the infinite flanges (shown as dotted lines AB, CD in the figure). Now the joint effect of the two flanges may be treated as the total effect produced by the two flanges acting separately. As shown above, when the flange OP only is operating the plate can be treated as divided into two equivalent disks by AB, that to the right having a conductance of $2R\sqrt{2}$ and that to the left (hatched vertically) a conductance of 2R. Let us now further consider the effect of the flange OQ :—the thin lamina CD will divide the plate into four sections, each of which can be treated as a quarter-circle of equivalent area. Assuming no mutual interference between the quadrants, the semicircle beneath the line CD will be unaffected by the presence of the infinite flange, and their conductances will remain as R for CB and $R\sqrt{2}$ for BD. Now the quadrants above CD (hatched horizontally) will suffer from the restraining influence of the flange OQ, which will reduce their conductivities to $1/\sqrt{2}$ of their unimpeded values, *i. e.*, to $R/\sqrt{2}$ for CA and $R\sqrt{2}/\sqrt{2}(=R)$ for AD. Thus the total conductance of the four quadrants will be

$$R\sqrt{2}+2R+R/\sqrt{2}=\frac{R(1+\sqrt{2})^2}{\sqrt{2}}.$$

The effect of these flange systems on the conductance of the equivalent disk, and the resultant effects on the end-corrections, calculated as Area/Conductance, are set out in Table I.

The equivalent values for the end-corrections, in terms of the length of the pipe-side, determined by the present writer [2] for square organ-pipes, when modified by the flange systems under discussion are also recorded in Table I., together with the value deduced from the experiments for the unflanged pipe. It can be seen that the ratios of the theoretical values and of the experimental values are in excellent agreement, justifying the assumptions of the present note.

Thus the difference in end-correction produced when the plane flange is placed against one side of the pipe and when it is placed at the end of the pipe normal to the pipe axis, observed experimentally, is shown to be in accordance with theory.

§ 3. *Orifices in Diaphragms.*

In recent years the acoustic impedance due to orifices in diaphragms forming the "open ends" of pipes has been investigated [5], [6], [7]. Experiments have been performed with the diaphragms surrounded by large

flanges and unflanged. The latter must be regarded as somewhat unsatisfactory, as the orifice is surrounded by a small flange, which becomes proportionally much smaller with increase in the orifice size.

An ideal open pipe would have an antinode at the open end. Practice falls short of the ideal, and the antinode is placed at a distance of $0{\cdot}58\,r$, where r is the pipe radius. By surrounding the pipe with an infinite horizontal flange the antinode is moved to a distance of $0{\cdot}82\,r$. Now

TABLE I.

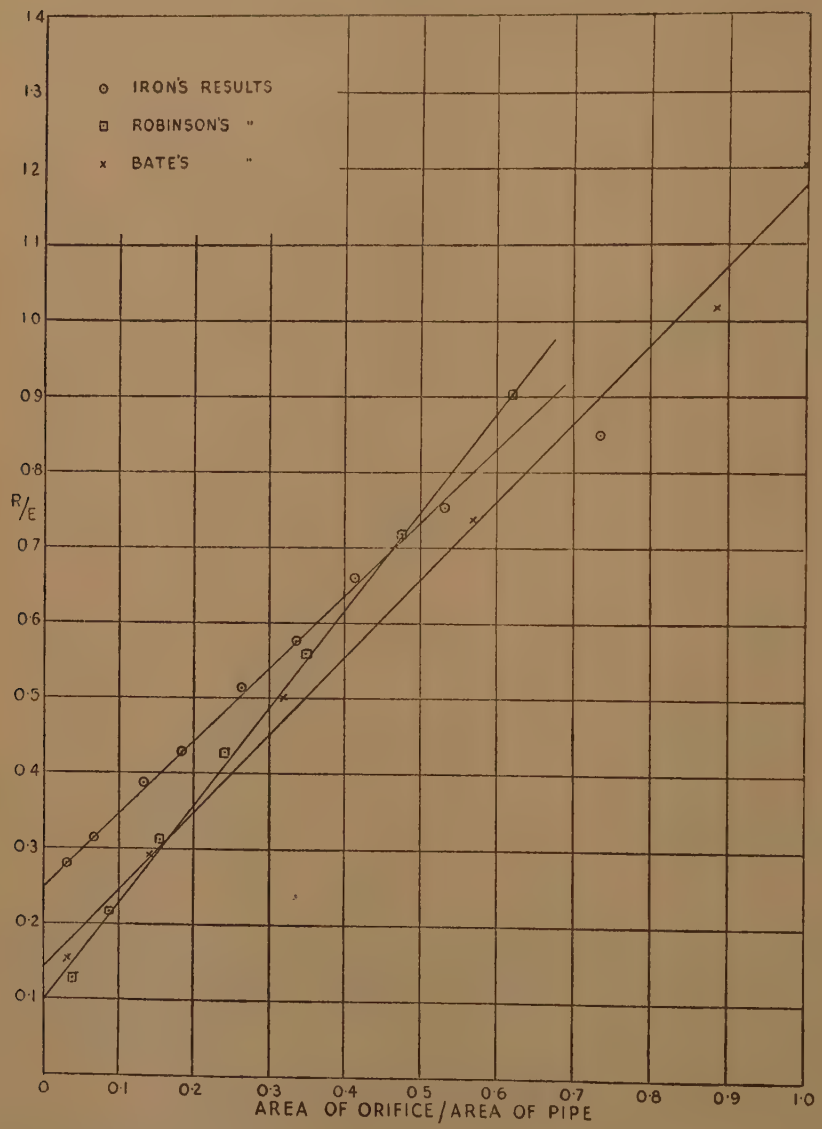
	Unflanged pipe.	Infinite plane flange normal to axis of pipe.	Infinite plane flange parallel to axis on one side of pipe.	Infinite plane flanges parallel to axis on two sides of pipe.
Equivalent conductance	$4\sqrt{2}\mathrm{R}$	$4\mathrm{R}$	$4\mathrm{R}(1+\sqrt{2})/2$	$\mathrm{R}(1+\sqrt{2})^2/\sqrt{2}$
Ratio of conductances ...	$\sqrt{2}$	1	$(1+\sqrt{2})/2$	$(1+\sqrt{2})^2/4\sqrt{2}$
Calculated end-correction ..	$\pi\mathrm{R}/4\sqrt{2}$	$\pi\mathrm{R}/4$	$\pi\mathrm{R}/2(1+\sqrt{2})$	$4\sqrt{2}\pi\mathrm{R}/(1+\sqrt{2})^2$
Ratio of calculated end-corrections	1	1·41	1·17	1·37
Experimental end-correction for square pipe.	0·62 B	0·85 B	0·74 B	0·87 B
Ratio of experimental end-corrections ..	1	1·37	1·19	1·40

decrease in the area of the orifice has the effect of further moving the antinode away from the end of the pipe, with consequent increase in the wave-length of the note sounded, until in the limit of the closed pipe the frequency is approximately halved and the "end-correction" to the original open pipe approaches the half wave-length of the fundamental note or the quarter wave-length of the note sounded. Thus, while the open-end-corrections are normally taken as independent of the wave-length, the equivalent correction for the closed end is dependent on the wave-length of the note emitted.

Bate [3] has suggested that the conductance C of an orifice can be expressed as

$$C = 2^{d/D}.d,$$

Fig. 4.

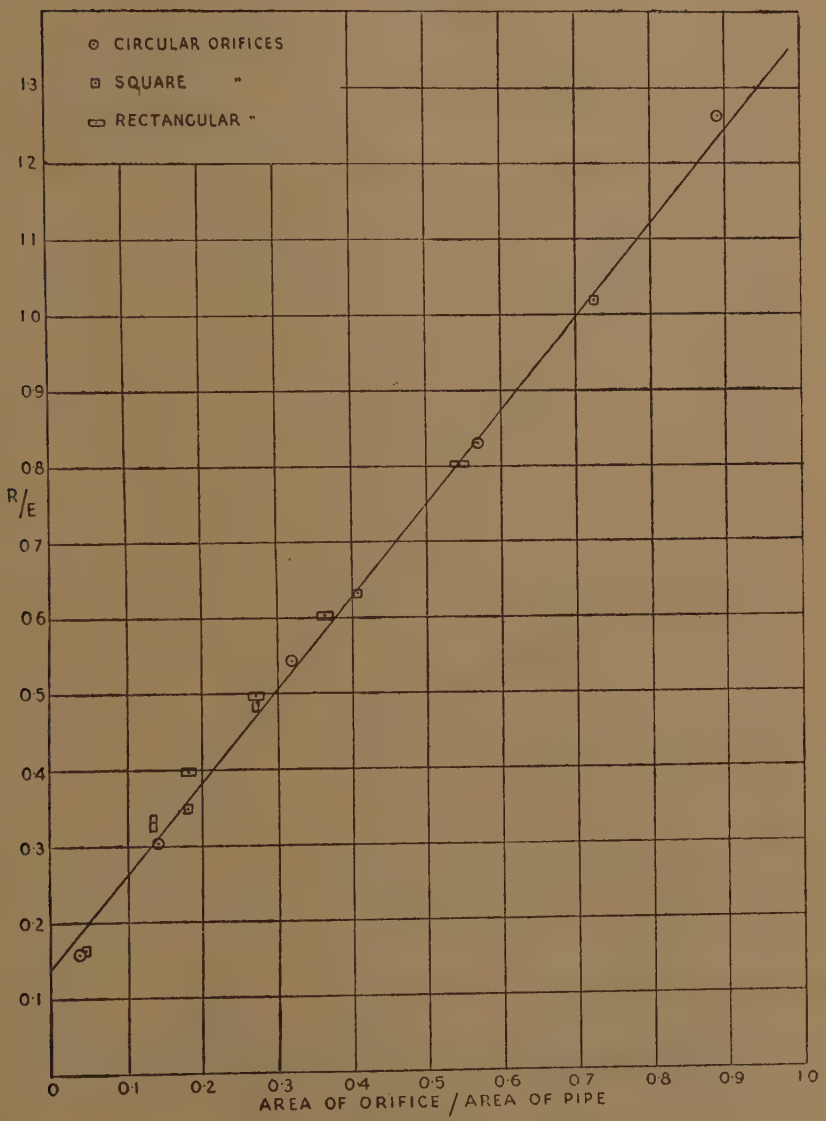


where d and D are the diameters of the orifice and pipe respectively. This expression was put forward for both flanged and unflanged pipes.

Robinson's experiments [7] have led him to suggest that

$$C=\alpha d(1+d/D)^{\beta},$$

Fig. 5.

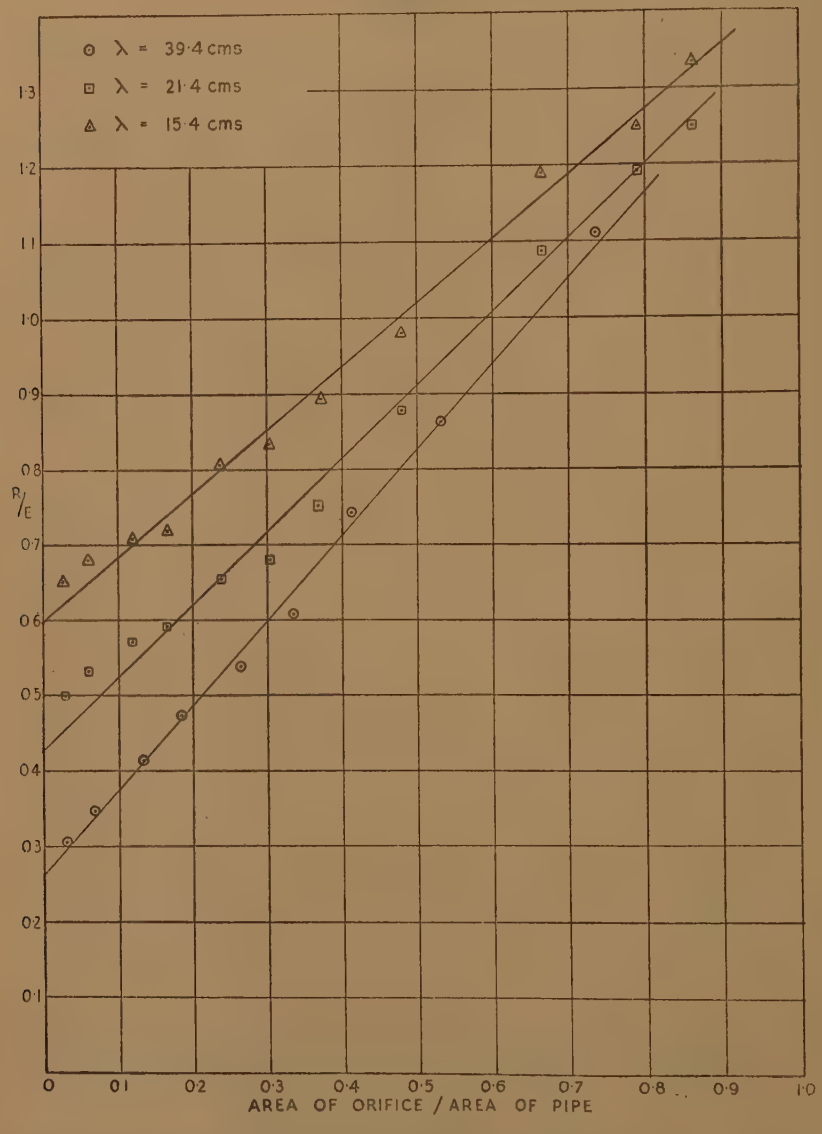


where C, d, and D have the same meaning as above and α and β are constants.

However, it seems that the experimental results can be expressed in

a simpler manner, the end-correction being inversely proportional to the area of the orifice. For direct comparison of results the end-correction

Fig. 6.



is expressed in terms of the pipe radius and the area of the orifice as a fraction of the area of the pipe.

The flanged pipe results are shown in fig. 4, representing the results

obtained by Irons [6] in Kundt's tube experiments in a pipe 4·7 cm. diameter for a wave-length of 39·4 cm., the mean values obtained by Bate [3] with a tuned organ-pipe of diameter 5·3 cm. over a range of frequencies, and the mean values obtained by Robinson [7] on a pipe of 5·08 cm. diameter, also over a range of frequencies, by a simple acoustic bridge method. It can be seen that each set of experiments is well represented by a linear relation, and the agreement between the observers is good in view of the different experimental methods adopted and the difficulty of the determinations.

Now Bate's results [3] for unflanged pipes given in fig. 5 show that the linear relation holds whether the pipes are flanged or not, although the slope of the lines will be different. The figure further shows that the relation can be extended to square and rectangular orifices.

Finally, in fig. 6 the results obtained by Irons [6] for unflanged pipes for different wave-lengths show the way in which the intercept varies with the wave-length, the values tabulated below indicating the excellent agreement.

TABLE II.

λ (cm.).	$4R/\lambda$.	Intercept.
39·4	0·24	0·26
21·4	0·44	0·44
15·4	0·61	0·60

Hence the end-correction can be expressed by the empirical relation

$$\frac{R}{E} = \frac{4R}{\lambda} + k\frac{a}{A},$$

where a is the area of the orifice, A that of the pipe, and k a constant. Representative values for the slopes and intercepts are given below.

TABLE III.

	Observer.	Wave-length.	Intercept.	Slope.
Flanged.	Bate.	Mean value.	0·14	1·03
,,	Robinson.	,, ,,	0·10	1·30
,,	Irons.	39·4 cm.	0·25	0·96
Unflanged.	Bate.	Mean value.	0·14	1·23
,,	Irons.	39·4 cm.	0·26	1·13
,,	,,	21·4 cm.	0·44	0·93
,,	,,	15·4 cm.	0·60	0·84

It would seem that the end-correction of the orifice would be best represented by the relations

$$\frac{R}{E}=\frac{4R}{\lambda}+\left(\frac{1}{0{\cdot}82}-\frac{4R}{\lambda}\right)\frac{a}{A} \text{ for flanged pipes}$$

and

$$\frac{R}{E}=\frac{4R}{\lambda}+\left(\frac{1}{0{\cdot}58}-\frac{4R}{\lambda}\right)\frac{a}{A} \text{ for unflanged pipes,}$$

which satisfy the limiting conditions for a, and, further, when λ approaches infinity give the end-correction as independent of the wave-length, which is in accordance with theoretical assumptions. The results do not satisfy these expressions particularly well, because of the differences between the experimental values for the open- and flanged-end-corrections and the classical values inserted in the formulæ.

However, it can be concluded that the end-correction of an orifice in a diaphragm both flanged and unflanged can be represented by an expression of this type within the accuracy of present determinations.

Hence the conductance of an orifice, which is given by area/end-correction is

$$C=\frac{A}{R}\left\{\frac{4R}{\lambda}+\left(\frac{1}{\mu}-\frac{4R}{\lambda}\right)\frac{a}{A}\right\},$$

where μ represents 0·82 or 0·58 according to the case considered,

$$=\frac{4\pi R^2}{\lambda}+\left(\frac{1}{\mu}-\frac{4R}{\lambda}\right)\frac{r^2}{R},$$

and thus when λ approaches infinity the conductivity is dimensionally represented by the radius of the orifice, which is in accordance with theory.

Acknowledgment.

I desire to express my thanks to Dr. Bate for an opportunity of discussing these problems, and to Dr. A. D. Crow and the Ordnance Board for permission to publish this paper.

References.

(1) Bate and Wilson, Phil. Mag. xxvi. p. 752 (1938).
(2) Higgs and Tyte, Phil. Mag. iv. p. 1099 (1927).
(3) Bate, Phil. Mag. x. p. 617 (1930).
(4) Gray, Matthews, and MacRobert, ' Bessel Functions,' p. 141.
(5) Bate, Phil. Mag. xxiv. p. 453 (1937).
(6) Irons, Phil. Mag. v. p. 588 (1928) ; vii. p. 873 (1929).
(7) Robinson, Proc. Phys. Soc. xlvi. p. 772 (1934).

XVII. *Solution of the Equation* $f'(x)=f(1/x)$.

By Ludwik Silberstein, Ph.D.*

[Received April 3, 1940.]

The equation reads: the derivative of a function at x is equal to the value of the function at $1/x$. Thus, unless $x=1$, the proposed equation is not an ordinary differential equation, but a kind of hystero-differential equation.

To satisfy it, put

$$f=x^m+\lambda x^n,$$

where m, n, λ are constants. Then

$$f'(x)=mx^{m-1}+\lambda nx^{n-1}=x^{-m}+\lambda x^{-n},$$

whence

$$\lambda=m,\ n=1-m,$$

$$\lambda=\frac{1}{n}=m=\frac{1}{1-m}.$$

Thus the complete solution of our equation is

$$f(x)=a(x^m+mx^{1/m}), \quad \ldots\ldots\ldots \quad (1)$$

where a is arbitrary and m is a root of the quadratic

$$m^2-m+1=0. \quad \ldots\ldots\ldots \quad (2)$$

Put $x=e^t$. Then, since the roots of (2) are $m_1=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $m_2=\frac{1}{m_1}$,

$$\frac{1}{a}f(t)=e^{mt}+me^{t/m}=e^{t/2}\left\{e^{\frac{\sqrt{3}}{2}it}+me^{-\frac{\sqrt{3}}{2}it}\right\},$$

or, after simple reductions,

$$f(t)=ae^{t/2}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}t+\sin\frac{\sqrt{3}}{2}t\right). \quad \ldots\ldots \quad (3)$$

In terms of x itself,

$$f(x)=a\sqrt{x}\left\{\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\log x\right)+\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\}.$$

---

* Communicated by the Author.

Since

$$\sqrt{3}\cos\frac{\sqrt{3}}{2}t+\sin\frac{\sqrt{3}}{2}t=2\cos\left(\frac{\sqrt{3}}{2}t-\frac{\pi}{3}\right),$$

we may write also

$$f(x)=a\sqrt{x}\cos\left(\frac{\sqrt{3}}{2}\log x-\frac{\pi}{3}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This, with an arbitrary constant coefficient a, is the complete solution of the equation $f'(x)=f(1/x)$.

For $$x=1, \quad f(1)=a\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}a,$$

so that

$$f(x)=\frac{2}{\sqrt{3}}f(1).\sqrt{x}\cos\left(\frac{\sqrt{3}}{2}\log x-\frac{\pi}{3}\right).$$

The function is completely determined by its (arbitrarily given) value at $x=1$.

Rochester, N.Y.
March 21, 1940.

XVIII. *The Fermi-Dirac Functions.*

By J. L. B. Cooper, D.Phil.*

[Received December 14, 1939.]

I.

The Fermi-Dirac functions occur in the quantum-mechanical Fermi-Dirac statistics †. Tables of these functions were drawn up recently by McDougall and Stoner ‡. The purpose of this article is to derive their main properties by a method simpler than those used hitherto ; this method makes use of the fact that the expression for the functions is a *Faltung*, and employs the Laplace Transform to deal with them. No originality is claimed for any of the formulæ of this article except formula (2) ; but its interest lies in the great ease with which the classical properties can be deduced from this formula.

The functions are defined for $\mathrm{R}(h)>-1$ by

$$\mathrm{F}_k(t)=\int_0^\infty \frac{x^k\,dx}{e^{x-t}+1}\,dx. \qquad \ldots\ldots\ldots \quad (1)$$

The important cases in practice are $k=\frac{1}{2}$, $k=\frac{3}{2}$.

Let $w=u+iv$; for $0<v<1$, we get

$$\int_{-\infty}^{\infty} e^{-wt}\mathrm{F}_k(t)\,dt=\frac{\Gamma(k+1)}{w^{k+1}}\frac{\pi}{\sin \pi w},$$

and so, by the inversion formula for the Laplace Transform,

$$\mathrm{F}_k(t)=\frac{\Gamma(k+1)}{2i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{wt}\,dw}{w^{k+1}\sin \pi w}, \qquad \ldots\ldots \quad (2)$$

where $0<\alpha<1$.

This formula is proved for $\mathrm{R}(k)>-1$; but the right-hand of 2 is analytic for all k, and hence gives an analytic continuation of $\mathrm{F}_k(t)$, regular save for poles at poles of $\Gamma(k+1)$. We can differentiate it under the integral sign, so

$$\mathrm{F}_k'(t)=k\mathrm{F}_{k-1}(t).$$

In general

$$\mathrm{F}_k^{\,r}(t)=k(k-1)\ldots(k-r+1)\mathrm{F}_{k-r}(t).$$

---

* Communicated by the Author.

† *E. g.*, see Sommerfeld, *Zeitschrift für Physik*, xlvii. p. 1 (1928).

‡ Phil. Trans. Roy. Soc. London, A, No. 773, vol. cclxxxvii. (1938).

From 1, using a well known integral for $\zeta(k)$,

$$F_k(0)=(1-2^{-k})\Gamma(k+1)\zeta(k+1),$$

and this holds for all k by analytic continuation.
Hence

$$F_k^r(0)=(1-2^{-k+1})\Gamma(k+1)\zeta(k-r+1).$$

2. *The Asymptotic Expansion* *.

The asymptotic expansion of the Fermi-Dirac functions is easily found from the expansion of their Laplace Transform near $w=0$. We have †

$$\frac{\pi w}{\sin \pi w}=1+\sum_{n+1}^{\infty} c_n w^{2n},$$

where $$c_n=2(2^{n-1}-1)B_n\pi^{2n}/(2n)!$$

Write $$\frac{\pi}{w^{k+1}\sin \pi w}-\frac{1}{w^{k+2}}\left\{1+\sum_{m=1}^{n} C_m w^{2m}\right\}=\psi(w),$$

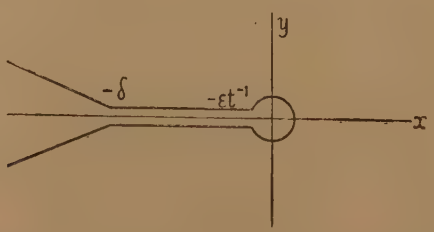
so that as $w\to 0$,

$$\psi(w)=O(w^{2n-k}).$$

The transform of $\psi(w)$,

$$\Psi(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\psi(w)e^{wt}\,dw$$

can be estimated for $t>0$ by transforming the contour of integration into the contour consisting of a circle of radius ϵt^{-1} about the origin, lines above and below the real axis connecting the point $-\epsilon t^{-1}$ to the point $-\delta$, and lines branching at angles $\pm\theta$, $0<\theta<\frac{\pi}{2}$, to the negative real axis from $-|\delta|$ $(\delta<1)$.



The integral around the circle $|w|=\epsilon t^{-1}$ is $O[(\epsilon t^{-1})^{2n-k+1}]$, and so tends to zero as $\epsilon\to 0$.

The integral along the branching lines is $O(e^{-\delta t})$.

---

* See Sommerfeld, *loc. cit.*

† Chrystal's 'Algebra,' ii. p. 383.

The integral along the lines from $-\epsilon t^{-1}$ to $-\delta$ is less than

$$A\int_{\epsilon t^{-1}}^{\delta} w^{2n-k}e^{-tw}\,dw = At^{-2n+k+1}\int_{\epsilon}^{+\delta} w^{2n-k}e^{-w}\,dw,$$

and making $\epsilon \to 0$, we see that this is $0(t^{-2n+k+1})$; altogether therefore,

$$\Psi(t) = 0(t^{-2n+k+1}).$$

Hence

$$\frac{1}{2\pi i}\int \frac{e^{wt}\,dw}{w^{h+1}\sin \pi w} = \frac{t^{h+1}}{\Gamma(h+1)} + \sum_{1}^{n} \frac{c_m t^{h+1}}{\Gamma(h+2-2m)}\frac{1}{t^{2m}} + O(t^{h-2m+1}).$$

So we get the asymptotic expansion

$$F_k(t) \approx \frac{t^{k+1}}{k+1} + t^{k+1}\Gamma(k+1)\sum_{m=1}^{\infty} \frac{c_m t^{-2m}}{\Gamma(k-2m+2)}.$$

XIX. *Partial Derivatives of Derivatives.*

By W. S. Kimball, Ph.D., Robert College, Istanbul, Turkey *.

[Received April 21, 1939.]

Part I.--The Partial Derivatives of the Slope of a Straight Line.

Introduction.

The method of differentials previously used to get the new critical equations in the Calculus of Variations [1] and the derivatives of line integrals by parallel displacement [2] is here extended to include second and higher partial derivatives of line integrals having a slope y' in the vector integrand as in the Calculus of Variations proper.

But attention is restricted here in Part I. to that class of such integrals whose path of integration is *rectilinear* at the upper and lower ends of these parallel path displacements. As formerly, these parallel displacements are measured by the variations $\delta x=h$ and $dy=k$ of the *independent* variables (x, y) anywhere on the path subjected between arbitrary limits to a parallel displacement. The slope y' thus experiences no variation or partial differentiation except at the upper and lower limits of the parallel displacement.

In order to meet sufficiency requirements for partial differentiation a new definition for a derivative is found to be necessary. Definition: The derivative of y with respect to x is the quotient of $dy_1=Ln$, by $dx=\Delta x=n$, where n is a mathematical symbol representing unit positive magnitude for the increment of the independent variable and $L=\underset{\Delta x\to 0}{\text{limit}}\dfrac{\Delta y}{\Delta x}$ is the usual limit concerned with a derivative:

$$\frac{dy_1}{n}=L=\underset{\Delta x\to 0}{\text{limit}}\frac{\Delta y}{\Delta x}. \quad \ldots\ldots\ldots \quad (35)$$

It is a kind of trigonometric "accident" that (35) equals the ratio of $dy=L\Delta x$ by $dx=\Delta x$ for *any* sized differential triangle having slope of its hypotenuse given by $L=\tan\alpha$. The new definition excludes all of these "accidentally" admitted similar triangles, and prescribes the *unit* base triangle of fig. 4 as the only one whose sides can represent the derivative elements in the numerator and denominator of the ratio (35).

---

* Communicated by the Author.

This differs no whit from the classical derivative except that it particularizes those differentials defined in the usual way by insisting on the ratio of $dy_1 = Ln$ by *unit* positive magnitude $dx = n$ for the increment of x. This should enhance the clarity of a derivative as a *rate* of change, or change in y per *unit* change in x. This definition leads to the simple operational rule : The partial derivatives with respect to x and y of the slope $p = \frac{dy}{dx}$ of a straight line are given by two steps—(a) Differentiate partially with respect to the corresponding differential ; (b) Set the differential of the independent variable x equal to $n = 1$ unit. Thus $p_x = -\frac{p}{n}$ and $p_y = \frac{1}{n}$, and higher partials with respect to y, are all zero.

Special features of the new partial derivatives are :—

(a) *Uniqueness.*—The method of parallel displacements, together with the concept of a derivative as a *rate* of change, leads directly to the new formulas.

(b) *Functions of Units.*—The new partial derivatives are functions of units in contrast with classical derivatives which represent the same physical or geometrical rate, regardless of units chosen.

(c) *Mathematical Consistency.*—For example, the partial rate of change of $\frac{\partial J}{\partial x} = \frac{\rho}{\sqrt{1+p^2}}$, where ρ is density and p is slope, is given by

$$\frac{\rho_x}{\sqrt{1+p^2}} + \rho \frac{\partial}{\partial x} \frac{1}{\sqrt{1+p^2}},$$

where the second term is essential to a complete and consistent formula where the length element can vary as well as the density.

(d) *Failure of Classical Differentiation.*—No challenge to the unquestioned correctness of classical differentiation is contained in the present article. Its failure lies in that it balks *completely* before the problem of supplying the partial rates of change of a slope p, as required above by mathematical consistency with a directness that can not be evaded, involving variations as essential as those of the density.

§ 1. *The Partial Derivatives of the Length of a Straight Line from the Origin to the Variable Point* (x, y).

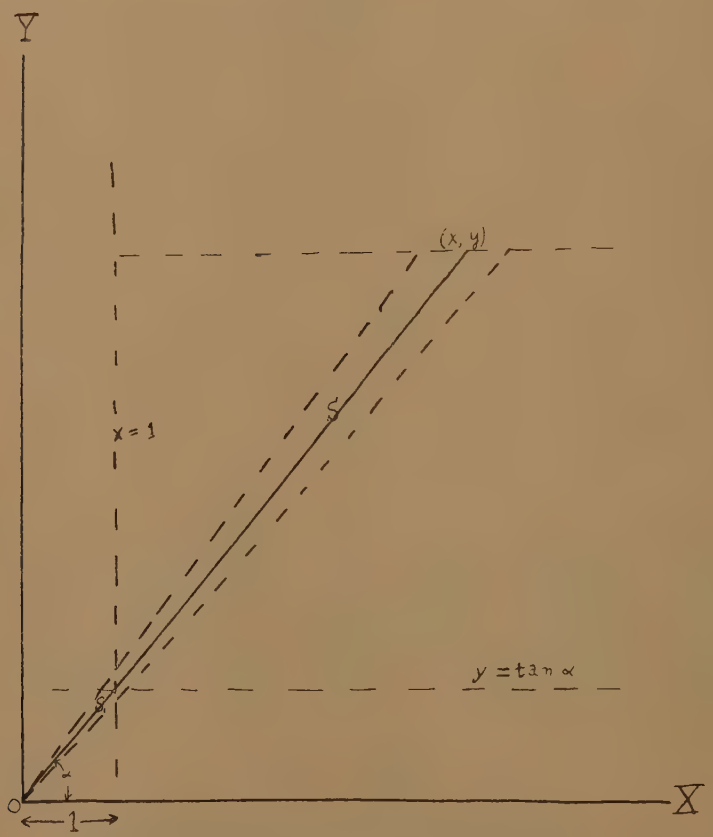
The length of a straight line from the origin to (x, y) may be expressed as $s = \sqrt{x^2+y^2}$. Its first and second partial derivatives with respect to x are

$$\frac{\partial s}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos \alpha, \quad \ldots \ldots \ldots \quad (1)$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{x} \sin^2 \alpha \cos \alpha = \frac{1}{s} \sin^2 \alpha. \quad \ldots \ldots \quad (2)$$

In taking these partials we are, geometrically speaking, letting the point (x, y) move along the straight lines $y=$constant, while the straight distance from the origin is measured along chords from the origin to (x, y), three of which are shown in fig. 1. Note that the first partial (1) is of zero dimensions and expressible as a trigonometric function of α, the angle of inclination of the line from the origin to (x, y), and hence is

Fig. 1.



The partial variations of S_1 and S.

independent of the length s for a given α. In contrast to this, the second partial (2) varies inversely with x for a given angle α, being of dimensions L^{-1}. Thus (1) is independent of how long s is, whereas (2) depends on how long s is, for a given angle of inclination α. In case we wish to consider only the rates of change of s_1, the length of s corresponding to unit magnitude of x, *i. e.*, the part of s whose projection on OX is unity as

indicated by fig. 1, then we are selecting a particular case of (2) so chosen that (x, y) will lie on the intersection of the lines $x=1$ and $y=\tan\alpha$ (the line value of $\tan\alpha$). This particular case of (2) is obtained by setting x equal to unity above, whereas in (1) the setting of x equal to unity makes no difference because this first partial remains unchanged for different x's and a given angle α.

This last situation is the one that presents itself when one attempts to differentiate line integrals in the calculus of variations.

§ 2. *The Partial Derivatives of the Length of a Straight Line with Fixed End Points by the Parallel Displacement Method.*

It might seem from the above title that the partial or any other derivatives of a straight line with fixed end points must necessarily be zero. The procedure here, however, involves deflecting the line from rectilinearity by parallel displacements of its path into the broken line segments AQ, QR, RM, ML, LB, as shown in fig. 2. Differentiation is then taken with respect to h and k, the parallel displacements, and then we set $h=k=0$, whereby the broken line becomes straight once more. Just as ordinarily the derivative of y at a place (x, y) on a curve is the rate of change of y in passing through that point, so here the partial derivatives of the length of a straight line are really the rates of change of the broken line's length at the position $h=k=0$, where the broken line is rectilinear.

We consider the length of an initially straight line from a fixed (x_0, y_0) to a fixed (a, b), and made up of separate segments summed

$$s=\sum_{i=1}^{m}\sqrt{\Delta x_i+\Delta y_i}=\Sigma\Delta s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

taken as a function of m pairs of variables $(\Delta x_i, \Delta y_i)$ used to define any (x_i, y_i) on the straight line from one end point to the other:

$$x_i=x_0+\sum_{j=1}^{i}\Delta x_j; \quad y_i=y_0+\sum_{j=1}^{i}\Delta y_j. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the constant upper limit (a, b) we have $i=m$ in (4), and hence

$$a=x_0+\sum_{j=1}^{m}\Delta x_j; \quad b=y_0+\sum_{j=1}^{m}\Delta y_j. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

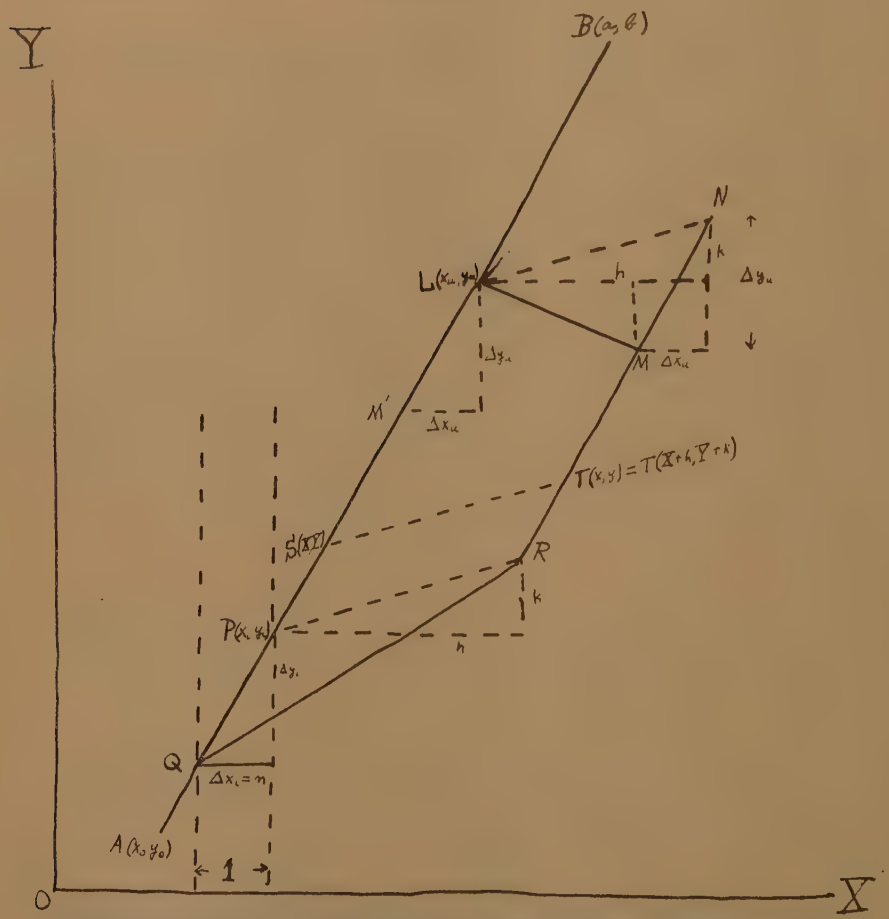
Eqs. (5) put a single restriction on the $m\Delta x$'s and another on the $m\Delta y$'s. Hence there are only $m-1$ independent variables in each set. We follow the same practice previously used in parallel displacement differentiation [2], *i. e.*, we let one pair of variables $(\Delta x_i, \Delta y_i)$ take on independent increments (h, k), and we further permit one and only one other pair to be dependent variables $(\Delta x_u, \Delta y_u)$, shown in fig. 2 as of higher index $(u>i)$ than the

first pair, and having increments $(-h, -k)$ because of their dependence on $(\Delta x_i, \Delta y_i)$:

$$\left.\begin{aligned}\delta\Delta x_i &= h = -\delta\Delta x_u \\ \delta\Delta y_i &= k = -\delta\Delta y_u\end{aligned}\right\}; \quad \text{and hence} \quad \frac{\partial\Delta x_u}{\partial\Delta x_i} = \frac{\partial\Delta y_u}{\partial\Delta y_i} = -1. \quad . \quad . \quad (6)$$

These *independent* increments (h, k) given to $(\Delta x_i, \Delta y_i)$ will, of course, change the sum (3) from a rectilinear length to that of a broken line

Fig. 2.



The broken line AQ, QR, RM, ML, LB, which is the varied length of the straight line AB under parallel displacements h and k.

such as given when the straight section QL is replaced by the three sections QR, RM, and ML. Note especially that the parallel displaced section RM is equal and parallel to its original $\mathrm{PM}' = \sum_{j=i+1}^{u-1} \Delta s_j$, being the sum of the same unchanged original interval variables included *between* $(\Delta x_i, \Delta y_i)$ and $(\Delta x_u, \Delta y_u)$. These last pairs, however, take on variations

that account *completely* for the changes in (3) whereby PQ changes to QR and M'L to LM :

$$\left.\begin{aligned} \Delta s_u(h,\,k) &= \sqrt{(\Delta x_u - h)^2 + (\Delta y_u - k)^2} = \mathrm{LM}, \\ \Delta s_i(h,\,k) &= \sqrt{(\Delta x_i + h)^2 + (\Delta y_i + k)^2} = \mathrm{QR}. \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

Using (7) we might write the varied form of (3), which now is a broken line in five sections,

$$s(h,\,k) = \sum_{j=1}^{i-1} \Delta s_j + \Delta s_i(h,\,k) + \sum_{j=i+1}^{u-1} \Delta s_j + \Delta s_u(h,\,k) + \sum_{j=u+1}^{m} \Delta s_j, \quad . \quad . \quad (8)$$

where the middle term on the right is evidently the unchanged length PM', displaced to its new parallel position RM shown in fig. 2.

Consider (X, Y) for the point S, where capital letters are taken to refer to coordinates of any point on the original straight section PL, and then $(x,\,y) = (\mathrm{X}+h,\ \mathrm{Y}+k)$ will be the coordinates of its displaced position T on the parallel displaced section RN. Any two such points on the section PL and RN will determine h and k under *parallel displacements*, and hence we may refer to (8), being thus obtained as a variant of (3) by parallel displacements, as a *two point* function determined by S at (X, Y) and T at $(x,\,y)$, since these are enough to determine $(h,\,k)$ and hence (8), in the situation here considered, *i. e.*, once the end points $(x_i y_i)$ and $(x_u y_u)$ of the parallel displacement and the original line are chosen. Thus we may indicate the functional dependence of (8) :

$$\left.\begin{aligned} &s(h,\,k) = s(x-\mathrm{X},\ y-\mathrm{Y}), \\ &\left.\begin{aligned} x-\mathrm{X} = h = \delta\Delta x_i = -\delta\Delta x_u, \\ y-\mathrm{Y} = k = \delta\Delta y_i = -\delta\Delta y_u, \end{aligned}\right\} \\ &\qquad \text{and with (X, Y) fixed} \left\{\begin{aligned} \delta x &= \delta h \\ \delta y &= \delta k \end{aligned}\right\}; \end{aligned}\right\} \quad . \quad . \quad (9)$$

and we note how the increments $(h,\,k)$ of the independent variables $(x,\,y)$ of T are the same (with suitable sign) as the increments of the interval variables Δx and Δy at the limits of the parallel displacement as given by (6).

If we take the partials of (8) with respect to h and k keeping (X, Y) fixed * we get also the partials with respect to x and y according to (9)

$$\left.\begin{aligned} \frac{\partial s_i^u}{\partial x}(x-\mathrm{X},\ y-\mathrm{Y}) &= \frac{\partial s_i^u}{\partial h}(h,\,k) = \frac{\Delta x_i + h}{\sqrt{(\Delta x_i + h)^2 + (\Delta y_i + k)^2}} \\ &\qquad - \frac{\Delta x_u - h}{\sqrt{(\Delta x_u - h)^2 + (\Delta y_u - k)^2}}, \\ \frac{\partial s_i^u}{\partial y}(x-\mathrm{X},\ y-\mathrm{Y}) &= \frac{\partial s_i^u}{\partial k}(h,\,k) = \frac{\Delta y_i + k}{\sqrt{(\Delta x_i + h)^2 + (\Delta y_i + k)^2}} \\ &\qquad - \frac{\Delta y_u - k}{\sqrt{(\Delta x_u - h)^2 + (\Delta y_u - k)^2}}. \end{aligned}\right\} \quad . \quad (10)$$

* So as to make use of (9) and obtain partials with respect to x and y.

And again the second partials are

$$\left.\begin{aligned}
\frac{\partial^2 s_i^u}{\partial x^2}(x-X,\, y-Y) &= \frac{\partial^2 s_i^u}{\partial h^2}(h,\, k) = \frac{(\Delta y_i + k)^2}{((\Delta x_i + h)^2 + (\Delta y_i + k)^2)^{3/2}} \\
&\quad + \frac{(\Delta y_u - k)^2}{((\Delta x_u - h)^2 + (\Delta y_u - k)^2)^{3/2}}, \\
\frac{\partial^2 s_i^u}{\partial x\, \partial y}(x-X,\, y-Y) &= \frac{\partial^2 s_i^u}{\partial h\, \partial k}(h,\, k) = -\frac{(\Delta x_i + h)(\Delta y_i + k)}{((\Delta x_i + h)^2 + (\Delta y_i + k)^2)^{3/2}} \\
&\quad - \frac{(\Delta x_u - h)(\Delta y_u - k)}{((\Delta x_u - h)^2 + (\Delta y_u - k)^2)^{3/2}}, \\
\frac{\partial^2 s_i^u}{\partial y^2}(x-X,\, y-Y) &= \frac{\partial^2 s_i^u}{\partial k^2}(h,\, k) = \frac{(\Delta x_i + h)^2}{((\Delta x_i + h)^2 + (\Delta y_i + k)^2)^{3/2}} \\
&\quad + \frac{(\Delta x_u - h)^2}{((\Delta x_u - h)^2 + (\Delta y_u - k)^2)^{3/2}}.
\end{aligned}\right\} \quad (11)$$

Now set $h=k=0$; then (10) and (11) become

$$\left.\begin{aligned}
\frac{\partial s_i^u}{\partial x} &= \frac{\partial s_i^u}{\partial h}(0,\, 0) = \frac{\Delta x_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}} - \frac{\Delta x_u}{\sqrt{\Delta x_u^2 + \Delta y_u^2}} \\
&= \frac{1}{\sqrt{1+p^2}} - \frac{1}{\sqrt{1+p^2}} = \cos\alpha - \cos\alpha, \\
\frac{\partial s_i^u}{\partial y} &= \frac{\partial s_i^u}{\partial k}(0,\, 0) = \frac{\Delta y_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}} - \frac{\Delta y_u}{\sqrt{\Delta x_u^2 + \Delta y_u^2}} \\
&= \frac{p}{\sqrt{1+p^2}} - \frac{p}{\sqrt{1+p^2}} = \sin\alpha - \sin\alpha.
\end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned}
\frac{\partial^2 s_i^u}{\partial x^2} &= \frac{\partial^2 s_i^u}{\partial h^2}(0,\, 0) = \frac{\Delta y_i^2}{(\Delta x_i^2 + \Delta y_i^2)^{3/2}} + \frac{\Delta y_u^2}{(\Delta x_u^2 + \Delta y_u^2)^{3/2}} \\
&= \frac{p^2}{\Delta x_i (1+p^2)^{3/2}} + \frac{p^2}{\Delta x_u (1+p^2)^{3/2}}, \\
\frac{\partial^2 s_i^u}{\partial x\, \partial y} &= \frac{\partial^2 s_i^u}{\partial h\, \partial k}(0\ \ 0) = -\frac{\Delta x_i \Delta y_i}{(\Delta x_i^2 + \Delta y_i^2)^{3/2}} - \frac{\Delta x_u \Delta y_u}{(\Delta x_u^2 + \Delta y_u^2)^{3/2}} \\
&= -\frac{p}{\Delta x_i (1+p^2)^{3/2}} - \frac{p}{\Delta x_u (1+p^2)^{3/2}}, \\
\frac{\partial^2 s_i^u}{\partial y^2} &= \frac{\partial^2 s_i^u}{\partial k^2}(0,\, 0) = \frac{\Delta x_i^2}{(\Delta x_i^2 + \Delta y_i^2)^{3/2}} + \frac{\Delta x_u^2}{(\Delta x_u^2 + \Delta y_u^2)^{3/2}} \\
&= \frac{1}{\Delta x_i (1+p^2)^{3/2}} + \frac{1}{\Delta x_u (1+p^2)^{3/2}}.
\end{aligned}\right\} \quad (13)$$

Here on the right we have introduced p for the slope of this straight line :

$$p=\frac{\Delta y_i}{\Delta x_i}=\frac{\Delta y_u}{\Delta x_u}=\ldots \quad =\tan\alpha.(h=k=0). \quad (14)$$

We employ p in this article for the slope of a straight line, reserving y' to represent the slope of a curve as in Part II.

The terms of (12) are the same as those in (1) above and are independent of how long Δs_i and Δs_u are. By contrast, the various terms of (13) are like those of (2) in that for a given slope $p=\tan\alpha$ they are *not* uniquely determined but depend on the lengths of the interval variables, being inversely proportional to Δx_i and Δx_u. A unique expression for the second and higher partials is obtained if we set Δx_i and Δx_u equal to one unit in the x direction, as we did with x in (2). This means that we are putting a restriction on the partition intervals along the x-axis such that at P and L in fig. 2 Δs shall be included between the vertical parallel lines unit distance apart, *i. e.*, the projection of Δs on the x axis becomes unity at P and L and Δs becomes in this particular way the change in s per (or corresponding to) *unit* change in x, since its projection, Δx, is unity. Then if we set $\Delta x=n=1$ unit, and $\Delta y=np$, eqs. (7) take the form

$$\mathrm{QR}=\sqrt{(n+h)^2+(np+k)^2}\,; \quad \mathrm{ML}=\sqrt{(n-h)^2+(np-k)^2}, \quad . \quad (7\,a)$$

and similar replacements could be introduced in all of (10) to (13). Then the right members of (12) would be the same as shown, being like (1) independent of the lengths of (7), but (13) by contrast is like (2) and depends on Δx, which, when we set equal to $n=1$ unit, gives

$$\left.\begin{aligned}
\frac{\partial^2 s_i^u}{\partial x^2}&=\frac{\partial^2 s_i^u}{\partial h^2}(0,0)=\frac{p^2}{n(1+p^2)^{3/2}}+\frac{p^2}{n(1+p^2)^{3/2}}=\frac{2p^2}{n(1+p^2)^{3/2}},\\
\frac{\partial^2 s_i^u}{\partial x\,\partial y}&=\frac{\partial^2 s_i^u}{\partial h\,\partial k}(0,0)=-\frac{p}{n(1+p^2)^{3/2}}-\frac{p}{n(1+p^2)^{3/2}}=-\frac{2p}{n(1+p^2)^{3/2}},\\
\frac{\partial^2 s_i^u}{\partial y^2}&=\frac{\partial^2 s_i^u}{\partial k^2}(0,0)=\frac{1}{n(1+p^2)^{3/2}}+\frac{1}{n(1+p^2)^{3/2}}=\frac{2}{n(1+p^2)^{3/2}}.
\end{aligned}\right\}. \quad (15)$$

Since it is a kind of norm, we have used n to represent this unit interval in the x direction, and since it is dimensionally a length, ordinarily, this symbol is retained in the formulas to give dimensional balance, where p is of zero dimensions. Eqs. (15) would be wrong dimensionally without n, which can not generally be set equal to a pure number 1, though equal to it in magnitude, but must be retained in (15) to indicate 1 cm., 1 yard, 1 pound, or whatever physical or geometrical units are measured along the x axis, being a mathematical symbol for any positive unit. Although the form of (15) is unchanged by change of units, the essential presence of n reveals the fundamental distinction between these new derivatives

in this article and the classical ones, in that these here are *not* physically and geometrically invariant under change of units as classical * derivatives are, being like (2) in this respect. Thus with $n=1$ foot (15) are three times their value for $n=1$ yard. See (43) to (46) below.

§ 3. *Why are the new Derivatives Functions of the Unit n, whereas Classical Derivatives are Independent of Units? Answer: The Actual Magnitudes of the Dependent Variables (the Problem itself) depends on Units.*

The rôle of the unit n which must appear explicitly in (15) if the right members are to express at all completely their actual dependence on units is the central theme of the present article. We may accordingly note at this point how this actual analytical dependence of (15) on the units appears as a genuine physical and geometrical dependence as soon as we introduce a corresponding physical or geometrical situation into the analytic equations. And therein lies the contrast between the new derivatives in this article such as (15) and the familiar classical derivatives which are all physically and geometrically independent of units.

Suppose, for example, that Δs, extending from P to Q in fig. 2, was 5 feet long, and, as shown in fig. 3, had a slope $p=\tan\alpha=\frac{4}{3}$. Then the corresponding terms in (12) and (13) will be given by

$$\left.\begin{aligned} \frac{\partial \Delta s}{\partial x} &= \frac{1}{\sqrt{1+p^2}} = \cos\alpha = \frac{3}{5} \text{ of a foot per foot,} \\ \frac{\partial^2 \Delta s}{\partial x^2} &= \frac{p^2}{\Delta x(1+p^2)^{3/2}} = \frac{16}{125} \text{ of a foot per foot per foot.} \quad (\Delta x=3). \end{aligned}\right\} \quad (16)$$

If we change the units from feet to yards the *same* triangle QPW in fig. 3 becomes a unit base triangle in the new units, and the above partial derivatives of its hypotenuse are

$$\left.\begin{aligned} \frac{\partial \Delta s}{\partial x} &= \frac{1}{\sqrt{1+p^2}} = \cos\alpha = \frac{3}{5} \text{ of a yard per yard.} \\ \frac{\partial^2 \Delta s}{\partial x^2} &= \frac{p^2}{\Delta x(1+p^2)^{3/2}} = \frac{48}{125} \text{ of a yard per yard per yard.} \quad (\Delta x=1). \end{aligned}\right\} \quad (17)$$

The second partials (17) (as well as the first) represent the same physical rate of change as (16), using yards as well as using feet, since obviously the change per yard in the rate 3/5 ft./ft.=3/5 yd./yd. will be three times

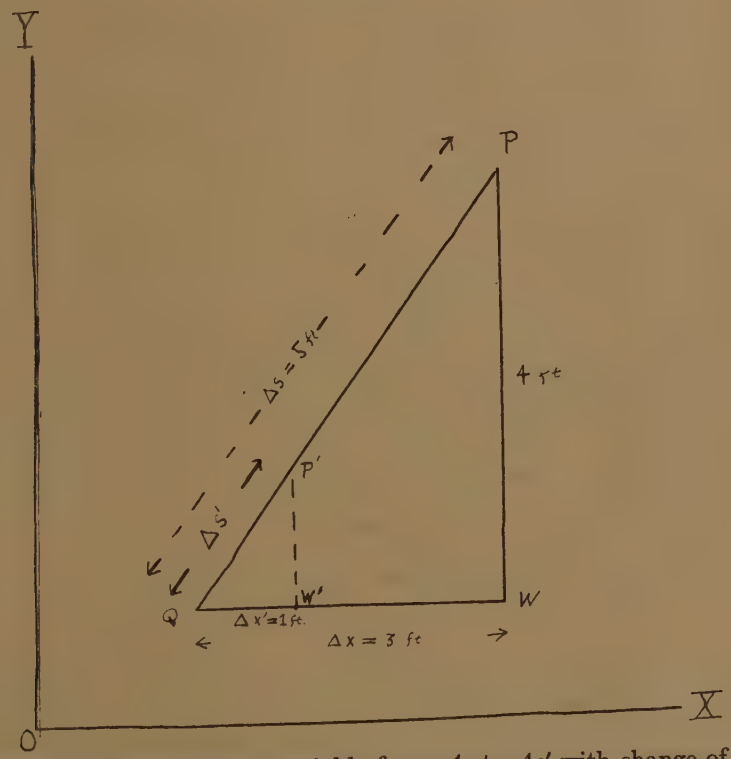
---

* Thus the slope is 6 at (3, 9) on $y=x^2$, using feet, or at (1, 3) on $y=3x^2$ using yards (the same point). Likewise, the second derivatives are geometrically the same, *i. e.*, 2 units per foot, or its equal, 6 units per yard.

the change it takes on per foot. Thus the physical and geometrical equality of the second partials above illustrates the corresponding independence of units in classical differentiation, which serves to furnish engineering with unique and correct second and higher rates of change regardless of units.

Suppose now we consider $\Delta s'$, the hypotenuse of the smaller triangle QP'W' in fig. 3 with base 1 ft. The first partials will be the same as

Fig. 3.



The change of the dependent variable from Δs to $\Delta s'$ with change of units.

before, but the second partials, using first feet and then yards for units as above, will give

$$\frac{\partial^2 \Delta s'}{\partial x^2} = \frac{p^2}{\Delta x(1+p^2)^{3/2}} = \frac{48}{125}$$

of a foot per foot per foot = 144/125 yards per yard per yard. . . (18)

Since for both units we are differentiating the same hypotenuse QP', the derivatives are necessarily the same thing physically, a conclusion

that is seen to be checked numerically. Note further that in (18) the derivatives are three times the values of the second partials of (16) and (17) for the larger triangle, as, of course, they would be according to (2).

If we introduce a constant linear density $\rho = 1\text{♯/ft.}$, or one lb. per foot, then the hypotenuse QP of fig. 3 becomes $\Delta M = \rho \Delta s = 5\text{♯} = 5$ lbs., and (16) become

$$\left.\begin{aligned} \frac{\partial \Delta M}{\partial x} &= \frac{\rho}{\sqrt{1+p^2}} = \frac{3}{5}\text{♯/ft.} = \frac{3}{5}\text{ of a pound per foot,} \\ \frac{\partial^2 \Delta M}{\partial x^2} &= \frac{\rho\, p^2}{\Delta x (1+p^2)^{3/2}} = \frac{16}{125}\text{♯/ft.}^2 = \frac{16}{125}\text{ of a pound per foot per foot.} \end{aligned}\right\} \quad . \; . \; . \quad (19)$$

For the same mass of the length QP, using yards, the density will be $\rho = 3\text{♯/yd.}$, so that instead of (17) we have

$$\left.\begin{aligned} \frac{\partial \Delta M}{\partial x} &= \frac{\rho}{\sqrt{1+p^2}} = \frac{9}{5}\text{♯/yd.} = \frac{9}{5}\text{ lbs. per yard,} \\ \frac{\partial^2 \Delta M}{\partial x^2} &= \frac{\rho\, p^2}{\Delta x (1+p^2)^{3/2}} = \frac{144}{125}\text{♯/yd.}^2 = \frac{144}{125}\text{ lbs. per yard per yard.} \end{aligned}\right\} . \quad (20)$$

Note again that (20) are precisely the same physical rates of change as (19), just as (16) and (17) were the same, each being a physical and geometrical equality that is not to be obscured by the change of units.

But again, if we consider $\Delta M' = \rho \Delta s'$ in the small triangle QP′W′ of fig. 3, the first derivatives are the same as those of (19) and (20), but the second partials, using $\rho = 1\text{♯}$ with feet and $\rho = 3\text{♯}$ with yards, and $\Delta x' = 1$ ft. $= 1/3$ yd., give

$$\frac{\partial^2 \Delta M'}{\partial x^2} = \frac{\rho\, p^2}{\Delta x' (1+p^2)^{3/2}} = \frac{48}{125}\text{♯/ft.}^2 = \frac{432}{125}\text{♯/yd.}^2. \quad . \; . \; (21$$

As with (16) to (20), change of units does not interfere with the physical equality of the two expressions for derivatives in (21), but these are seen to be three times the second partials in (19) and (20) because of the short length and the smaller mass $\Delta M' = \rho \Delta s'$, according to (2).

Now the central idea of the present article is that we deliberately *select* the only second partials above that refer to *unit* base triangles, *i. e.*, those using yards with the big triangle and those using feet with the small triangle. We thus make the *size* of the triangle a function of the units, in that we consider only those triangles with unit base, and with choice of ΔM and Δs determined in the same way. Differentiation of ΔM or $\Delta s = \text{QP}$ is a *different* problem from differentiation of $\Delta M'$ or $\Delta s' = \text{QP}'$. Thus the selection of our problem is controlled by the units chosen, the guide being in case of (15) and fig. 3 to choose a Δs having

unit projection on the x axis, and accordingly (17) and (20) are selected when yards are the units, whereas (18) and (21) are selected when feet are the units.

This choice of unity for the value of Δx is *not* arbitrary, but is forced upon us by the concept of a rate of change, as the change per unit interval in the independent variable. The crucial point is that Δx, Δy, and Δs are finite variables like the x, y, and s of (1) and (2), and are not the differentials concerned in the differentiation process of (10) to (13). These latter are the differentials h and k given by (6) and (9) or their increments. On the other hand, Δx, Δy, and Δs are related finite interval variables whose finite size is quite unconcerned with the limiting processes of (10) to (13), and hence we are required to set $\Delta x = n$, the unit interval, if we want the Δy and Δs to be changes per *unit* change in x, and give a *rate* of *change* in that sense. Likewise $p = \tan\alpha = \dfrac{\Delta y}{\Delta x}$ is a rate or slope by virtue of the set-up which antedates and is entirely distinct from the derivative limiting processes (10) to (13) and the differentials functioning as such in these formulas.

By introducing $\Delta x_i = \Delta x_u = n$ we are specifying that the derivatives (15) or (12) are determined by interval variables Δs such as those included between the vertical parallel lines unit distance apart such as shown in fig. 2, and the Δy's also determined in a corresponding way, and any change of units involves a corresponding change of Δs and Δy, whereby the problem itself is changed as with fig. 3 due to changes in these dependent variables. Thus the derivatives (15) are functions of the variable unit n.

In the foregoing treatment we have taken the finite h and k to be the increments of $(\Delta x_i, \Delta y_i)$ and $(\Delta x_u, \Delta y_u)$, and these increments bore complete responsibility for the independent variations of x and y. The above procedure, however, amounts to nothing more than successive differentiation with respect to $(\Delta x_i, \Delta y_i)$ and $(\Delta x_u, \Delta y_u)$, and taking account of the signs according to (9), and the further fact that such differentiation is the same (according to (9)) as differentiation with respect to (x, y) coordinates of any point, T, on the parallel displaced length from P to L, using (x, y) as independent variables, being the same method used previously [2] in differentiation by parallel displacement.

§ 4. *Units.*

Since the rôle of the unit n is fundamental in this paper, and appears in the new definition of a derivative below, we proceed to examine its nature in more detail.

It is well known that it takes the product of two factors to specify

a distance, one of which is a numerical multiplier and the other a unit. Thus for the distance of one mile we have

$$D=5280n_1=1760n_2=320n_3=1n_4=xn, \quad . \quad . \quad . \quad . \quad (22)$$

where n_1, n_2, n_3, and n_4 represent unit foot, yard, rod, and mile. The present treatment requires a symbol, n, that will represent all of these units as well as any other units. Hence we may note that the introduction of this generalized unit n is like the step from arithmetic to algebra. Just as x represents a set of numbers, so this n here represents a set of units, and we see how n as well as x is a "variable" or mathematical symbol that takes on a set of values.

Analysis ordinarily neglects units, which are usually not represented, because a given unit multiplied by all variables in a problem often leaves the result unaffected. But to be explicit (2) should use (22) for x, and be written $\frac{\partial^2 s}{\partial x^2}=\frac{p^2}{xn(1+p^2)^{3/2}}$, which, in turn, if we are unrestricted in choice of units, can be written like a term of (15), $\frac{\partial^2 s}{\partial x^2}=\frac{p^2}{n(1+p^2)^{3/2}}$, which expresses the same result as (2), and with equal justification, because the distance $xn=D$ can be given by a single properly chosen unit n just as well as by a properly chosen x, as a multiplier for some arbitrary unseen unit; and this n is the algebraic symbol that replaces x in (2) and Δx in (13) in the rôle of the variable that may take on the set of values usable for units.

§ 5. *The First Partial Derivatives of the Slope of a Straight Line at the Upper and Lower Limits of a Parallel Displacement. How the Mathematical Definition of a Derivative fails and the Physicist's prevails.*

The slopes of LM and QR given by (7) are shown in fig. 2 as the varied slopes at the limits * of a finite parallel displacement :

$$\frac{\Delta y_u-k}{\Delta x_u-h}\equiv p+\frac{hp-k}{\Delta x_u-h}=p+\frac{hp-k}{n\left(1-\frac{h}{n}\right)} \quad \text{(upper limit)}, \quad . \quad . \quad (23)$$

$$\frac{\Delta y_i+k}{\Delta x_i+h}\equiv p-\frac{hp-k}{\Delta x_i+h}=p-\frac{hp-k}{n\left(1+\frac{h}{n}\right)} \quad \text{(lower limit)}. \quad . \quad . \quad (24)$$

Here we have set $\Delta x_i=\Delta x_u=n$ for precisely the same reason as with (15), *i. e.*, because we want Δy which depends on Δx to be the change

* The upper and lower ends of the displaced segment in fig. 2 become integration limits when (3) and (30) are replaced by integrals like (36) and (46).

in y per unit change in x, just as before we wanted Δs and Δy to be the changes corresponding to unit change in x. From the above we have for the slope increments at the limits of a parallel displacement

$$\Delta p(h,\ k)=\pm\frac{(hp-k)}{n\left(1\mp\frac{h}{n}\right)}, \qquad \ldots\ldots\ (25)$$

where the upper and lower signs refer to the upper and lower limits of a parallel displacement of amount $(h,\ k)$. Since (9) holds for (23) and (24) as well as for (8), we may take the partials of (23) and (24) with respect to h and k with the understanding that these give, as before, the partials with respect to x and y for any point $(x,\ y)=(\mathrm{X}+h,\ \mathrm{Y}+k)$ on the displaced path (line segment) between the limits of the parallel displacement (see fig. 2).

$$\left.\begin{aligned}\frac{\partial p}{\partial x}(x-\mathrm{X},\ y-\mathrm{Y})&=\frac{\partial p}{\partial h}(h,\ k)=\pm\frac{\Delta y\mp k}{(\Delta x\mp h)^2}=\pm\frac{1}{n\left(1\mp\frac{h}{n}\right)}\left(p\pm\frac{hp-k}{n\left(1\mp\frac{h}{n}\right)}\right),\\ \frac{\partial p}{\partial y}(x-\mathrm{X},\ y-\mathrm{Y})&=\frac{\partial p}{\partial k}(h,\ k)=\mp\frac{1}{(\Delta x\mp h)}=\mp\frac{1}{n\left(1\mp\frac{h}{n}\right)},\end{aligned}\right\} \qquad \ldots\ (26)$$

where, again, the upper and lower signs refer to the upper and lower limits of the parallel displacement, being the derivatives of (23) or (24) respectively.

When we set $h=k=0$ (26) become

$$\left.\begin{aligned}p_x&=\frac{\partial p}{\partial h}(0,\ 0)=\pm\frac{\Delta y}{\Delta x^2}=\pm\frac{p}{n}\\ p_y&=\frac{\partial p}{\partial k}(0,\ 0)=\mp\frac{1}{\Delta x}=\mp\frac{1}{n}\end{aligned}\right\}\quad p=\frac{\Delta y}{\Delta x},\quad \Delta x=n, \qquad \ldots\ (27)$$

giving the partial derivatives of the slope of a straight line such as shown in fig. 2 at the upper and lower limits of a parallel displacement, and taken with respect to x and y for any point on the line between these limits (since $h=k=0$).

Comparison of the middle members of (23) to (27), which are explicit in Δx, with the right members, where $\Delta x=n$, shows a marked advantage of the physicist's definition of a derivative over the mathematician's. Thus by mathematics (14) gives $p=\dfrac{\Delta y}{\Delta x}=\operatorname{limit}_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ for the derivative, the limit being equal to the ratio $\dfrac{\Delta y}{\Delta x}$ of the finite increments since we deal with a straight line. Thus (23) and (24), using an indeterminate

Δx, give the variants of a true derivative by mathematics, and the middle members of (26) and (27) give the partial derivatives of a true derivative by mathematics, even though these are not uniquely defined, since Δx may vary arbitrarily. On the other hand, the physicist expects a derivative of a derivative to be a rate of change of a rate, and this points immediately to the right members of (23) to (27), using $\Delta x=n$ as the only correct forms usable as variants of a rate or rates of change of a rate. Thus, using C.G.S. units, the middle members of (27) are the changes per cm. (in the limiting calculus sense) in the change Δy per (or divided by) change Δx in the abscissa (not unity in general); whereas to be the rate of change of a rate we must have changes per cm. (in the derivative sense) in the change Δy corresponding to *unit* abscissa interval or changes per cm.^{2}, requiring $\Delta x=n$ as in the right members of (23) to (27).

If we use (27) to form the total derivative of p in the direction of the path or straight line we have

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=p_x+pp_y=\pm\left(\frac{p}{n}-\frac{p}{n}\right)=0, \quad . \quad . \quad . \quad . \quad (28)$$

and for the total differential, using (9),

$$\delta p=p_x\delta x+p_y\delta y=\pm\frac{1}{n}(p\delta x-\delta y)=\pm\frac{1}{n}(hp-k), \quad . \quad . \quad . \quad (29)$$

which is the same as (25) to first order terms in h and k.

We may use (14), and write (3) in the form

$$s=\sum_{i=1}^{m}\sqrt{1+p^2}\,\Delta x_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and differentiate, treating $p=\frac{\Delta y_i}{\Delta x_i}$ and $p=\frac{\Delta y_u}{\Delta x_u}$ at the limits of a parallel displacement as functions of x and y according to (27), and noting according to (9) that

$$\left.\begin{aligned}\frac{\partial\Delta x_u}{\partial x}=\frac{\partial\Delta y_u}{\partial y}&=-1 \quad \text{(upper limit)},\\ \frac{\partial\Delta x_i}{\partial x}=\frac{\partial\Delta y_i}{\partial y}&=1 \quad \text{(lower limit)}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Then the first partials of (30), $\frac{\partial s_i^u}{\partial x}$ and $\frac{\partial s_i^u}{\partial y}$, will be the same as (12), using $\Delta x_i=\Delta x_u=n$, and the second partials will be given by (15), thus demonstrating how (30) can be treated under parallel displacement differentiation as an implicit function of x and y for any point on the line PL of fig. 2 between the chosen limits of the parallel displacement.

§ 6. *Higher Partial Derivatives of the Slope of a Straight Line at the Limits of a Parallel Displacement.*

Like the second partials (2), (11), and (13) above, we might take third and higher partial derivatives of dimensions L^{-2} and higher negative powers of L, which, like (11) and (13), would vary for a given $p=\frac{\Delta y}{\Delta x}=\tan\alpha$ inversely now as the second and higher powers of Δx, but would be uniquely determined in case we set $\Delta x=n$, as formerly.

Here, again, this choice of unity for Δx would operate automatically to determine values of the dependent Δs and Δy (see fig. 2) corresponding to unit change in the abscissa interval $\Delta x=n$, whereby Δs and Δy are made to serve as dependent variables in a *rate* of change.

Likewise when we take higher partials of (23), (24), (26), and (27) we are obtaining true partials of a derivative according to mathematics $\left(\underset{\Delta x\to 0}{\text{limit}}\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta x}\right)$, even when we leave Δx undetermined, but these will not be the corresponding rates of change of a true rate as required by physics unless we set $\Delta x=n=$unit abscissa.

Thus, differentiating (26) and (27) with respect to h and k, we have

$$\left.\begin{aligned}
&\frac{\partial^2 p}{\partial x^2}(x-\mathrm{X},\ y-\mathrm{Y})=\frac{\partial^2 p}{\partial h^2}=2\frac{\Delta y\mp k}{(\Delta x\mp h)3}=\frac{2}{n^2\left(1\mp\frac{h}{n}\right)^2}\left(p\pm\frac{hp-k}{n\left(1\mp\frac{h}{n}\right)}\right),\\
&\frac{\partial^2 p}{\partial x\,\partial y}(x-\mathrm{X},\ y-\mathrm{Y})=\frac{\partial^2 p}{\partial h\,\partial k}=-\frac{1}{(\Delta x\mp h)^2}=-\frac{1}{n^2\left(1\mp\frac{h}{n}\right)^2};\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\frac{\partial^2 p}{\partial k^2}=\frac{\partial^2 p}{\partial y^2}=0,\\
&\frac{\partial^m p}{\partial x^m}=\frac{\partial^m p}{\partial h^m}=(\pm 1)^m\frac{m!}{n^m\left(1\mp\frac{h}{n}\right)^m}\left(p\pm\frac{hp-k}{n\left(1\mp\frac{h}{n}\right)}\right),\\
&\frac{\partial^m p}{\partial x^{m-1}\partial y}=\frac{\partial^m p}{\partial h^{m-1}\partial k}=-(\pm 1)^m\frac{(m-1)!}{n^m\left(1\mp\frac{h}{n}\right)^m};\quad\frac{\partial^m p}{\partial y^m}=\frac{\partial^m p}{\partial k^m}=0,\quad(m>1).
\end{aligned}\right\}\quad\ldots\quad(32)$$

These are all partial derivatives of the slopes of the lines LM and QR shown in fig. 2 and having length (7), and are taken with respect to x and y for any point in the parallel displaced line segment RN, being the same as the partials with respect to h and k according to (9).

The important special case comes from setting $h=k=0$, whereby we get the partial derivatives of the slope of a straight line at the limits

of a parallel displacement, taken with respect to (x, y) for any point on the line between these limits :

$$\left.\begin{aligned} p_x &= \frac{\partial p}{\partial h}(0,\,0) = \pm\frac{p}{n}, \\ p_{xx} &= \frac{\partial^2 p}{\partial h^2}(0,\,0) = \frac{2p}{n^2}, \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial^m p}{\partial x^m} &= \frac{\partial^m p}{\partial h^m}(0,\,0) = (\pm 1)^m \frac{m!}{n^m} p. \end{aligned}\right\} \quad \begin{aligned} &\Delta x = n = 1 \text{ unit}, \\ &p = \frac{\Delta p}{\Delta x} \end{aligned} \qquad \cdots\cdots \quad (33)$$

$$\left.\begin{aligned} p_y &= \frac{\partial p}{\partial k}(0,\,0) = \mp\frac{1}{n}, \\ p_{xy} &= \frac{\partial^2 p}{\partial h\,\partial k}(0,\,0) = -\frac{1}{n^2}, \\ \frac{\partial^m p}{\partial x^{m-1}\partial y} &= \frac{\partial^m p}{\partial h^{m-1}\partial k}(0,\,0) = -(\pm 1)^m \frac{(m-1)!}{n^m}, \\ \frac{\partial^m p}{\partial y^m} &= \frac{\partial^m p}{\partial k^m} = 0, \quad m > 1. \end{aligned}\right\} \quad \Delta x = n = 1 \text{ unit}. \qquad (34)$$

Here the upper and lower signs refer to upper and lower limits of a parallel displacement, and we note that even order partials of p have the same sign at each limit and vice versa with odd order derivatives. The minus sign introduced by (9) each time we differentiate p at the upper limit results in all orders of upper limit derivatives having the same sign, being positive for (33) and negative for (34). Note also that the dimensions of these derivatives is L^{-m} where m is the order, and that this mth order derivative has m for the power of n in its denominator. The basic difference between (27), (33), and (34) and the classical derivatives is that these new ones are functions of the unit n, whereas the classical derivatives are not, just as (15) depend on the unit magnitude whereas (1) and (12) do not, as illustrated by (49) to (51) below.

§ 7. *A new Definition for a Derivative including sufficient as well as necessary Requirements.*

In the past the physicist's and the mathematician's definitions of a derivative have been found equivalent, but the method of partial differentiation by parallel path displacements points to a distinction and decides in favour of the former.

It is a kind of trigonometric accident, as it were, for this straight line case (likewise when we use differentials for interval variables between points along the *straight* tangent line to a *curve*) that $p = \tan\alpha = \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{limit}} \frac{\Delta y}{\Delta x}$ is independent of the separate sizes of Δx and Δy (and likewise with the

zero dimensional expressions like (1), (10), and (12) above), and this " accident " is tied up with the equivalence between the above definition of a derivative and its definition as a rate of change. We now see that under parallel path displacements this " accident " no longer obtains, and we come to the parting of the ways, and are obliged to leave by the wayside the mathematician's definition of a derivative which has fallen down, and proceed according to the physicist's, on which rests the superstructure of applied science, and whose position is thus impregnable.

This breakdown of the mathematical definition of a derivative involves the distinction between *necessary* and *sufficient* requirements upon a ratio for it to be the derivative. The mathematician's definition is necessary but not sufficient to provide that (23) and (24) shall be variations of a *rate* p and uniquely determined by the finite parallel displacements h and k, and likewise with the partial derivatives of p given by (26), (27), and (32) to (34). It seems due solely to the above trigonometric accident that the mathematical definition of the derivative has hitherto escaped challenge, and its exact behaviour and aspect as a limit has somewhat obscured its essential nature and importance as a rate. The following is suggested as a definition adequate to meet sufficient as well as necessary requirements for a derivative.

Definition.—The derivative of y with respect to x is the quotient of $dy_1=\mathrm{L}n$ by $dx=\Delta x=n$, where n is a mathematical symbol representing unit positive magnitude for the increment of the independent variable, and $\mathrm{L}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ is the limit of the quotient $\Delta y/\Delta x$ as Δx approaches zero. It is given by the formula

$$\frac{dy_1}{n}=\mathrm{L}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}, \text{ with } \Delta x=n=1 \text{ unit.} \quad . \quad . \quad . \quad (35)$$

This definition assumes the existence of the limit which is the usual one heretofore identified with the derivative which here also depends upon L for its existence, and it is like the old derivative in that it is of the same dimensions as the quotient of y divided by x, usually of zero dimensions, and equal in magnitude to the limit in question.

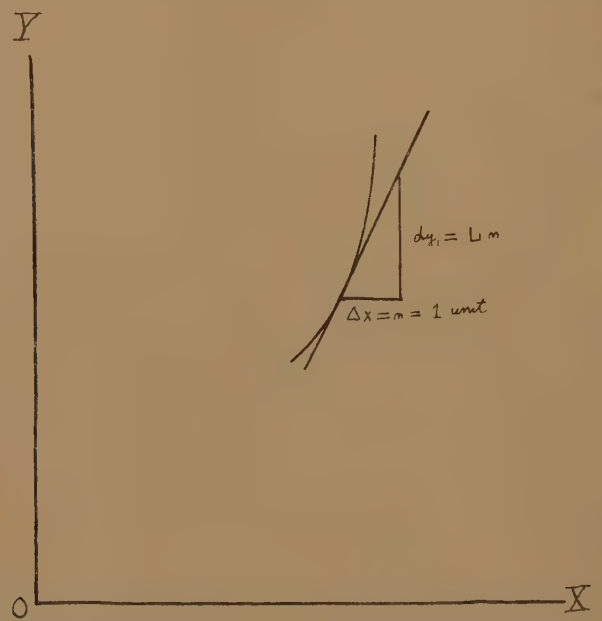
It differs from the previous derivative in that it postulates a separate numerator $dy_1=\mathrm{L}n$ and denominator $\Delta x=n=1$ unit for its essential parts, which function under partial differentiation as particular values of the *variable* differentials $dy=\mathrm{L}dx$ and $dx=\Delta x$, defined in the usual way.

Like previous derivatives (35) may be any variable function of x and y, $f(x, y)=\mathrm{L}$, but it goes beyond previous definitions in that it specifies precisely how it depends on x and y under parallel displacement variations h and k, by virtue of being the ratio of particular values of $dx=\Delta x$ and $dy=\mathrm{L}dx$ which have these same h and k for their increments.

The unit n is especially to be noted. As a mathematical symbol

it represents any conceivable unit usable with the independent variable, from centimetres to calories, or zero dimensional units, but always is taken as unit positive magnitude, and nevertheless must stay in the mathematical formulas whose value like (33) and (34) depends on the units. It properly is called a *variable*, since it is a mathematical symbol that takes on a set of values, the set in question being the totality of units usable in the independent variable. Its introduction to represent any or all of these is like the step from arithmetic to algebra where x is used to represent a set of numbers. Eqs. (50) and (51) illustrate derivatives

Fig. 4.



The unit base triangle, the ratio of whose sides equals the derivative, $\frac{dy_1}{n} = L$.

that are *functions* of this " variable " unit n, and show the changes that occur in the physical rate of change in shifting from one unit to another in a given problem.

Geometrically (35) may be represented as a slope of a curve, or the ratio of the two sides of a *unit base* triangle, with hypotenuse taken along the tangent to the curve as in fig. 4, where the particular values $dy_1 = Ln$ and $\Delta x = n = 1$ unit for the differentials are shown as the sides of the triangle.

Physically (35) is the change dy_1 in y that would occur over unit interval in the x direction at the rate given by $L = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, kept constant

over this unit interval $\Delta x=n$. Thus for the first time the true derivative is required to be the change dy_1 divided by the actual unit change ($\Delta x=n$) in x, and a rate of change in this true sense, a physical quantity equal by trigonometric accident to the ratio of the *variable* differentials $dx=\Delta x$ and $dy=\mathrm{L}dx$ for any sized differential triangle including those other than the unit base triangle shown in fig. 4. Formula (35) is, however, a strictly analytical definition, containing no physics or geometry, until these are introduced by the substitution of particular units in the usual way.

Equations given below show how the new definition of a derivative is required by mathematical consistency as well as by the requirements of partial differentiation as above.

§ 8. *The Uniqueness of the Increments and Derivatives of the Slope of a Straight Line under Parallel Displacements.*

The uniqueness of the slope increments $\pm(hp-k)/n\left(1\mp\frac{h}{n}\right)$ and derivatives (33) and (34) is especially to be emphasized. All of these formulas are very direct and immediate results of two simple operations so basic as to be almost a matter of definition : (*a*) The first is the corner stone of differentiation by parallel displacements as carried out here and in the three previous articles, whereby all interval variables remain constant except the two pairs (Δx_u, Δy_u) and (Δx_i, Δy_i) at the upper and lower limits of a parallel displacement, which latter is hence measured by the increments of these interval variables according to (9). (*b*) Secondly the operation of setting Δx equal to one unit is as basic as the definition of a derivative in use by physicists, and is required if the ratio with this denominator Δx is to function under partial differentiation as a rate of change according to the requirements of engineering and applied mathematics, as shown above.

These two basic requirements point directly to the slope increments and derivatives (23) to (34) of $p(h, k)=p(x-\mathrm{X}, y-\mathrm{Y})$ as a function of x and y under parallel displacements.

§ 9. *The Directional Derivative, the Partial Derivatives, and Total Differential of a Line Integral at a Place (x_i, y_i) on the Path of Integration, compared with Differentiation by Parallel Displacement.*

The line integral

$$\mathrm{J}=\int_{x_0}^{a} f(x, y, y')\,dx\equiv\int_{x_0 y_0}^{a, b}(f-y'f_{y'})\,dx+f_{y'}\,dy=\int \mathrm{M}\,dx+\mathrm{N}\,dy\ . \quad (36)$$

along a curved path $y=\phi(x)$ for an interval corresponding to unit change in x at the point $(x_i y_i)$ lying on $y=\phi(x)$ would have the value, assuming the indicated constant integrands,

$$\left.\begin{aligned} J_1 &= \int_{x_iy_i}^{x_i+1,\, y_i+y_i'} (f-y_i'f_{y'})_i\,dx + f_{y'}(x_i, y_i, y_i')\,dy = \frac{dJ}{dx}(x_i, y_i) \\ &\equiv \Big[\{f(x_iy_i, y_i') - y_i'f_{y'}(x_iy_iy_i')\}x\Big]_x^{x_i+1} + \Big[f_{y'}(x_i, y_iy_i')\,y\Big]_{y_i}^{y_i+y_i'} \\ &\qquad\qquad \equiv f(x_i, y_iy_i'). \end{aligned}\right\} \quad . \quad (37)$$

Since this is the change along the integration path in J that would take place per unit change in x in case the rate were maintained at the constant value it has at (x_iy_i) over the entire unit interval, this expression is by definition the derivative or *rate* of change of J at (x_iy_i). And, being taken along the path, it is the total or *directional* derivative of J with respect to x, the direction being along the path $y=\phi(x)$ at (x_iy_i), using $y_i'=\phi'(x_i)$.

If either dx or dy were zero (37) would degenerate into the two special cases which are, of course, the two partial derivatives of J with respect to x and y at (x_iy_i) :

$$\left.\begin{aligned} J_{1x} &= \frac{\partial J}{\partial x} = f(x_iy_iy_i') - y_i'f_{y'}(x_iy_iy_i') = M(x_i, y_i, y_i') \\ J_{1y} &= \frac{\partial J}{\partial y} = f_{y'}(x_iy_iy_i') = N(x_i, y_i, y_i') \end{aligned}\right\} \text{at } (x_iy_i). \quad . \quad (38)$$

If y_i' in the upper *limit* of (37) were replaced by $q=\dfrac{\partial y}{\partial x}$ then the evaluation of (37) would yield $\dfrac{\partial J}{\partial x} + q\dfrac{\partial J}{dy}$, using (38), *i. e.*, the total derivative in the direction arc tan q at (x_iy_i), and the total differential at this point is

$$\delta J = (f-y_i'f_{y'})_i\,\delta x + f_{y'}(x_i, y_i, y_i')\,\delta y \text{ at } (x_iy_i), \quad . \quad . \quad . \quad (39)$$

where the Greek symbols are used to indicate independent variations not necessarily along the path of integration.

We may compare (37), (38), and (39) with the closed circuit partial derivatives of a line integral obtained by the parallel path displacements,

$$\left.\begin{aligned} \frac{\partial J}{\partial x} &= M(x_iy_iy_i') + \int_{x_iy_i}^{x_uy_u} M\,dx + N_x\,dy - M(x_uy_uy_u') \\ &\qquad = (f-y'f_{y'})^i + \int_{x_i}^{x_u} f_x\,dx - (f-y'f_{y'})^u, \\ \frac{\partial J}{\partial y} &= N(x_iy_iy_i') + \int_{x_iy_i}^{x_uy_u} M_y\,dx + N_y\,dy - N(x_uy_uy_u') \\ &\qquad = f_{y'}(x_iy_iy_i') + \int_{x_i}^{x_u} f_y\,dx - f_{y'}(x_uy_uy_u'), \end{aligned}\right\} \quad . \quad (40)$$

and we note that (40) includes (38), since the former are closed circuit line integrals comprising the difference between changes in J per unit change in x and y at (x_iy_i) and at (x_uy_u) combined with the integral

term which is the difference in J for these limits along the original path $y=\phi(x)$ and along its parallel displaced extent per *unit* displacement. Recall also that (40) are obtainable (*a*) by differentiation of a sum like (3) or (30) *before* passing to the limit as in integration, or (*b*) *after* integration by parallel path displacements as in Part II. of the previous [2] article. Recall * also that the area derivative of (36) taken with respect to $d\sigma=\delta x\,dy-\delta y\,dx$ adjoining the path is given by taking the total derivative of each of (40) with respect to y and x with variable upper and variable lower limit respectively in the two equations, yielding:

$$\frac{dJ}{d\sigma}=\frac{d}{dy}\frac{\partial J}{\partial x}=-\frac{d}{dx}\frac{\partial J}{\partial y}=\frac{d}{dx}f_{y'}-f_y,$$

which, when set equal to zero, gives Euler's equations.

§ 10. *First Partial Derivatives by Parallel Displacement of a Calculus of Variations Integral whose Path is Rectilinear at the Upper and and Lower Limits.*

We consider a *special* case of (36) where the path of integration $y=\phi(x)$ is *straight* at the upper and lower limits Q and P of a parallel displacement as shown in fig. 5, and having the *constant* slopes p_u and p_i at these limits, and in general the slope $y'=\phi'(x)=Y'$. In this article we use p for the *constant* slope of a straight line, in contrast to y' for a curve. Now employ the method of differentiation by parallel path displacements, and define our " varied " integral as (36) but taken along the three section path PS, SR, and RQ:

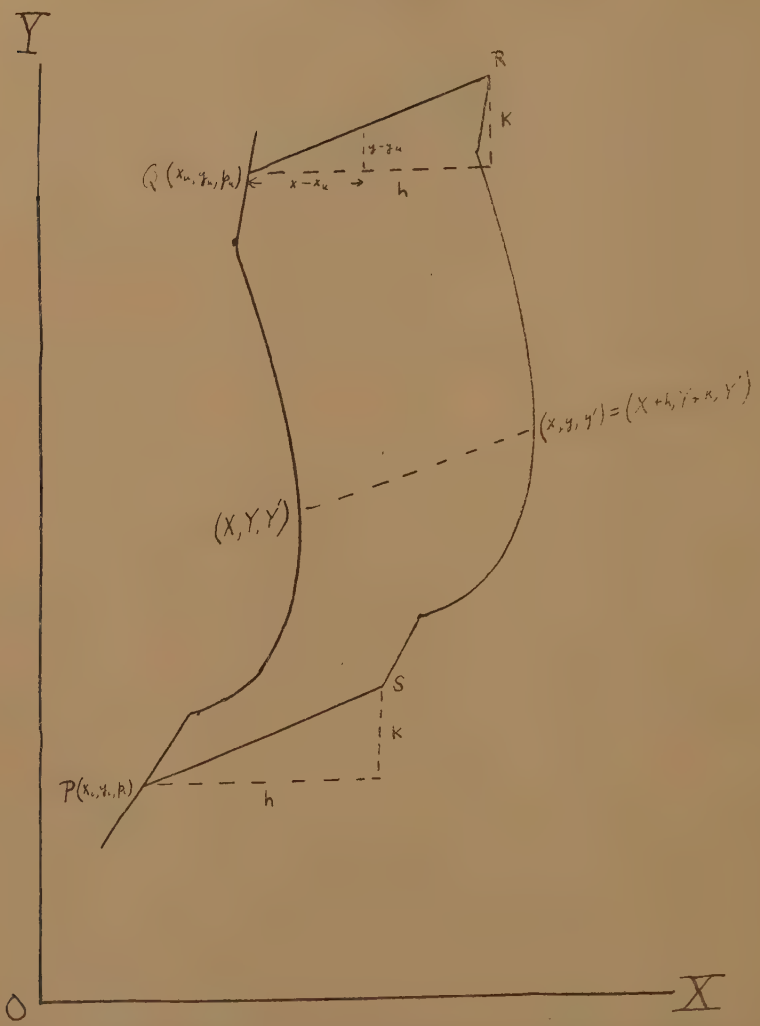
$$\begin{aligned}J_i^u(h,k)=J_i^u(x-X,y-Y)=&\int_{x_iy_i}^{x_i+h,\,y_i+k}M\left[x,y,p_i-\frac{p_i(x-x_i)-(y-y_i)}{n\left(1+\frac{x-x_i}{n}\right)}\right]\delta x\\&+N\left[x,y,p_i-\frac{p_i(x-x_i)-(y-y_i)}{n\left(1+\frac{x-x_i}{n}\right)}\right]\delta y\\&+\int_{x_ix_i}^{x_uy_u}M(X+h,Y+k,Y')\,dX+N(X+h,Y+k,Y')\,dY\\&-\int_{y_uy_u}^{x_u+h,\,y_u+k}M\left[x,y,p_u+\frac{p_u(x-x_u)-(y-y_u)}{n\left(1-\frac{x-x_u}{n}\right)}\right]\delta x\\&+N\left[x,y,p_u+\frac{p_u(x-x_u)-(y-y_u)}{n\left(1-\frac{x-x_u}{n}\right)}\right]\delta y.\end{aligned}\quad(41)$$

The second integral of (41) is over SR, the parallel displaced path $y=\phi(x-h)+k$ satisfied by $(X+h, Y+k)$, where (X, Y) are taken to be the coordinate values satisfying the original path equation $y=\phi(x)$, and we

* See eqs. (60) and (76) in reference (2).

note that the slope is of course unchanged by parallel displacement.

Fig. 5.



Parallel displacement of a path that is rectilinear at the limits of the displacement.

The first and third integrals of (41) are those along PS and RQ, the straight line end pieces of the displacement circuit having equations

$$\left.\begin{aligned} y-y_u &= \frac{k}{h}(x-x_u), \\ y-y_i &= \frac{k}{h}(x-x_i). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (42)$$

Note especially that the third arguments here are given by using $(x-x_u,\ y-y_u)$ for $(h,\ k)$ in (23) and $(x-x_i,\ y-y_i)$ for $(h,\ k)$ in (24), since, for a varied point $(x,\ y)$ on (42), we have the varied slopes (23) and (24) of the *straight* path segments whose original values $p_u=\left(\frac{dy}{dx}\right)_u=\frac{\Delta y_u}{\Delta x_u}$ and $p_i=\left(\frac{dy}{dx}\right)_i=\frac{\Delta y_i}{\Delta x_i}$ take on increments like those of (23) and (24) shown in fig. 2, but for displacements $(x-x_u,\ y-y_u)$ and $(x-x_i,\ y-y_i)$ instead of h and k. Note also especially that the differentials involved in p_i and p_u, the derivatives (slopes) at P and Q can be replaced by the finite interval variables Δx and Δy of (23) and (24) and *vice versa*, since for straight line paths as provided herein at P and Q there is no difference between differentials and finite interval variables, the limiting aspect of the former being out of the picture and treated as past history.

The partial derivatives of (41) with respect to h and k are the same as with respect to x and y, keeping (X, Y) fixed:

$$\left.\begin{aligned}
\frac{\partial \mathrm{J}_i^u}{\partial x}=\frac{\partial \mathrm{J}_i^u}{\partial h}(h,\ k)&=\mathrm{M}\left[x_i+h,\ y_i+k,\ p_i-\frac{hp_i-k}{n\left(1+\frac{h}{n}\right)}\right]\\
&+\int_{x_iy_i}^{x_uy_u}\mathrm{M}_x(\mathrm{X}+h,\ \mathrm{Y}+k,\ \mathrm{Y}')\,d\mathrm{X}+\mathrm{N}_x(\mathrm{X}+h,\ \mathrm{Y}+k,\ \mathrm{Y}')\,d\mathrm{Y}\\
&-\mathrm{M}\left[x_u+h,\ y_u+k,\ p_u+\frac{hp_u-k}{n\left(1-\frac{h}{n}\right)}\right],\\
\frac{\partial \mathrm{J}_i^u}{\partial y}=\frac{\partial \mathrm{J}_i^u}{\partial k}(h,\ k)&=\mathrm{N}\left[x_i+h,\ y_i+k,\ p_i-\frac{hp_i-k}{n\left(1+\frac{h}{n}\right)}\right]\\
&+\int_{x_iy_i}^{x_uy_u}\mathrm{M}_y(\mathrm{X}+h,\ \mathrm{Y}+k,\ \mathrm{Y}')\,d\mathrm{X}+\mathrm{N}_y(\mathrm{X}+h,\ \mathrm{Y}+k,\ \mathrm{Y}')\,d\mathrm{Y}\\
&-\mathrm{N}\left[x_u+h,\ y_u+k,\ p_u+\frac{hp_u-k}{n\left(1-\frac{h}{n}\right)}\right],
\end{aligned}\right\}\quad(43)$$

which we see are the same as (40) when $h=k=0$. Also note how (41) and (43) are two point functions of $(x,\ y)$ independent variables and (X, Y) related by $y=\phi(x)$, since these two points are enough to determine the straight line parallel displacements $h=x-\mathrm{X}$, $k=y-\mathrm{Y}$, as well as the expressions (41) and (43) themselves, once the original integral with its path and limits is given.

Here again the third argument at the upper and lower limits is given by (23) and (24), using $\Delta x=n=1$ unit, which is required to make this third argument the variation of a *rate* such as p_u or p_i under parallel displacements h and k.

§ 11. *Second Partial Derivatives when the Path is Rectilinear at the Limits.*

The second partial derivatives of (41) are also the same when taken with respect to $h=x-X$ and $k=y-Y$ as for (x, y), independent variables, keeping (X, Y) fixed and differentiating (43):

$$\left.\begin{aligned}
\frac{\partial^2 J_i^u}{\partial x^2} = \frac{\partial^2 J_i^u(h,k)}{\partial h^2} = -\Bigg[M_x \left\{ x+h, y+k, p \pm \frac{hp-k}{n\left(1 \mp \frac{h}{n}\right)} \right\} \\
\pm \left\{ \frac{p}{n\left(1 \mp \frac{h}{n}\right)} \pm \frac{hp-k}{n^2\left(1 \mp \frac{h}{n}\right)^2} \right\} M_{y'} \Bigg]_i^u \\
+ \int_{x_i y_i}^{x_u y_u} M_{xx}(X+h, Y+k, Y')\, dX + N_{xx}(X+h, Y+k, Y')\, dY, \\
\frac{\partial^2 J_i^u}{\partial y \partial x} = \frac{\partial^2 J_i^u(h,k)}{\partial k \partial h} = -\Bigg[M_y \left\{ x+h, y+k, p \pm \frac{hp-k}{n\left(1 \mp \frac{h}{n}\right)} \right\} \\
\mp \frac{1}{n\left(1 \mp \frac{h}{n}\right)} M_{y'} \Bigg]_i^u \\
+ \int_{x_i y_i}^{x_u y_u} M_{xy}(X+h, Y+k, Y')\, dX + N_{xy}(X+h, Y+k, Y')\, dY, \\
\frac{\partial^2 J_i^u}{\partial x \partial y} = \frac{\partial^2 J_i^u(h,k)}{\partial h \partial k} = -\Bigg[N_x \left\{ x+h, y+k, p \pm \frac{hp-k}{n\left(1 \mp \frac{h}{n}\right)} \right\} \\
\pm \left\{ \frac{p}{n\left(1 \mp \frac{h}{n}\right)} \pm \frac{hp-k}{n^2\left(1 \mp \frac{h}{n}\right)^2} \right\} N_{y'} \Bigg]_i^u \\
+ \int_{x_i y_i}^{x_u y_u} M_{xy}(X+h, Y+k, Y')\, dX + N_{xy}(X+h, Y+k, Y')\, dY, \\
\frac{\partial^2 J_i^u}{\partial y^2} = \frac{\partial^2 J_i^u(h,k)}{\partial k^2} = -\Bigg[N_y \left\{ x+h, y+k, p \pm \frac{hp-k}{n\left(1 \mp \frac{h}{n}\right)} \right\} \\
\mp \frac{1}{n\left(1 \mp \frac{h}{n}\right)} N_{y'} \Bigg]_i^u \\
+ \int_{x_i y_i}^{x_u y_u} M_{yy}(X+h, Y+k, Y')\, dX + N_{yy}(X+h, Y+k, Y')\, dY.
\end{aligned}\right\} \quad (44)$$

Here the upper and lower signs refer to the upper and lower limits respectively, and the subscripts indicate partial differentiation, and the

derivatives with respect to y' are multiplied by $\frac{\partial p}{\partial h}=\frac{\partial p}{\partial x}$ or $\frac{\partial p}{\partial k}=\frac{\partial p}{\partial y}$ according to (26), where p is the *constant** value of y' at the limits as per notation here used. Note especially that no partials of y' come into the integral terms which are derivatives of the part of (41) from S to R obtained by *parallel* path displacements, and with hence Y′ unchanged by h and k, which are only added to the position coordinates.

When we set $h=k=0$ (x, y) will lie on the original path of integration $y=\phi(x)$, and will coincide with (X, Y), its undisplaced position. Then, when we introduce $M=f-y'f_{y'}$ and $N=f_{y'}$ from (36),

$$\left.\begin{aligned}
\frac{\partial^2 J_i^u}{\partial x^2}&=\frac{\partial^2 J_i^u(0,0)}{\partial h^2}=-\Big[f_x-p(f_{xy'}\pm\frac{p}{n}f_{y'y'})\Big]_i^u+\int_{x_i}^{x_u}f_{xx}dx,\\
\frac{\partial^2 J_i^u}{\partial y\,\partial x}&=\frac{\partial^2 J_i^u(0,0)}{\partial k\,\partial h}=-\Big[f_y-p(f_{y'y'}\mp\frac{1}{n}f_{y'y'})\Big]_i^u+\int_{x_i}^{x_u}f_{xy}dx,\\
\frac{\partial^2 J_i^u}{\partial x\,\partial y}&=\frac{\partial^2 J_i^u(0,0)}{\partial h\,\partial k}=-\Big[f_{xy'}\pm\frac{p}{n}f_{y'y}\Big]_i^u+\int_{x_i}^{x_u}f_{xy}dx,\\
\frac{\partial^2 J_i^u}{\partial y^2}&=\frac{\partial^2 J_i^u(0,0)}{\partial k^2}=-\Big[f_{yy'}\mp\frac{1}{n}f_{y'y'}\Big]_i^u+\int_{x_i}^{x_u}f_{yy}dx,
\end{aligned}\right\}\quad(45)$$

where the path at the limit is rectilinear.

Note that $f_{y'y'}$ is multiplied by $p_x=\pm\frac{p}{n}$ or $p_y=\mp\frac{1}{n}$, given by (27), and that we must choose $dx=\Delta x=n$, as before, if the term in question is to be a rate of change of a rate as required by the left members, as well as all the other terms on the right which are second partials by standard calculus methods, and accordingly changes per cm.^{2} using C.G.S. units.

This becomes more clear if we take a special case. Consider the mass of a wire of length L and linear density $\rho(x, y)$:

$$J=\int_0^L\rho(x,y)\,ds=\int\rho(x,y)\sqrt{1+y'^2}\,dx=\int\rho(x,y)\Big(\frac{dx+y'\,dy}{\sqrt{1+y'^2}}\Big).\quad(46)$$

Here $\quad M=\frac{\rho(xy)}{\sqrt{1+y'^2}};\quad N=\frac{\rho(x,y)y'}{\sqrt{1+y'^2}};\quad f=\rho(x,y)\sqrt{1+y'^2},$

and the first of (45) becomes

$$\frac{\partial^2 J_i^u}{\partial x^2}=\Big[\frac{\rho_x}{\sqrt{1+p^2}}+\frac{\rho p^2}{n(1+p^2)^{3/2}}\Big]^i+\int_{x_i}^{x_u}\rho_{xx}\sqrt{1+y'^2}\,dx-\Big[\frac{\rho_x}{\sqrt{1+p^2}}-\frac{\rho p^2}{n(1+p^2)^{3/2}}\Big]^u.\quad(47)$$

---

* With total derivative zero according to (28) for any straight line.

Now the terms having partials of ρ are in each case a part of the changes per cm. in the change in J per cm. For example, look at the first term. We saw in § 9 by actually carrying out the integration as per (37) and (38) that

$$J_{1x} = \frac{\partial J}{\partial x} = \mathbf{M} = \frac{\rho}{\sqrt{1+p^2}}$$

is the partial change in J per *unit* interval in the x direction in the calculus sense, or the change in J per cm. Accordingly the second partial or change in J per cm.^{2} (so far as variations in ρ are concerned) is given by $\rho_x/\sqrt{1+p^2}$ as per (47). The second term, however, is exactly like the first term of (15) above except for the density factor ρ, and like that of (15) cannot represent the rate of change of the change in the length of s (times density) per *unit* length along the x axis (as included between the vertical dotted lines of fig. 2)—that is, the change in J per cm.^{2}, unless we set $dx=\Delta x=n$ as done with (13) to get (15).

Thus *consistency* of the various terms of (47) and (45), as well as the plain meaning for physics and engineering of a second derivative as a rate of change of a rate, requires the new definition for a derivative given in § 7, whereby the special values of the differentials dy_1 and $dx=\Delta x=n$ are singled out as essential in the ratio $\frac{dy_1}{n} = \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{limit}} \frac{\Delta y}{\Delta x} = y'$, which is the same by trigonometric accident (being of zero dimensions) for the ratios of less significant and variable values of dy and dx.

§ 12. *The* True *Derivatives of a Line Integral at a Point on a Rectilinear Path and the True Partials of the Slope at this Point are the same as if this Point were the Lower Limit of a Parallel Displacement.*

Eqs. (23), (24), and (32) show that the minus sign introduced with each derivative at the upper limit with respect to h and k comes from the minus sign in front of h or k, and hence is not truly a part of the derivative strictly as such taken with respect to x, y, h, and k. Leaving out these minus signs, the derivatives taken partially with respect to x and y are given by those at the *lower* limit, which thus are the *true* derivatives of p the slope of a straight line at the point (x_i, y_i) on the line (formerly the lower limit). Likewise for *true* derivatives of functions of p treated as implicit functions of x and y.

Note that all the *true* derivatives of p (those at the lower limit) given by (33) and (34) are obtainable by differentiating partially with respect to its differentials and then setting $dx=n=$one unit. Hence the

Rule: The partial derivatives of the slope $p=\frac{dy}{dx}$ of a straight line with respect to x and y are given by two steps :

(1) Differentiate partially with respect to the differential corresponding to the variable with respect to which differentiation is taken.

(2) Set the increment dx of the variable taken as the independent one for the slope p, equal to n=one unit, according to the new requirement in defining a derivative.

Thus we may take the first partials of (46) at $(x_i, y_i) = (4n, 4\sqrt{3}n)$ on the straight integration path $y=x\sqrt{3}$. If the linear density is $\rho=\frac{x\sharp}{n^2}=x$ lb./ft., the mass of 16 feet of wire is

$$J=\frac{\sharp}{n^2}\int_0^{8n} x\sqrt{1+p^2}\,dx=2\,\frac{\sharp}{n^2}\,\frac{x^2}{2}\Big]_0^{8n}=64\sharp=64 \text{ lb.} \quad . \quad . \quad (48)$$

Then the first partials of (48) at $(x_i, y_i)=(4n, 4\sqrt{3}n)$, where the density is $\rho=\frac{4\sharp}{n}$=four pounds per foot, are

$$\left.\begin{aligned}\frac{\partial J(x_iy_i)}{\partial x}&=M(x_iy_i)=\frac{\rho_i}{\sqrt{1+p^2}}=\frac{x_i\sharp}{2n^2}=\frac{2\sharp}{n}=2 \text{ lb./ft.},\\ \frac{\partial J(x_iy_i)}{\partial y}&=N(x_iy_i)=\frac{\rho_i p}{\sqrt{1+p^2}}=\frac{\sqrt{3}x_i\sharp}{2n^2}=\frac{2\sqrt{3}\sharp}{n}=2\sqrt{3} \text{ lb./ft.}\end{aligned}\right\} \quad . \quad . \quad (49)$$

Note that these are obtainable by application of Rule I. for differentiation with respect to differentials of a line integral as applied at the *lower* limit, *i. e.*, drop the integral sign from $J=\int\rho\sqrt{dx^2+dy^2}$ and differentiate partially with respect to dx or dy as though these were x and y.

For the second derivatives we have by application of the usual rules plus the Rule of this section (or by (45)):

$$\left.\begin{aligned}\frac{\partial^2 J}{\partial x^2}&=\frac{\rho_x}{\sqrt{1+p^2}}+\frac{\rho p^2}{n(1+p^2)^{3/2}}=\frac{\sharp}{2n^2}+\frac{3x_i\sharp}{8n^3}=\frac{2\sharp}{n^2}=2 \text{ lb./ft.}^2,\\ \frac{\partial}{\partial y}\frac{\partial J}{\partial x}&=\frac{-\rho p}{n(1+p^2)^{3/2}}=\frac{-\sqrt{3}x_i\sharp}{8n^3}=-\frac{\sqrt{3}\sharp}{2n^2}=-\frac{\sqrt{3}}{2} \text{ lb./ft.}^2,\\ \frac{\partial}{\partial x}\frac{\partial J}{\partial y}&=\frac{\rho_x p}{\sqrt{1+p^2}}-\frac{\rho p}{n(1+p^2)^{3/2}}=\frac{\sqrt{3}\sharp}{2n^2}-\frac{\sqrt{3}x_i\sharp}{8n^3}=0 \text{ lb./ft.}^2,\\ \frac{\partial^2 J}{\partial y^2}&=\frac{\rho}{n(1+p^2)^{3/2}}=\frac{x_i\sharp}{8n^3}=\frac{\sharp}{2n^2}=\tfrac{1}{2} \text{ lb./ft.}^2.\end{aligned}\right\} \quad . \quad (50)$$

The second and third of (50) would be equal if $y=\sqrt{3}x$ were an extremal arc, which, of course, it is not, because of the variable density $\rho=kx$ Also note that except for the terms in ρ_x all terms in (50) are obtainable by differentiating (49) with respect to dx or dy, using $p=\frac{dy}{dx}$, and then setting $dx=n$, according to the Rule of this section.

§ 13. *The new Derivatives* vs. *the Classical Derivatives.*

If we change the length units from feet to yards in connexion with the same wire considered above, with mass given by (48), we have for the density $\rho = 9x\sharp/n^2$, and the mid-point of the wire used in (49) and (50) now has coordinates $(x_i,\ y_i) = \left(\frac{4n}{3},\ \frac{4\sqrt{3}n}{3}\right)$. With these coordinates and this density (49) will show three times the numerical value shown previously. But this is physically the same as (49), because 2 and $2\sqrt{3}$ lb. per foot are the same rates of change as 6 and $6\sqrt{3}$ lb. per yard. And this sameness results from (49)'s being like (1) and (12), independent of the length Δs involved in differentiation, and hence free from n except where n is part of ρ.

On using yards instead of feet in the second derivative we have, instead of (50),

$$\left.\begin{aligned}
\frac{\partial^2 J}{\partial x^2} &= \frac{9\sharp}{2n^2} + \frac{27x_i\sharp}{8n^3} = \frac{9\sharp}{n^2} = 9 \text{ lb./yd.}^2, \\
\frac{\partial^2 J}{\partial y \partial x} &= -\frac{9\sqrt{3}x_i\sharp}{8n^3} = -\frac{3\sqrt{3}\sharp}{2n^2} = -\tfrac{3}{2}\sqrt{3} \text{ lb./yd.}^2, \\
\frac{\partial^2 J}{\partial x \partial y} &= \frac{9\sharp\sqrt{3}}{2n^2} - \frac{9\sqrt{3}x_i\sharp}{8n^3} = \frac{3\sqrt{3}\sharp}{n^2} = 3\sqrt{3} \text{ lb./yd.}^2, \\
\frac{\partial^2 J}{\partial y^2} &= \frac{9\sharp x_i}{8n^3} = \frac{3\sharp}{2n^2} = \tfrac{3}{2} \text{ lb./yd.}^2,
\end{aligned}\right\} \quad . \quad . \quad (51)$$

in which we note that the terms in ρ_x (terms representing classical derivatives of (49)) appear to have nine times their value given by (50), whereas those terms involving ρ/n are shown only three times as great numerically as the corresponding terms of (50). In comparing the new derivative terms of (51) with the classical derivative terms we may emphasize four viewpoints :—

(a) *Functional Dependence on n.*—The new derivatives are functions of the unit n, whereas the classical derivatives are not (see (15) to (21) above). Comparison of (50) and (51) shows how this is. Thus $\frac{\rho_x}{\sqrt{1+p^2}} = \frac{1}{2}\sharp/\text{ft.}^2$ in (50) is the same physically as $9/2\sharp/\text{yd.}^2$ given by (51), being the classical derivative of this same term. But the other term of $\frac{\partial^2 J}{\partial x^2}$ in (50) : $3/2\sharp/\text{ft.}^2$ is not the same physically as $9/2\sharp/\text{yd.}^2 = \frac{27\sharp x_i}{8n^3}$ for the point in question which corresponds to it in (51), and the discrepancy comes from the factor $\frac{\partial^2 s}{\partial x^2} = \frac{p^2}{n(1+p^2)^{3/2}} = \frac{\partial^2 \Delta s_i}{\partial x^2}$

(see fig. (3)). This factor is like a term of (15), and is one-third as great with $n=1$ yd. and a proportionately long Δs_i as shown in fig. 3 as it is for $n=1$ ft. and a short Δs_i, so that in (51) it is only $9/2$#/yd.^{2}, instead of $(27/2)$#/yd.^{2}, that it would have to be in order to equal physically the $3/2$#/ft.^{2} of (50). Thus all of the new derivative terms in (50), *i. e.*, those having the factor ρ/n, are physically three times as great in (50) as the corresponding terms in (51). The check on this is that such terms in (51) are only three times numerically those of (50), but along with yd.^{2}, in the denominator instead of ft.^{2}. Furthermore, this is a result that is bound to occur as soon as we go beyond the trigonometric accident whereby $\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{limit}} \frac{\Delta y}{\Delta x} = \tan \alpha$ has the same value for any $dx = \Delta x$ that we choose to take, and employ partial differentiation formulas like (2), (13), and (15), whose validity as rates of change of a rate depends upon making the dependent variables correspond to unit interval in the independent one, by choosing $dx = \Delta x = 1$ unit. See (16) to (21) in this connection.

(b) *Rates of Change of a Rate.*—Both the new and the classical terms are true second derivatives in being rates of change of a rate or changes in J (in the limiting calculus sense) per squared unit of the independent variable. In the classical term this is so by trigonometric accident, since $\frac{\rho_x}{\sqrt{1+p^2}} = \frac{1}{\sqrt{1+p^2}} \underset{\Delta x \to 0}{\text{limit}} \frac{\Delta \rho}{\Delta x}$ is the same for any ratio of $d\rho$ by dx (the usual differentials) as it is for the special $d\rho_1 \div n$. With the new derivatives, however, we must choose $\Delta x = n = 1$ unit if $\rho \Delta s_i$ is to be the change in J due to unit change in x, and a rate of change in that sense, as appeared from (15). That we must have such an increment ΔJ due to *unit* change in x became clear in connexion with the *first* partials by actually carrying out the integration between the limits x_i and $x_i + 1$ as in (37) and (38).

(c) *Mathematical Consistency requires the new Derivative Terms.*—A glance at fig. 6 shows that under change of path from the rectilinear QL to the broken line QR, RM, and ML there will be changes in length to affect the mass (48) as well as changes in density, and corresponding rates of change indicated by the second partials at $h = k = 0$ as well as for other values of h and k. Thus to call $\frac{\rho_x}{\sqrt{1+p^2}}$ the only term in the second partial of J at (x_i, y_i) (as if the path length were constant) is obviously incorrect where the path length takes the varied form given by (7), $\Delta s_i = \sqrt{(\Delta x_i + h)^2 + (\Delta y_i + k)^2}$. Thus to be consistent, and have a change of length term also represent a rate of change of a rate, we must employ second partials of this Δs_i as in (50) and (51), *i. e.*, the new

derivatives which set $\Delta x=n=1$ unit so as to make Δs and hence ΔJ the actual changes per *unit* change in x.

(d) *Failure of Classical Differentiation.*—The present treatment attributes *no* incorrect results to *any* classical derivatives. Beyond any question here all results given by classical differentiation are correct. The inadequacy under discussion here refers to the complete failure of the classical differentiation to function *at all* when it comes to taking partials of the slope or derivative p in an integral like (48). The first partials (49) are clearly like (1) and (12), being given by

$$\frac{\rho_i}{\sqrt{1+p^2}}=\rho_i\cos\alpha,\ \frac{\rho_i p}{\sqrt{1+p^2}}=\rho_i\sin\alpha,$$

and it is equally obvious that when a varied path such as that of fig. 6 or QRML swings through the rectilinear position corresponding to $h=k=0$ the rates $\frac{\partial J}{\partial x}$ and $\frac{\partial J}{\partial y}$ given by (49) will themselves have a rate of change, and yet classical differentiation can provide only the $\rho_x\cos\alpha$ and $\rho_x\sin\alpha$ terms. It *balks completely* before rates of change of the other factors $\frac{1}{\sqrt{1+p^2}}$ and $\frac{p}{\sqrt{1+p^2}}$, and the gist of the difficulty becomes apparent by the middle members of (26), (27), and (32), all of which are true partial derivatives of $p=\frac{\Delta y}{\Delta x}=\underset{\Delta x\to 0}{\text{limit}}\frac{\Delta y}{\Delta x}$ (or its varied form (23) or (24)) for any Δx arbitrarily chosen for this straight line derivative formula, and yet these middle members are indeterminate and hence meaningless. It thus appears that this above derivative formula by itself is insufficient for the demands of mathematical consistency as well as of physics, which require it to be supplemented by introducing $\Delta x=n=1$ unit, if (23) and (24) are to be variants of a rate $p=\frac{dy_1}{n}=\underset{\Delta x\to 0}{\text{limit}}\frac{\Delta y}{\Delta x}$ and (26) to (34) are to be its partial derivatives and rates of change.

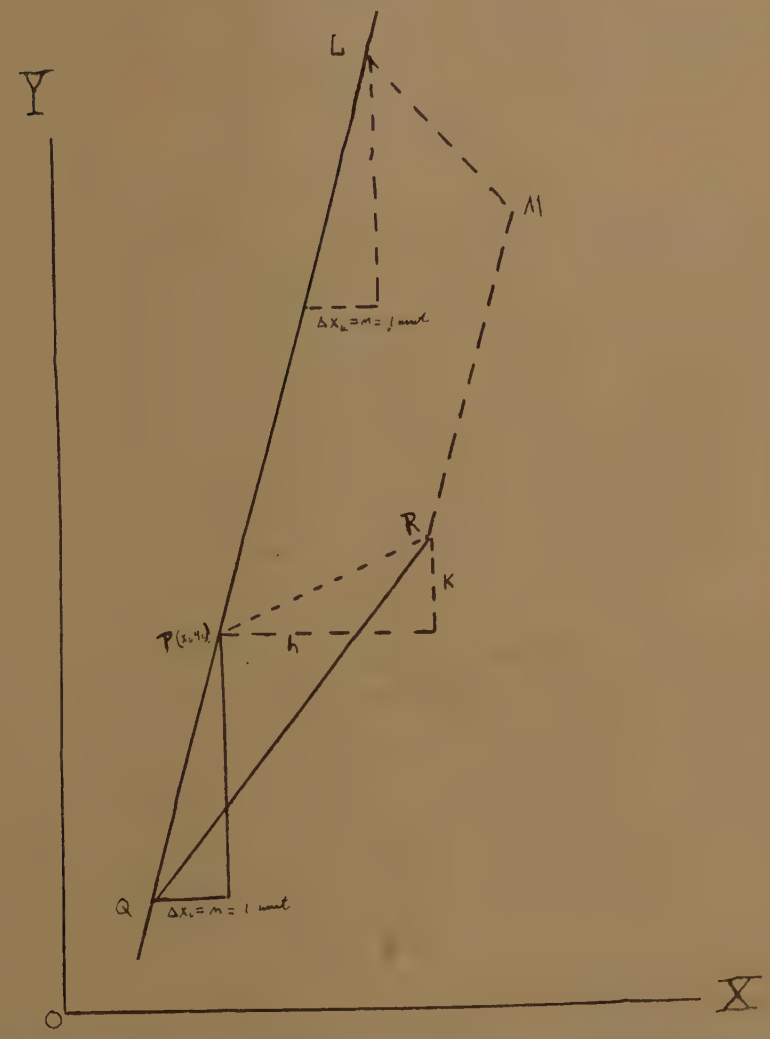
Finally, we may note how the corresponding geometrical situation points clearly to rates of change dependent on units, and hence classical differentiation, independent of units, is sure to fail. Thus the x partial rate of change of Δs given by (7) at (x_i, y_i) in fig. 6 is

$$\frac{\partial\Delta s}{\partial x}=\frac{\Delta x+h}{\sqrt{(\Delta x+h)^2+(\Delta y+k)^2}},$$

which reduces to $\frac{1}{\sqrt{1+p^2}}$ when $h=k=0$ and QR swings back to QP. But these derivatives as well as (7) itself may be made to depart as little as we please (by diminishing the size of the units taking $\Delta x=n=1$ unit)

from the corresponding rate and interval ds for an arbitrary curved path variation starting at P instead of a broken line variation from recti-

Fig. 6.



The increment $dJ_1 = \Delta s = \mathrm{QP}$ of the length integral per unit change in x, and its varied position $\mathrm{QR} = \Delta \mathrm{J}(h, k)$ given by (7).

linearity as in fig. 6. And during this diminution of unit size Δs diminishes too (thus changing the problem, *i. e.*, the dependent variable), but con-

tinues constantly to have unit projection and continues to be the change in s per unit change in x. The second partial,

$$\frac{\partial^2 \Delta s}{\partial x^2} = \frac{(\Delta y + k)^2}{((\Delta x + h)^2 + (\Delta y + k)^2)^{3/2}},$$

also may be made to depart as little as we please from the second partials of the ds for a curved path variation starting at P instead of fig. 6's broken line variation above given, which reduces to

$$\frac{\partial^2 \Delta s}{\partial x^2} = \frac{p^2}{\Delta x (1+p^2)^{3/2}}$$

when $h=k=0$. And, as before, taking $\Delta x = n = 1$ keeps Δs as the change in s per unit change in x. So it seems clear (because of the familiar approximate relations that may be set up between curves and broken lines) that the length s of a curved line, as well as the lengths of the straight and broken lines here treated, will have for its second partials at a point on its path expressions dependent on units, as surely as a variable unit n may legitimately replace x in (2) as discussed in § 4. Dependence on units for rates of change of a rate is hence for classical differentiation an insurmountable difficulty, intrinsic in the problem of finding the partial rates of change of a length s, with respect to x and y, because it prohibits any differentiation process providing results essentially independent of units as classical differentiation does.

References.

(1) W. S. Kimball, " New Basis Critical Equations and Methods in the Calculus of Variations," Phil. Mag. xxiii. p. 114 (1937).
(2) W. S. Kimball, " Differentiation of Line Integrals by Parallel Displacement," Phil. Mag. xxv. p. 1 (1938).

XX. *Some Acoustical Source-Observer Problems.*

By S. W. Redfearn *.

[Received February 13, 1940.]

Synopsis.

In view of the practical importance of the problem of the reduction of noise at a point due to an unavoidable source of sound situated at a second point, certain aspects forming a logical sequence have been selected for a theoretical treatment from as general a point of view as possible. The first relates to barriers, the second to the enclosure of the source, and the third to the enclosure of the point of observation. In all cases connected with a simple source of sound and a point of observation the Helmholtz reciprocal principle holds, and two applications of this are given, in connexion with the sound transmitted round a barrier, and in connexion with the problem of an enclosed source.

I. Introduction.

The relation between the intensity of a sound of a definite frequency at a given point in a field and the characteristics of a source as defined either by the nature of the limit of velocity potential as the source is approached, or by other means for extended sources such as the displacement or pressure, is governed by the physical properties and geometrical distribution of various regions of the field or objects within it, and it is the purpose of the present paper to discuss the theory of certain reasonably simple systems. One general proposition linking source and point of observation is the reciprocal theorem of Helmholtz, stating that the potential at O in a field due to a source at S is the same as that at S due to the same source situated at O. It must be noted that the similarity of sources for this purpose is defined, not by the energy output of a given source, but by the similarity of the potential function as it approaches the point where the source is situated. The energy output from a source depends very largely on the "impedance" into which it is working, though for all practical and effective purposes many systems have approximately the same impedance—and so equivalent sources

* Communicated by Professor G. P. Thomson, F.R.S.

have approximately the same energy output. The actual effect of such a factor is only appreciable in limiting cases, examples of which will be discussed later.

The consideration in this present paper of the reciprocal relation between source and point of observation will extend somewhat further than that laid down in Helmholtz's theorem, to the relation in actual fields with some simply defined configurations, notably fields in which the more obvious obstacles to the transmission of sound may be placed. Except in the simplest cases, rigid analysis of the systems is impossible, and certain approximate methods are available, some based on Kirchhoff's theorem, relating the potential at a point in a field to the potential and potential gradient over an enclosing surface, and some based on statistical considerations relative to energy distribution and propagation, notably those following the methods of Sabine, Jäger, and more recent workers, especially in the field of reverberation acoustics, where exact field distributions are impossible to work out theoretically, are specific to given cases only, and often for that reason do not throw much light on the totality of such similar problems.

The problems dealt with here as hinging round the general source-observer problem follow a logical sequence in that they relate directly to the practical methods available for minimizing the noise-level at any given point due to an unavoidable source situated at another point. In the first place, a simple barrier may be erected, so that the sound reaching the point of observation in consequence of diffraction is effective ; but this is considerable when the barrier and distances from it are not somewhat large compared with the wave-length of the sound concerned. Carrying such a process one step further, the barrier may be converted into a complete enclosure, made of material of given transmission, and surrounding the source, with the internal surfaces having a certain absorption for sound. Reciprocally from the point of view of the fundamental theorem the point of observation may be situated in a similar enclosure. Such enclosures may have apertures in them either open or offering greater transmission than the main body of the enclosure. The detailed discussion then begins with a study of the barrier problem.

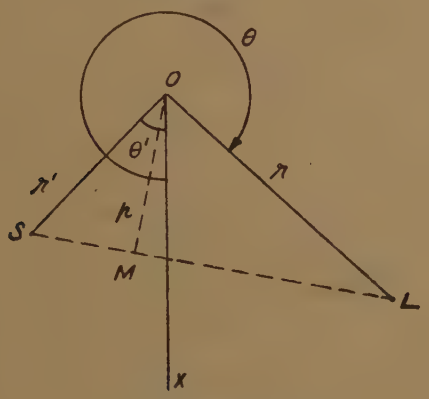
II. *Barrier Problem.*

As regards the question of a simple plane barrier, a solution of the wave equation $(\nabla^2+k^2)V=0$ for the velocity potential in a perfectly reflecting plane surface, where the condition $\frac{\partial V}{\partial s}=0$, is satisfied would provide an adequate model for the influence of a barrier of more concrete dimensions.

Exact solutions of the problem have been furnished for the case of an incident plane wave by Sommerfeld [1], and by Macdonald [2] for that of an incident cylindrical wave. Only approximate solutions based on certain assumptions regarding the distribution of normal velocity over the blank half-plane of the barrier have been given, notably by Lommel [3], for the case of a simple source at finite distance. Another form of approximate solution of a more convenient form is proposed here.

Let us consider a semi-infinite barrier and a source in its neighbourhood and consider a plane perpendicular to the barrier through the source as the principal plane, the representation of which is given in fig. 1, where OX represents the barrier and S the source, at $r'\theta'$ (or $r'\theta'z'$ if coordinates out of the principal plane are chosen), the condition at the barrier being $\frac{\partial V}{\partial \theta}=0$ for $\theta=0$ and $\theta=360°$.

Fig. 1.



Principal plane of barrier.

The point of observation L is considered to be $r\theta z$. The solution for such a case will involve $r', \theta', z', r, \theta, z$, and involve these quantities symmetrically as between the points L and S.

If we consider (as in Jeans, 'Magnetism and Electricity,' where the analogous problem of an electrostatic charge in the presence of a barrier is treated) that, in order to satisfy the conditions $\frac{\partial V}{\partial \theta}=0$ at $\theta=0$ and $\theta=2\pi$. the solution will have the form $F(r, r', z, z', \theta-\theta')+F(r, r', z, z', \theta+\theta')$. The expression $F(r, r', z, z', \theta)$ is an even function of θ.

In order to satisfy the reciprocal condition, the function must be of the form

$$F(rr', r+r', z+z', zz', \theta).$$

A solution has been given for the analogous problem of the temperature at any given point in space in the presence of an instantaneous point source of heat (Carslaw, 'Conduction of Heat') in the following form:

$$v=\frac{e^{\frac{-r^2+r'^2+\overline{z-z'}^2}{4\mathrm{K}t}}}{4\pi\ \mathrm{nK}t\ t}\Sigma\cos\frac{n\overline{\theta-\theta'}}{2}\,e^{\frac{-in\pi}{4}}\,\mathrm{J}_n\left(\frac{irr'}{2\mathrm{K}t}\right)$$

(added to the corresponding expression in $\theta+\theta'$).

If we substituted $t-t'$ for t in the above solution, and multiplied by $e^{ipt'}$, and then integrated with regard to t' from minus infinity to t, we should obtain the solution for a sinusoidal point source of heat in the neighbourhood of a barrier. Solutions in steady state of the equation of conduction should be identical with that of the wave equation if the substitution of k of the wave equation is made for the quantity $\frac{ip}{\mathrm{K}}$ in the equation of conduction.

Such operations as the above will not change the form in which the quantities r, r', z, z', and θ appear, so that we may say that the form of the function will be

$$v=\mathrm{F}\left\{rr',\ r^2+r'^2+\overline{z-z'}^2,\ \cos\frac{1}{2}\theta\right\},$$

which satisfies the reciprocal condition between r', θ', z', and r, θ, z. The solution given by Sommerfeld, when $r'=\infty$, is

$$v=e^{-ikr}\int_{-\infty}^{u}e^{\frac{i\pi v^2}{2}}dv+e^{ikr}\int_{-\infty}^{u_1}e^{-\frac{i\pi v^2}{2}}dv,$$

where $$u=\frac{2}{\pi}\sqrt{kr}\cos\frac{1}{2}(\theta-\theta'),\quad u_1=\frac{2}{\sqrt{\pi}}\sqrt{kr'}\cos\frac{1}{2}(\theta+\theta').$$

If we were to consider a solution of similar form, for the case of a source at finite distance, we should have, as the simplest analogous general form,

$$v=\frac{e^{ik\mathrm{R}}}{\mathrm{R}}\psi\left\{rr',\ r^2+r'^2+\overline{z-z'}^2,\ \cos\frac{1}{2}(\theta-\theta')\right\}$$
$$+\frac{e^{ik\mathrm{R}_1}}{\mathrm{R}_1}\psi\left\{rr',\ r^2+r'^2+\overline{z-z'}^2,\ \cos\frac{1}{2}(\theta+\theta')\right\},$$

where $$\mathrm{R}^2=r^2+r'^2+\overline{z-z'}^2-2rr'\cos(\theta-\theta')$$
$$\mathrm{R}_1{}^2=r^2+r'^2+\overline{z-z'}^2-2rr'\cos(\theta-\theta');$$

we may, if we choose, put ψ in the form

$$\psi\left\{\sqrt{\frac{rr'}{\mathrm{R}}}\cos\frac{1}{2}(\theta-\theta'),\ r^2+r'^2+\overline{z-z'}^2,\ \cos\frac{1}{2}(\theta-\theta')\right\},$$

and similarly for the corresponding function of $\theta-\theta'$. The simplest approximation which will reduce to the Sommerfeld solution for a source at infinite distance (plane wave) would therefore be

$$v=\frac{e^{ikR}}{R}\int_{-\infty}^{u} e^{-\frac{i\pi v^2}{2}}dv+\frac{e^{ikR_1}}{R_1}\int_{-\infty}^{u_1} e^{-\frac{i\pi v^2}{2}}dv,$$

where

$$u=\frac{2}{\sqrt{\pi}}\sqrt{\frac{krr'}{R}}\cos\frac{1}{2}(\theta-\theta'),$$

$$u_1=\frac{2}{\sqrt{\pi}}\sqrt{\frac{krr'}{R_1}}\cos\frac{1}{2}(\theta+\theta').$$

From the practical point of view of using this relation to present the results of the calculations of intensity within a sound "shadow," such a form of approximation is exceedingly convenient, for we may represent the varying distances from the barrier of source and observer in terms of a single quantity, if we confine our attention to the principal plane.

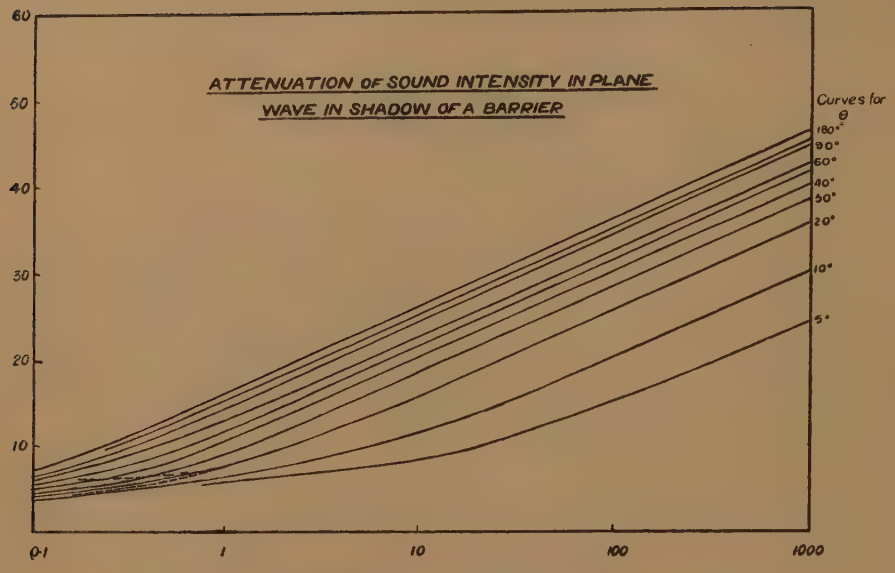
In fig. 1 in the triangle SOL, if we represent by p the perpendicular from O, the barrier top, on to SL the line of sight of the source from the point of observation, the area of the triangle gives the relation $\frac{rr'}{R}=-\frac{p}{\sin}(\theta-\theta')$, the negative sign arising since the angle $\theta-\theta'$ is the reflex angle SOL.

The quantity p is the effective height of the barrier top above the line of sight from the point of observation to source.

The curves of fig. 2 give in decibel measure the attenuation of the intensity at a point in the field over that in the undisturbed incident plane wave as calculated from the Sommerfeld formula. Inside the region of "shadow" the second term of the formula makes only a minor contribution to the intensity. This second term is the one depending on the actual position of the line of the barrier, the first being concerned only with the distance from the barrier top and the angular distance into the geometrical sound "shadow." The second term depends, furthermore, on the barrier surface conditions. For reflexion of the type giving $\frac{\partial V}{\partial r}=0$ it is positive, for the condition $V=0$ it is negative, and for a completely absorbent barrier, defined as one giving no reflexion, it is zero. Consequently, the value of the ratio of intensity in the "shadow" over that in the incident wave has been plotted against the distance in wavelengths from the barrier top, for varying angular distances into the "shadow," for the case of an absorbing barrier such as given by the first term only.

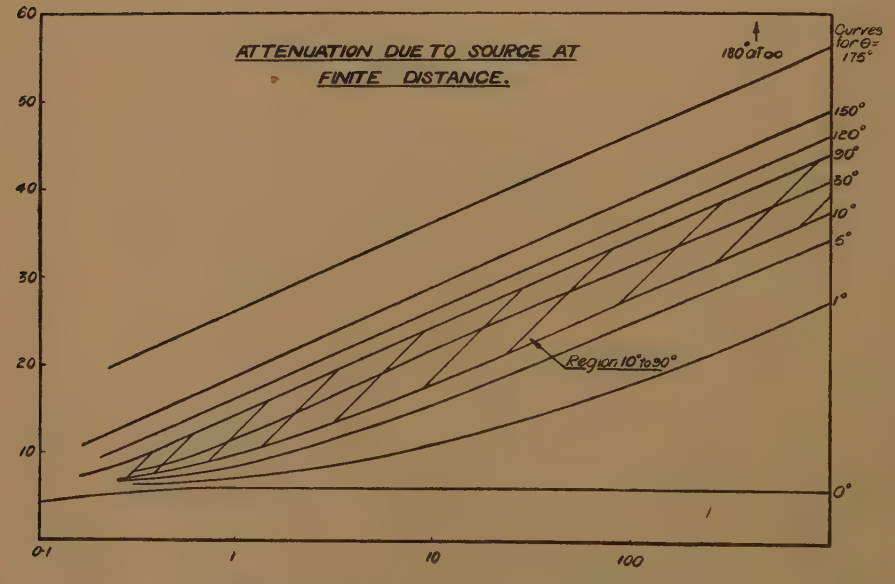
The diagram, fig. 3, translates these values into the terms applicable to a source at any distance, giving the attenuation in the principal

Fig. 2.



Wave-lengths from barrier top.

Fig. 3.



Effective barrier height in wave-lengths.

plane at any point in the "shadow" due to the barrier, over the intensity at the point due to the original spherical wave from the source. The curves are of attenuation against equivalent barrier height in wave-lengths, for varying angular distances into the sound "shadow," this latter variable, it may be noted, making but few decibels change for quite an appreciable change in angle.

III. *Enclosure of the Source.*

The next practical measure in minimizing the intensity of a sound at a specified point is by means of surrounding the source with some form of enclosure whose walls have a low transmission coefficient for sound energy. Let us suppose we have a source of energy output e_s, within an enclosure, and Q is the total energy incident on the enclosure walls internally, arising from the source and reverberation within the enclosure. Then the total external energy, with walls of a transmission coefficient β will be $e_x=\beta Q$. If the absorption coefficient for incident energy of the walls is α in the absence of transmission, and $\alpha+\beta$ with transmission, then for equilibrium within the enclosure

$$e_s=(\alpha+\beta)Q \quad \text{and} \quad e_x=\beta Q,$$

giving

$$\frac{e_x}{e_s}=\frac{\beta}{\alpha+\beta}.$$

So that these simple considerations lead to the results that if $\alpha=0$, $e_x=e_s$, *i. e.*, that the total external sound energy is different from that emitted internally only by the amount absorbed on the walls or in transmission, an obvious enough result, but related by the above simple formula to the absorption of the surfaces and the transmission through the walls.

The ratio e_x/e_s will, leaving on one side the peculiarities of distribution, be a measure of the reduction of intensity of the energy due to the enclosure of a source, provided always we keep the energy output from the source or sources constant. This may be by no means the case, as the act of enclosing a source of given characteristics may affect its energy output as compared with that in free space, and furthermore, the output may vary again with the enclosure absorption. Before returning to this point, however, it may be of interest to consider the modification of this simple formula for the case of non-uniformly distributed absorption and transmission, and absorption extra to that situated on the actual enclosure walls. As more detailed assumptions about the conditions inside the enclosure as regards the distribution of the energy and the mechanism of the building up of the final intensity are required, the results may be related to the Sabine formulæ for the intensity and time constant within the enclosure.

Let us assume that the energy from the source may be considered separate from that which has been diffusively reflected, the latter being assumed uniformly distributed both in position and direction of propagation, and assume that E_1 is the sound energy falling per unit area on any surface in the enclosure direct from the source, and E_0 that falling on the same surface element per unit area from the diffused sound.

Then for equilibrium within the enclosure

$$e_s = \int_s \alpha(E_0+E_1)\,dS$$

$$= E_0 \int_s \alpha\,dS + \int_s \alpha E_1\,ds$$

(α being the absorption coefficient including the effects of transmission).

If we define $\bar{\alpha}$ by $\bar{\alpha}\int_s E_1\,dS = \int_s \alpha E_1\,dS$

(S being the total surface area in the enclosure).

Then $e_s(1-\bar{\alpha}) = E_0 \Sigma\alpha S$, $\Sigma\alpha S$ being the usual way of writing the total absorption, written above $\int_s \alpha\,dS$. The quantity $\bar{\alpha}$ is the mean absorption coefficient weighted according to the intensity of the sound falling on it direct from the source.

Now if e_x is the amount transmitted outside, and S_1 is the total surface of the external walls, then

$$e_x = \int_{S_1} \beta(E_0+E_1)\,dS_1$$

$$= E_0 \int_{S_1} \beta\,dS_1 + \int_{S_1} \beta E_1\,dS_1$$

Thus, as above,

$\int_{S_1} \beta\,dS$ is written $\Sigma\beta S_1$, the "total-transmission,"

and

$$\bar{\beta}\int_{S_1} E_1\,dS_1 = \int_{S_1} \beta E_1\,dS.$$

Then we get

$$\frac{e_x}{e_s} = (1-\bar{\alpha})\frac{\Sigma\beta S_1}{\Sigma\alpha S} + \bar{\beta}.$$

This extends the simple formula given previously to the more general type of enclosure, and, provided the assumption of the invariability of e_s were to hold, will give a measure of the reduction of the external energy due to enclosure. It may be interesting before leaving this matter of the internal conditions in the enclosure with a constant energy source to relate the above formula to the Sabine law.

Let ρ=energy density of the reverberated sound, as distinct from that direct from the source.

Then in time dt, with an enclosure of volume V,

$$V\,d\rho = e_s\,dt - \int_S (E_0+E_1)\alpha\,dS\,.\,dt.$$

So, similarly,
$$\frac{d\rho}{dt} = \frac{e_s}{V}(1-\bar{\alpha}) - \frac{E_0}{V}\Sigma\alpha S.$$

Now, by a well-known formula, if c=velocity of sound $E_0 = \frac{\rho c}{4}$. So that
$$\rho = \frac{4e_s(1-\bar{\alpha})}{c\Sigma\alpha S}\{1-e^{-t/T_1}\},$$

where $T_1 = \frac{4V}{c\Sigma\alpha S}$, the Sabine formula for the time constant of decay and consequently of build-up of sound reverberated in an enclosure. Modifications have been proposed to the Sabine formula because it gives the result that the time constant of an enclosure with walls of unit absorption coefficient is finite. It may be noted in passing, however, that in the above model of the physical process whereby the direct and reverberated sound energy are considered separately, the Sabine law applies to the latter only, and the difficulty of the finite time constant for high absorptions does not arise owing to the fact that since under such conditions $\bar{\alpha}\to 1$, and so the intensity of the reverberated sound approaches zero. In practice, too, the distribution of reverberated sound even in an irregular enclosure, cannot attain uniformity at high absorption, since only one reflexion is effective, and the distribution is bound to be a specific function of the enclosure shape to a certain extent.

According to the formula for ρ, the energy-density of the reverberated sound, if e_s remains constant, the sound in the enclosure will " build-up " to an extent dependent on the total absorption within it. For instance, if e_s is directly incident from the source on the enclosure walls, then from this formula $E_0S_1 = \frac{e_s(1-\bar{\alpha})S_1}{\alpha S}$ will be incident from the reverberated sound, indicating a " build-up " of the order of $\frac{1}{\alpha'}$, where α' is the mean absorption coefficient relative to the external walls. For α' a small quantity, this indicates a build-up of an indefinitely large amount. That this is not universally true may be shown in the case of a hollow sphere with perfectly reflecting wall with a source at its centre when the quantity corresponding to the " reverberated " sound has the value, if the emitted sound is considered to be a wave of unit amplitude,

$$\frac{ka\tan ka+1}{\tan ka-ka}, \quad \text{where } k=\frac{2\pi}{\text{wave-length}}, \quad a=\text{radius of sphere.}$$

The natural modes of the sphere in spherical symmetry are given by $\tan ka = ka$, so that a large build-up is dependent on appropriate proximity of the frequency to a natural mode. Consequently, the assumption of a source of invariable energy output is open to question, as the energy emitted will tend to be a function of the dimensions and shape of the source-enclosure system, as well as of the total absorption within it. It is, however, for the purpose of generalizing the problem, the best practical one to make for the following reasons :—

(i.) It gives rise to conclusions in line with the facts of high build-up at low absorption for enclosures small compared with the wave-length of sound.

(ii.) In similar case, when the enclosure is large compared with the wave-length, a relatively high build-up results for low absorptions.

(iii.) When the natural frequency of one of the modes of the enclosure coincides with one of the frequency components of the sound emitted, high build-up results.

The greater the size and the more pronounced the irregularity of the enclosure boundary, the greater the number of such modes. This is an alternative statement of the usual assumption that uniform diffusion of the sound energy is dependent on a sufficient irregularity in the reflecting walls and other surfaces in the enclosure, and it will be shown later that this assumption involves an energy output independent of enclosure characteristics.

IV. *Enclosure of the Observer.*

The reciprocal problem to that of an enclosed source according to the Helmholtz principle is that of the enclosed observer, and this is the third practical form of obstacle to be met with interposed between observer and source. Let us consider the relation between the sound level due to a distant source at a point in free space and that inside an enclosure with an aperture facing the incident sound, having an effective area S. Let W be the energy incident per unit area on the aperture, which will, therefore, behave as a source of energy W_0S_1. As in Section (II.), we calculate the diffused energy density as

$$\rho = \frac{4WS_1(1-\bar{\alpha})}{c(S_1+S_0)}$$

if S_1+S_0=total enclosure absorption, S_1 due to the aperture, and S_0 due to the walls.

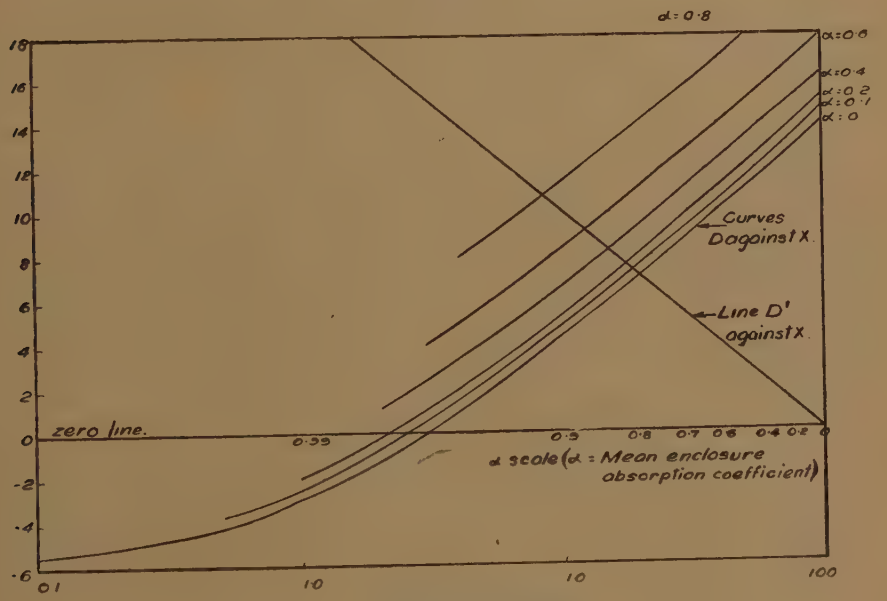
The energy density in the directed incident sound will be given by W/c.

Hence the ratio $\dfrac{\text{Energy density in incident sound}}{\text{Energy density in diffused component}}$

$$= \frac{4S_1}{S_1+S_2}(1-\alpha).$$

The direct sound from the aperture into the enclosure will, either (a) in the case of an aperture smaller in dimension than the wave-length of sound, radiate approximately spherically, or (b) be confined within definite beams, in which case a general statistical treatment is impossible. It

Fig. 4.



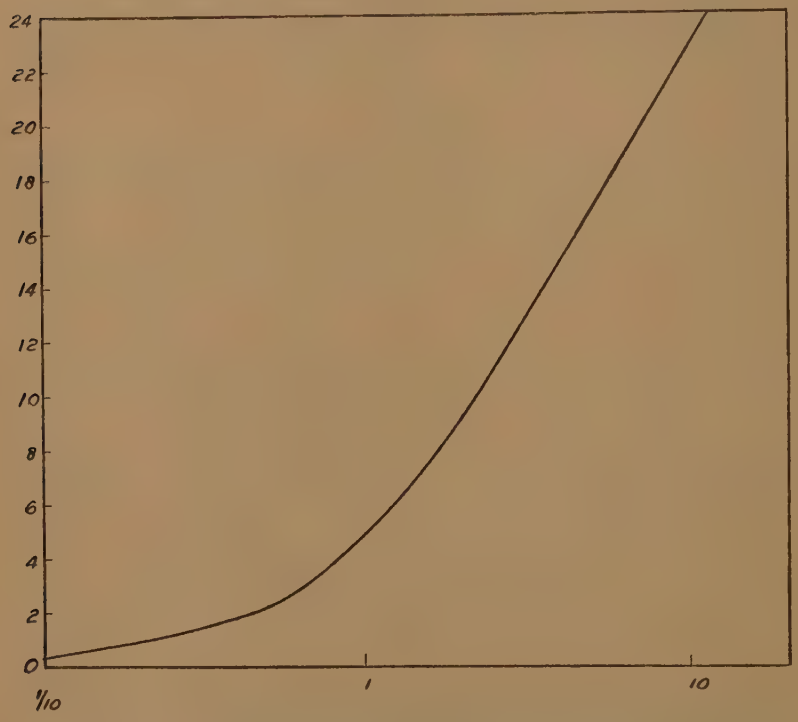
$$\times = \frac{\text{Total absorption}}{\text{Effective aperture area}}.$$

can be shown that the attenuation due to the spherical spreading of this energy with distance rapidly gives rise to an attenuation greater than that given by the enclosure absorption, leaving the component given by the above expression as the effective measure of the energy level within the main body of the enclosure as a whole, compared with that in the incident sound.

The curves of fig. 4 show the variation of the intensity level of the reverberated sound in the enclosure in terms (decibel measure) of that in the aperture, against the ratio of the total absorption in the enclosure

divided by the effective aperture area as given by the formula above, for various mean absorption coefficients of the enclosure. Fig. 5 gives for comparison purposes the decibel measure of a typical inverse square law distribution, namely, the formula for the potential or pressure function for the case of flow through a circular aperture, with distance from the aperture, the formula being, when wave motion effects are neglected,

Fig. 5.



r=Distance from aperture, in terms of the mean aperture radius.

$p=\frac{2p_0}{\pi}\tan^{-1}\frac{a}{\sqrt{r^2-a^2}}$. The values of p/p_0 in terms of r/a are given in decibel measure as showing that for most practical purposes the direct sound at appreciable distances from the aperture is small compared with the reverberated.

V. *Application of Helmholtz Theorem.*

Before concluding, let us consider the application of the reciprocal principle to the enclosure, external observer problem. Consider a source that in free space would emit an energy e_s. If placed within an enclosure

of given absorption and transmission characteristics, the intensity at an external point will by the principle be equal to that inside the enclosure due to the same source at the external point.

Consider an enclosure, part of whose absorption and all of whose transmission arise from uniformly distributed small apertures round the surface, of total effective area S_1. For apertures of small area, transmission of energy is independent of the angle of incidence of the sound, so that if R=distance of a source (large) from the enclosure that emits e_s energy per second when placed in free space, then the energy per second entering the enclosures from what are virtually sources on the walls$=\frac{e_s S_1}{4n\mathrm{R}^2}$, and if S_0=total absorption within including transmission, the resultant energy density at a point inside will be, by the foregoing,

$$\frac{4}{cS_0} S_1 \frac{e_s}{4n\mathrm{R}^2}.$$

(Owing to the distribution of the sources, both direct and reverberated energy are regarded as completely diffused within the enclosure, so the factor $(1-\bar{\alpha})$ is missing from this expression.)

If now we have the "same" source (as defined by the limit of the potential at the source) in the enclosure, the energy emitted by it per second would be a different quantity, say e'_s. Then if as before β is the transmission coefficient and α the absorption, the external energy would be

$$\frac{e'_s\beta}{\alpha+\beta}=\frac{e'_s\beta S}{S_0},$$

where S is the total enclosure surface. Then if radiation is approximately spherical, the energy density at distance R would be

$$\frac{e'_s\beta S}{cS_0}\cdot\frac{1}{4n\mathrm{R}^2}.$$

We may note in passing that since the apertures contributing to the quantity β transmit independently of the angle of incidence, the quantity βS will equal $2S_1$ for uniformly diffused radiation.

Now the Helmholtz theorem merely states that the velocity potentials or pressures at the two reciprocal points are equal. Since the average energy density around a point, as regards time and space, is proportional to the square of the pressure amplitudes, this suggests that, as regards order in any case, we may equate the energy densities instead of pressures, and so we obtain $e'_s=2e_s$, suggesting that the fact of enclosure is to double the energy emitted, in a similar way to the doubling of the emitted energy when a source is brought close to the reflecting wall.

This last section indicates that if we can speak of absorption and transmission coefficients for sound energy, independent of its phase and distribution, and of diffused sound energy, the question of the effect of an enclosure impedance on the energy emitted from a source does not arise, and that if we can obtain a perfect diffusion of the sound energy from a source, its emission is of the same order as in free space.

The problems discussed were suggested to the author by Mr. A. J. King and Mr. B. G. Churcher, to whom he wishes to express his thanks for assistance in preparing the paper. He also wishes to thank Dr. A. P. M. Fleming, Manager, Research and Education Departments, and Director of Metropolitan-Vickers Electrical Co. Ltd., for permission to publish the paper.

References.

(1) Sommerfeld, *Gött. Nach.* (1), p. 338 (1894) ; (1) p. 267 (1895). *Math. Ann.* xlvii. p. 317 (1896).
(2) Macdonald, Proc. Lond. Math. Soc. (2) xiv. (1915).
(3) Lommel, *Abh. der K. Bayer. Akad. der Wissen.* xv. pp. 233, 531 (1884–1896).

XXI. *Measurements of the Noise from Aerofoils and Streamline Wires.*

By W. F. Hilton, Ph.D., A.R.C.Sc., B.Sc., D.I.C., A.F.R.Ae.S.
(of the National Physical Laboratory) *.

[Received August 4, 1939.]

§1. *Introduction.*

The silencing of aircraft reduces to the fundamental problem of the noise made by the passage of an aerofoil section through air, which includes noise made when gliding, and airscrew noise. Exhaust noise can already be effectively silenced. In ordinary wind-tunnels the noise from the airscrew inducing the flow frustrates any attempt to measure or even to hear direct aerofoil noise. Thus much previous work has been concentrated on the more easily observed airscrew noise, which introduces difficulty of interpretation.

Compressed air induction tunnels, such as the 1 ft. diameter high speed tunnel at the National Physical Laboratory, shown in fig. 7, are suitable for the study of noise produced by airflow over streamline bodies, due to the absence of an actuating airscrew. The tunnel alone, with no model, makes a hissing sound which is easily distinguished from the musical tones produced by some aerofoils. As will be seen from figs. 1–3, the tunnel is even quieter at the top speed of about 900–1000 ft./sec., due to the formation of a shock-wave across the tunnel, which limits the speed and also prevents the noise from the compressed air jets from coming down the tunnel and out of the intake, the noise being literally blown up the tunnel. This is an exception to the very general rule that the noise produced by movement should increase with the speed of movement.

These experiments were originally undertaken in order to reduce the noise made by the streamline shapes used to support apparatus in the air-stream, but the results are of general interest.

§2. *Experimental Method.*

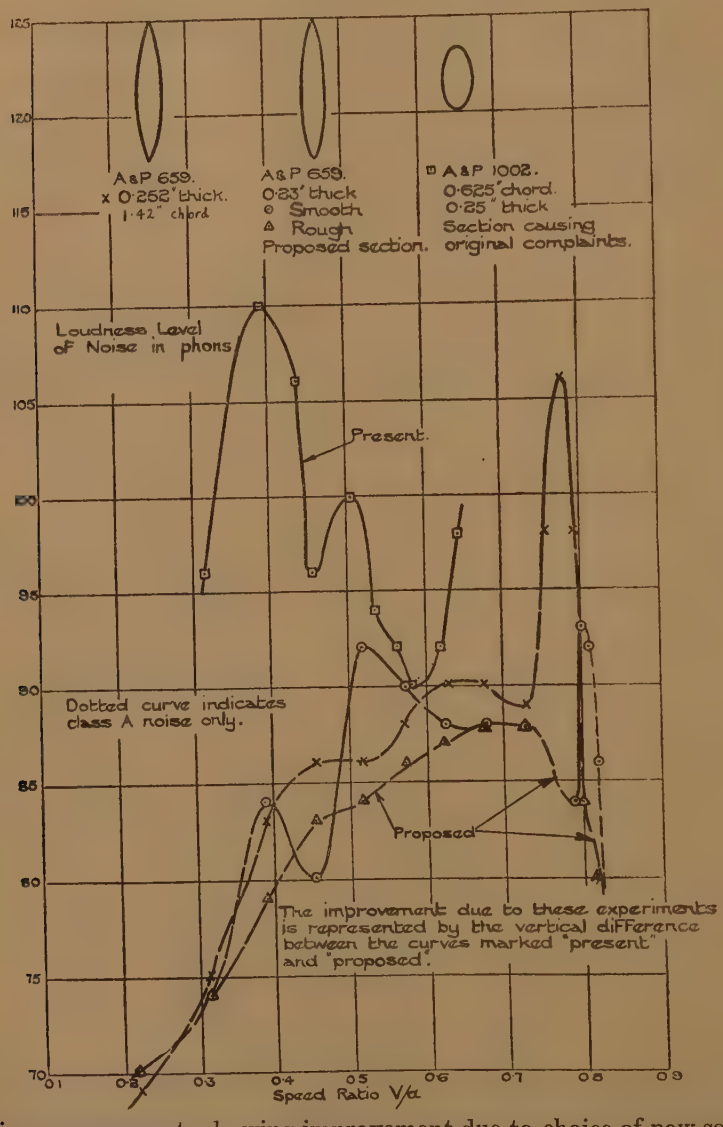
All sections tested extended the whole width of the tunnel (one foot) with encastré ends. In every case the noise was measured by the same observer (the author), using the same Barkhausen audiometer applied

---

* Communicated by E. F. Relf, F.R.S.

to each ear in turn, and adjusted to equality of loudness, except in the case of the very loudest sounds, which had to be estimated by the method

Fig. 1.

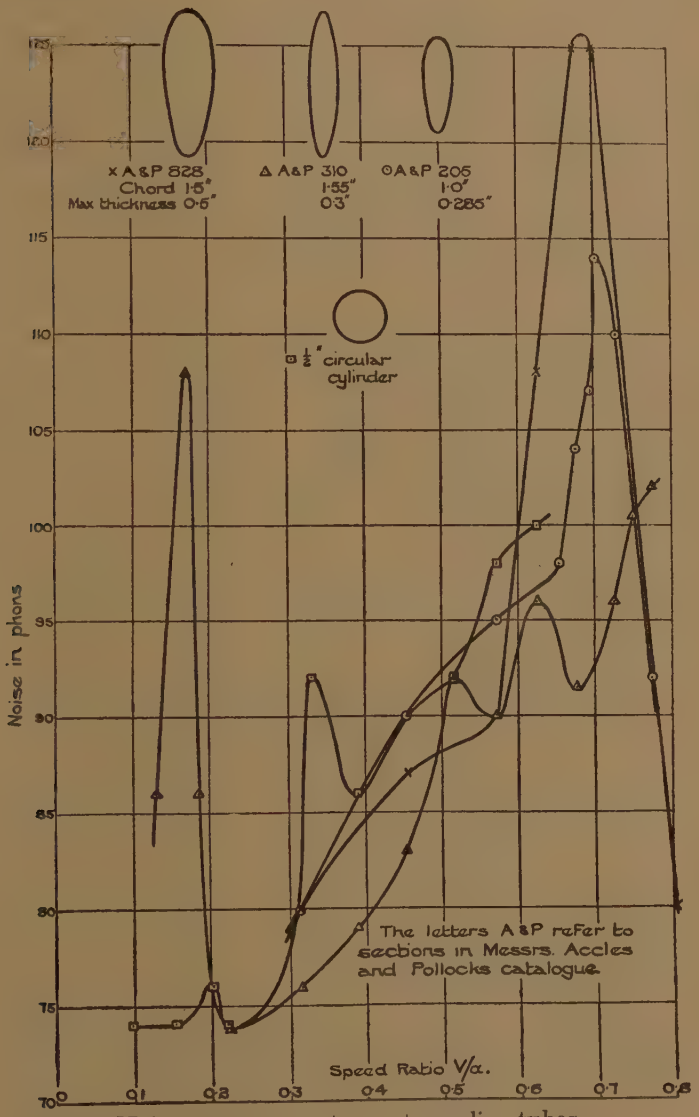


Noise measurements showing improvement due to choice of new section.

of masking the standard noise. The noise was measured at a fixed point in the building in which the high speed tunnel was situated. Some

measurements were also made outside the building, but seem to indicate nothing beyond what would be expected, except that it is not perhaps

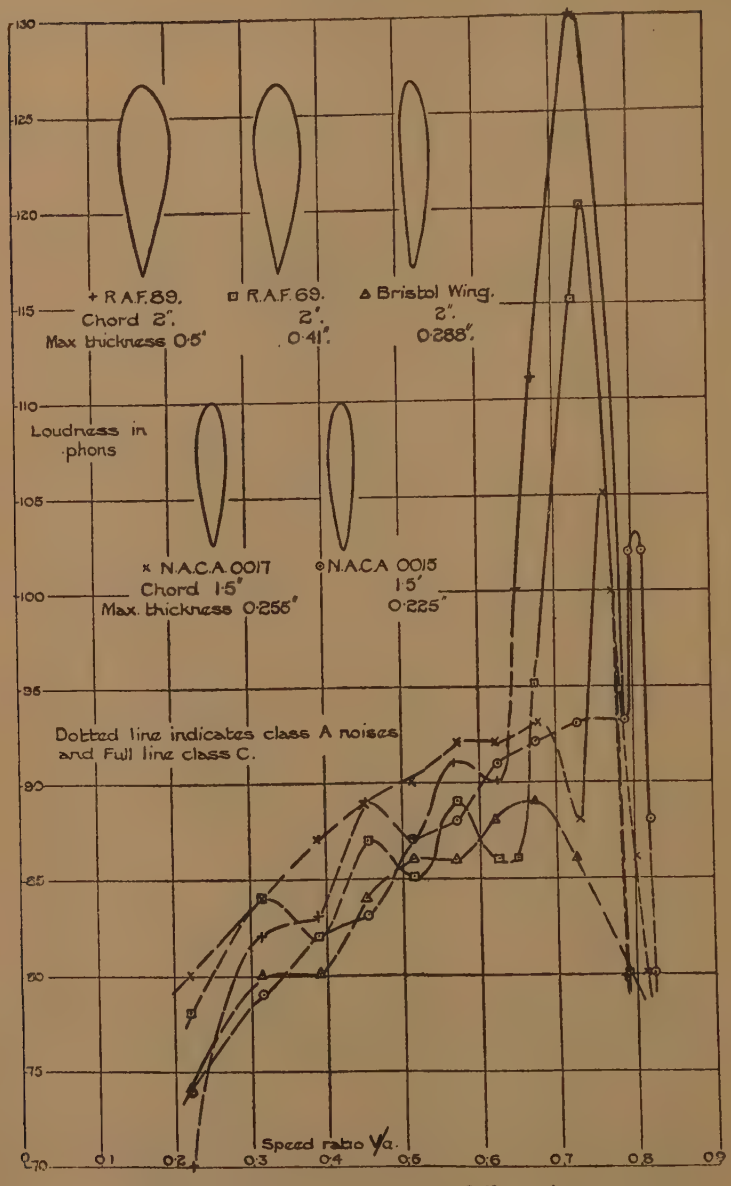
Fig. 2.



Noise measurements on streamline tubes.

obvious that the top-speed condition of the tunnel would still be relatively silent, as was found to be the case.

Fig. 3.



Noise measurements on aerofoil sections.

§3. *Classification of Results.*

Very early in these experiments it was found that the noises produced fell into three distinct classes :—

(*a*) A rushing noise, produced by the tunnel itself, which could be rendered practically inaudible by wearing cotton-wool earplugs with a pellet of plasticine in the centre. Without earplugs the noise was not unpleasant, but prolonged exposure (say 1 hour) was sometimes followed by singing noises in the head.

(*b*) The second class of noise was a high pitched, almost supersonic, whistling or screeching sound of very definite frequency, occurring at certain speeds in the range 0 to 770 ft./sec., and often made more unpleasant by violent beats having a frequency of about 5 per second. The noise was probably produced by regular fluctuations in the flow at critical speeds due possibly to the formation of a Kármán vortex street, and the beats may have been caused by slight variations of conditions along the span.

Although the noises of class (*b*) were not so loud as some falling into category (*c*) below, they were considered to be more objectionable by the majority of observers, due to their very high pitch. This raises an interesting physiological point, for if this whistling noise from a section was suppressed, most observers agreed that the noise ceased to be objectionable, even though there was no appreciable change of loudness as measured with an audiometer.

It was discovered that noises of this type could be eliminated by the simple procedure of roughening the surface of the section. For the purpose of rapid tests on metal sections, this was done by smearing the surface with seccotine, which dried quickly in the wind, and which could be washed off again at the conclusion of the experiment. For more permanent silencing of a section, it was found that a groove 0·005 in. deep a quarter chord back from the nose on both sides of the section was sufficient to silence class (*b*) noises entirely.

(*c*) The third class of noise was that made by sections of more than 15 per cent. thickness ratio (*i. e.*, thickness expressed as a percentage of chord), which emit with startling suddenness, at a speed of about 800 ft./sec., a very loud steady noise resembling that of a factory hooter. The loudness was beyond the range of the Barkhausen audiometer employed, but was estimated as about 120 phons, and the frequency as about 200 to 300 cycles/second, often falling a semitone or so with rising speed, and fading out altogether when top-speed was attained.

This type of noise was reduced by roughening the surface, as is described in (*b*) above, but was not normally eliminated by such treatment. The noise can be suppressed by decreasing the thickness ratio of the section employed to 15 per cent. or less. The various sections tested are shown together with the noise measurements in figs. 1 to 3. It was found that the 40 per cent. thick elliptic cylinder originally used as

a supporting bar for traversing pitot tubes made the most objectionable noise encountered in the whole series of experiments, and in fig. 1 it is compared with the most silent section found.

§ 4. *Other Observations.*

Before the method of roughening the surface to obtain silence was discovered, various other tests were made, and the results are given below.

1. *Effect of Material from which Section is made.*—Two sections, both N.A.C.A. 0020–63 *, 2 in. chord, identical except that one was made of brass and the other of Tufnol, were tested, and no measurable difference could be found in the noise produced.

2. *Scale Effect.*—The 2 in. brass 0020–63 section was compared with a similar section of steel having a chord of 1·2 in., and it was found that the speeds for the 1·2 in. chord section to produce the same noise as the 2 in., had to be some 3 per cent. higher. It should be noted that whilst the loudness of these sounds could not be measured with accuracy, the speed at which they occur is very sharply defined to within a few feet per second. Noises of class (*c*) are created with startling suddenness.

3. *Effect of Angle of Incidence.*—Only slow changes with incidence were observed. A change of 4° was sufficient to eliminate the very loud howling noise emitted by 0020–63 section, and to replace it by the screech normally found at higher speeds.

4. *Effect of Squeezing the Section.*—Streamline tubes can easily be squeezed between rollers to any reasonable extent. It was found that class (*c*) noises tended to become weaker and finally to disappear at a thickness ratio of about 16 per cent., whilst class (*b*) noises tended to increase unless the rolling was done carelessly, which would roughen the surface and consequently diminish the sound (see fig. 5).

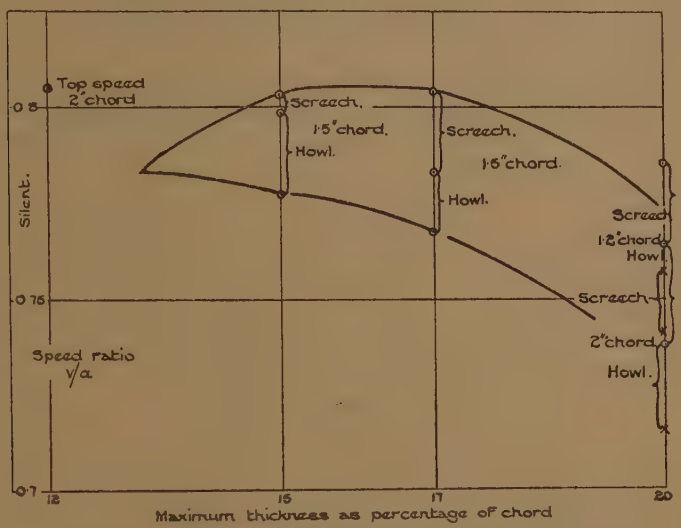
5. *Effect of Thickness Ratio.*—Four symmetrical sections calculated from a single basic section were tested, their maximum thickness being 12, 15, 17, and 20 per cent. of the chord respectively. The results are summarized in fig. 4, from which it will be seen that the class (*c*) noise persists rather below the limit of 16 per cent. thickness ratio found in other cases.

6. *Effect of adding a Sharp Trailing Edge.*—Little effect was observed when a sharp trailing edge was fitted to a streamline section, although it had been noticed that of seven aerofoils tested, not one emitted a class

* Details of section given in N.A.C.A. Report, no. 492 (1934).

(*b*) noise. Throughout this paper aerofoils are distinguished from streamline bars by having a sharp trailing edge.

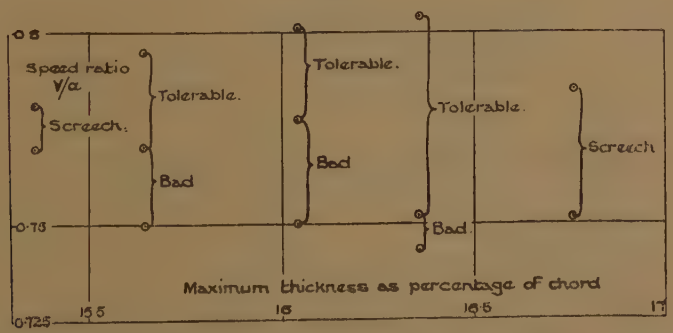
Fig. 4.



Effect of thickness ratio of aerofoils of 0020–63 class.

7. *Effect of Noise on Readings.*—In one particular experiment, the flow past the bar supporting a thermocouple in the tunnel was critical, a noise being generated at one time, and not at others. It was observed

Fig. 5.



Effect of squeezing A & P section 659.

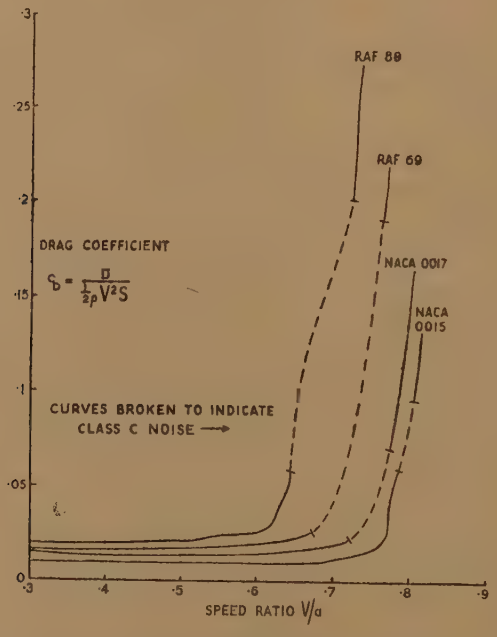
that the thermocouple was warmer with the noise than without it. Interference of noise with the readings makes it doubly important to eliminate the noise at its source. The effect may have been due to

mechanical vibration of the thermocouple, either direct, or due to the impinging of large amplitude sound waves. Similar interference with total head readings was also observed.

8. *Effect of Sharp Leading Edge.*—An aerofoil was tested trailing edge foremost, and gave rise to very prominent class (*b*) noises.

9. *Effect of Ordinary Stall.*—The breakaway and unsteady flow associated with the ordinary stall gave rise to very distinctive sounds, much louder than the unstalled flow. In a critical case (N.A.C.A. 0012–63, 12° incidence at a speed of 430 ft./sec.) the noise, and presumably also the

Fig. 6.



flow, alternated between stalled and unstalled in a very marked way.

This stalled noise really represents a fourth class of noise, and has been observed in gliding flight.

§5. *Origin of the Noise.*

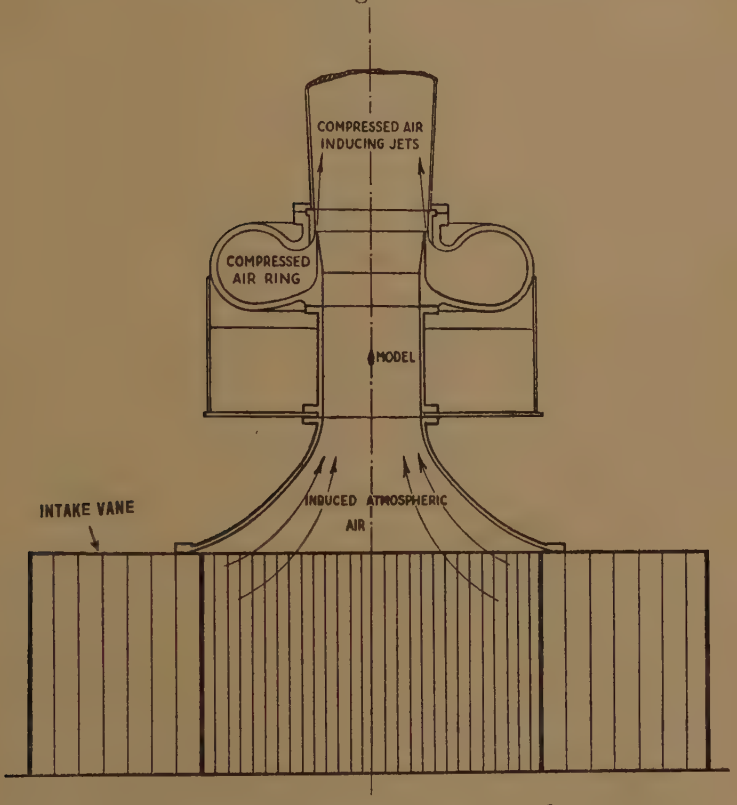
The possibility of the tunnel itself interacting and contributing to the noise must not be overloooked. The class (*a*) rushing noise made by the tunnel alone stops observation on class (*a*) aerofoil noise. This noise is probably associated with the boundary layer.

Class (*b*) noises have been observed in flight, as the singing of streamline wires and struts, and also from the radiator slats of some motor cars.

This would seem to show that these noises are produced independently of the tunnel.

Class (*c*) noises are produced at speeds at which the aerofoil forms a shock-wave, as may be seen from fig. 6, which shows the well-known increase of drag coefficient at the shock-stall, caused by the formation of a shock-wave. These curves are broken over the range in which class (*c*) noises were audible, and the connexion between the two pheno-

Fig. 7.



Section of high speed wind tunnel.

mena is fairly evident. It was found to be possible to look up the wind tunnel whilst it was running, by means of a mirror, and to see the shock-wave as a brownish coloured appendage to the aerofoil. This shock-wave, in the case of N.A.C.A. 0020–63 section, was observed to be in a state of vibration at exactly the speeds at which the class (*c*) noise was generated. Other aerofoils were not observed, but the connexion between the noise and the vibration of the shock-wave was conclusive.

A possible explanation is that the shock wave was critical, so that any

slight addition to its amplitude, due to a disturbance arriving from behind, increased its propagational speed, enabling it to move forward along the aerofoil surface into a region of lower velocity, and thus to propagate down the tunnel. The period of this series of events would give the frequency of the sound. With increasing tunnel speed, a greater amplitude would be required for the waves to escape, taking longer to build up and giving a lower frequency, until above a certain speed the waves could no longer escape. Photographs of shock-waves breaking away from an airscrew blade, and getting ahead of it, have been published by the author *. The distance between successive wavelets does not give the wave-length of the sound in free air, but allowance must be made for the decrease in propagational speed due to the air-stream. The tone wave-length in still air may thus be many times greater than that shown on a photograph.

It is not certain whether the comparative silence at top speed is due to the aerofoil generating no class (*c*) noise, or to the blanketing effect of a shock-wave across the tunnel.

Summary.

Observations and measurements of noise in the high speed wind tunnel at the National Physical Laboratory, show that the noises can be classified into three main classes, designated (*a*), (*b*), and (*c*) for convenience. Class (*a*) consists of rushing noises similar to those produced by the empty tunnel, which can be considerably reduced by using ordinary acoustic absorbents, or rendered unobjectionable to the experimenter by wearing special earplugs.

Noises of class (*b*) are whistling sounds of high, but very definite musical pitch, occurring at any speed below the shock-stall, and causing the greatest annoyance. It is found that this class of noise may be entirely eliminated by roughening the surface producing them. The singing noise made by exposed streamline wires on some aeroplanes could probably be eliminated by the method of surface roughening described in this report.

Class (*c*) noises occur on sections having a thickness greater than 15 per cent. of the chord, at about the shock-stalling speed. The noise is very loud (110 to 120 decibels), relatively low-pitched, and only partially silenced by roughening the surface.

In addition, observations are recorded on the effects of material (none), scale (small effect), incidence (small effect), thickness (pronounced effect on class (*c*)), and on sharp leading and trailing edges (marked effect on class (*b*)).

---

* See fig. 6, Proc. Roy. Soc. clxix. pp. 174–190 (1938). "The Photography of Airscrew Sound-waves." By W. F. Hilton.

XXII. *Some Measurements of High Frequency Permeability.*

By Willis Jackson, D.Sc.,
Electrotechnics Department, Manchester University *.

[Received July 10, 1940.]

Introduction.

The behaviour of magnetic materials at very high frequencies has attracted both theoretical and experimental attention over a long period, and the literature on the subject is considerable. Unfortunately, few accurate measurements have been carried out over a wide range of frequency by the same worker, and the constitution of the materials studied has usually been so indefinite that only limited fundamental progress has been made. It is not claimed that the measurements here presented constitute a contribution in this direction. They were made several years ago at a time when Dr. E. B. Moullin and the author had in mind a serious study of the subject. Unfortunately, circumstances did not permit a continuance of the work, and these preliminary results are presented now for what they are worth because of the increasing use of materials of the type tested within the range of frequency explored.

Form of the Test Specimens and the Method of Measurement.

The test specimens consisted of two straight round rods, about 18 in. in length and $\frac{1}{4}$ in. in diameter, of each of the following materials, "Mumetal," "Stalloy" (4 per cent. silicon), mild steel, and 6 per cent. tungsten steel hardened and unhardened. The surface of each was freed from contamination and irregularities as much as possible consistent with gentle handling. Each pair of rods was clamped in turn between copper blocks A, B, C, D (fig. 1), to form the sides of a rectangular coil 6 in. in width between rod centres. The ends of the coil consisted of copper rod of the same diameter.

The coil was connected across the terminals of a low-loss variable air condenser of capacitance range requisite for adjustment of the circuit so formed to resonance over the frequency range 10^6–10^7 cycles per sec. The coil was coupled magnetically to a small valve oscillator and the resonance curve at each test frequency delineated on a thermionic voltmeter, connected in parallel with the condenser, by varying the frequency

---

* Communicated by Dr. E. B. Moullin.

of the oscillator about the resonance value. The resistance of the circuit at the resonant frequency may be derived from this resonance curve through the equation

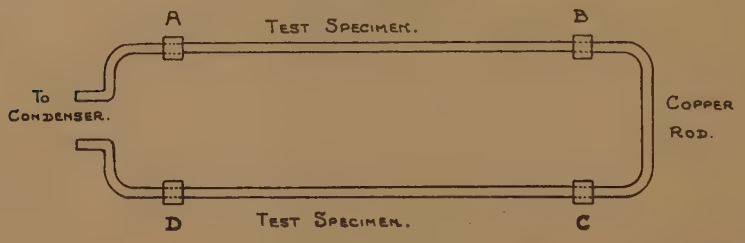
$$R = \frac{1}{2\pi f C} \frac{\Delta f}{f} \text{ ohms},$$

where $\Delta f/f$ is the fractional width of the resonance curve at $1/\sqrt{2}$ of its height, and C the tuning capacitance in farads. Full particulars of this method are given elsewhere [1].

This resistance value includes contributions due to the coil, the condenser, and the thermionic voltmeter, and if, as is desired, that due to the coil, and in particular that due to the test specimens, is to be separated, it is necessary to replace the latter by similar rods of accurately calculable resistance and of dimensions such that the resonant adjustment of the circuit, with C unchanged, is retained. This means, in effect, that the coil inductance must be the same in each case.

Fig. 1.

(Not to scale.)



The virtue of the coil construction described lies in the fact that the separation between the coil sides AB, CD, is sufficiently large to render the "proximity effect" between them negligibly small, so that they can be treated as isolated conductors.

The well-known relation [2] for the high-frequency resistance R_f of isolated conductors of circular section is as follows :—

$$R_f = R_0 \frac{\sqrt{2}Z + 1}{4}$$

where R_0 is the D.C. resistance and $Z = \sqrt{\frac{2\pi^2 d^2 f \mu}{\rho}}$, d being the diameter in cm., ρ the specific resistance in c.g.s. units, and μ the effective permeability of the material.

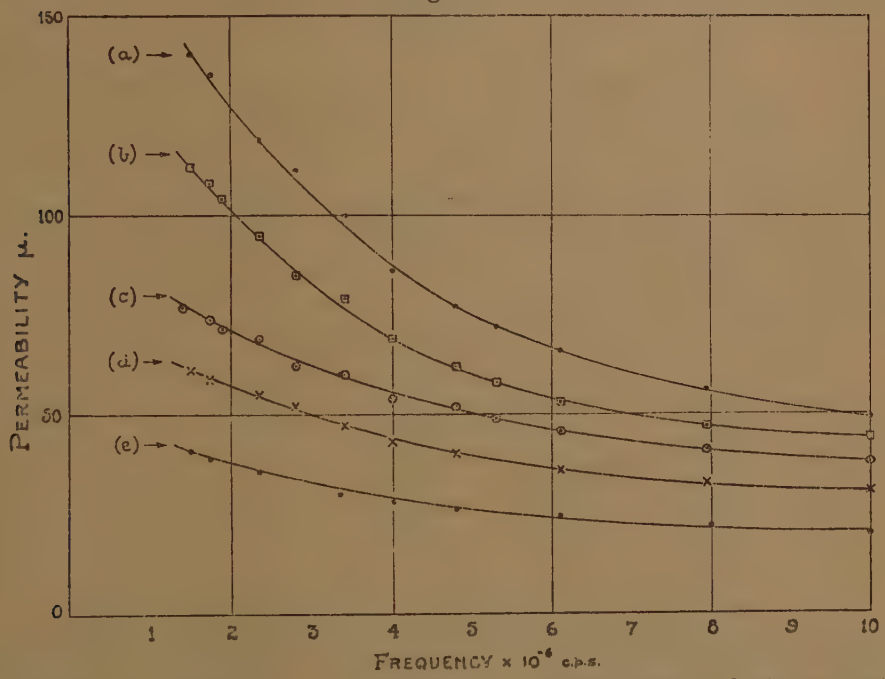
The high-frequency resistance of the rod specimens may be separated, and their high-frequency permeability calculated, therefore, by redetermining the circuit resistance with these rods replaced by copper rods of which the resistance is predeterminate, since for them $\mu = 1$. The

results obtained on the various test specimens by these means, over the frequency range 10^6–10^7 c.p.s., are given in fig. 2.

Discussion of the Results.

In view of the normal dependence of permeability upon the magnetizing force it is desirable to give some indication of the value of magnetizing force H to which these results relate. At the frequencies in question the current flowing through the rods will be concentrated in a surface skin of minute thickness, so that the significant value of H is that at the

Fig. 2.



Curves: (*a*) "Mumetal"; (*b*) "Stalloy"; (*c*) mild steel; (*d*) 6 per cent. tungsten steel (unhardened); (*e*) 6 per cent. tungsten steel (hardened).

surface. The current i was not measured directly, but could be derived from the measured voltage set up across the circuit, and the required value of H is then calculable as $\frac{4i}{d}$. Calculation shows that the range of magnetizing force to which the rods were subjected was about 0·03 c.g.s. units at the 10^6 c.p.s. end to 0·003 at the 10^7 c.p.s. end of the frequency band covered.

The possibility of the above changes in H exercising a significant effect on the observed change of permeability with the frequency of measure-

ment is indicated by the series of low-frequency permeability measurements, given in Table I., which were carried out for the author by Mr. A. M. Armour of the Research Department, Metropolitan Vickers. These measurements reveal for the Mumetal rods a decrease of permeability from about 30,000 to 4000 for a change of H from 0·03 to 0·003, and for the Stalloy rods a corresponding fall of μ from about 1600 to 4–500.

It would seem reasonable to infer from these data that the rate of decrease of permeability with frequency shown in fig. 2, for the two materials Mumetal and Stalloy in particular, is less than would have been the case had H remained constant throughout the frequency range. The correct-

Table I.

Magnetizing force H (c.g.s. units).	Permeability.				
	Mumetal.	Stalloy.	Mild steel.	Tungsten steel (unhardened).	Tungsten steel (hardened).
0·0025	4000	—	—	—	—
0·005	4600	500	—	—	—
0·01	8000	700	—	—	—
0·02	27000	1400	100	—	—
0·025	—	1560	105	70·0	39·2
0·05	—	1840	106	69·8	41·2
0·1	—	2450	113	70·0	41·2
0·25	—	—	123	76·0	43·6
0·5	—	—	136	75·8	45·0
1·0	—	—	—	75·6	45·1

ness or otherwise of such a deduction would have been determined directly had it been possible to carry out measurements at a single frequency over a range of circuit currents. On the other hand, measurements by Dr. C. Dannatt [3] on laminæ of Mumetal and 4 per cent. silicon steel at a frequency of 10^6 c.p.s. have shown that the "intrinsic" permeability (expressed normally as μ_0+vH, where μ_0 is the "initial" permeability) is practically independent of H between 0·02 and 0·5 c.g.s. units. It seems probable that the "intrinsic" permeability calculated by Dannatt is the same quantity as the permeability plotted in fig. 2, so that by inference the frequency variations there depicted may be taken as being a pure frequency effect.

Dannatt's measurements extended up to 10^7 c.p.s., and a comparison of his results on Mumetal and 4 per cent. silicon steel with those of fig. 2 on Mumetal and Stalloy, between 10^6 and 10^7 cycles, is drawn in Table II.

It will be seen that Dannatt's permeability values are substantially higher than the author's and that the rate of decrease with frequency is much smaller. An interesting effect shown in Dannatt's results, of which, however, there is no evidence in the author's, is that whereas for frequencies below about $1{\cdot}5\times10^6$ c.p.s. his Mumetal had a higher permeability than his 4 per cent. silicon steel, above this frequency the reverse is the case.

The ultimate high-frequency permeability values obtained by Dannatt were very much lower than the initial permeabilities of the materials. That the same applies in the present measurements is revealed by reference to the permeability values corresponding to the lowest values of H given in Table I.

TABLE II.

Frequency c.p.s.	Permeability.			
	Mumetal.		Stalloy.	4 per cent. silicon steel.
	Fig. 2.	Dannatt.	Fig. 2.	Dannatt.
10^6	—	291	—	256
2×10^6	128	228	101	237
5×10^6	86	182	60	225
10^7	49	163	44	218

The general conclusion drawn by Dannatt in discussing his results is that the main influence in the change of permeability with frequency above 10^5 c.p.s. is that of varying demagnetization effects in the magnetic domains within the thin flux-penetrated surface layer of the material. Unfortunately, the author's present results are too restricted to permit of entry into this discussion, but it is hoped to return to it at a later date.

The work recorded was carried out in the Engineering Laboratory, Oxford, and are published with deep gratitude to Dr. E. B. Moullin.

References.

(1) E. B. Moullin, 'Radio Frequency Measurements,' 2nd Edn. p. 273; W. Jackson, Proc. Inst. Radio Engineers, xxii. p. 957 (1934).
(2) E. B. Moullin, *loc. cit.* p. 238.
(3) C. Dannatt, Jour. Inst. Elec. Engs. lxxix p. 667 (1936).

XXIII. *Castigliano's Principle of Minimum Strain-Energy, and the Conditions of Compatibility for Strain.*

By R. V. Southwell *.

[Received June 18, 1940.]

[*Note.*—The following paper was written in 1938 as a contribution to a volume which some friends of Professor S. P. Timoshenko were preparing in commemoration of his sixtieth birthday on 22 December, 1938. This volume, edited by a committee which consisted of J. M. Lessells (Chairman), J. P. Den Hartog, G. B. Karelitz, R. Paterson and H. M. Westergaard, was published in New York by the Macmillan Company. I am greatly indebted to the Company for permission to reprint the paper, without charge, in the pages of the 'Philosophical Magazine'. What follows is the original paper without modification of any kind.—R. V. S.]

1. In the three stress-equations of equilibrium, viz.

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0, \ldots, \text{etc.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we can with complete generality write

$$Y_z=-\frac{\partial^2\chi_1}{\partial y\,\partial z}, \quad Z_x=-\frac{\partial^2\chi_2}{\partial z\,\partial x}, \quad X_y=-\frac{\partial^2\chi_3}{\partial x\,\partial y}. \quad . \quad . \quad . \quad (2)$$

Then they take the forms

$$\frac{\partial}{\partial x}\left[X_x-\frac{\partial^2\chi_3}{\partial y^2}-\frac{\partial^2\chi_2}{\partial z^2}\right]=0, \ldots, \text{etc.}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and are satisfied if we write

$$X_x=\frac{\partial^2\chi_3}{\partial y^2}+\frac{\partial^2\chi_2}{\partial z^2}, \quad Y_y=\frac{\partial^2\chi_1}{\partial z^2}+\frac{\partial^2\chi_3}{\partial x^2}, \quad Z_z=\frac{\partial^2\chi_2}{\partial x^2}+\frac{\partial^2\chi_1}{\partial y^2}. \quad . \quad (4)$$

χ_1, χ_2, χ_3 are *Maxwell's stress functions* †.

2. Alternatively we can with complete generality write

$$X_x=\frac{\partial^2\psi_1}{\partial y\,\partial z}, \; Y_y=\frac{\partial^2\psi_2}{\partial z\,\partial x}, \; Z_z=\frac{\partial^2\psi_3}{\partial x\,\partial y}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

---

* Communicated by the Author.

† Love, A. E. H., 'The Mathematical Theory of Elasticity', § 56 (e).

Then they take the forms

$$\frac{\partial^2\psi_1}{\partial x\,\partial y\,\partial z}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0,\ \ldots,\text{ etc.},\quad .\ .\ .\ .\ .\ .\quad (6)$$

so that if we write further (and again with complete generality)

$$Y_z=-\frac{1}{2}\frac{\partial\phi_1}{\partial x},\quad Z_x=-\frac{1}{2}\frac{\partial\phi_2}{\partial y},\quad X_y=-\frac{1}{2}\frac{\partial\phi_3}{\partial z},\quad .\ .\ .\quad (7)$$

it follows that ϕ_1, ϕ_2, ϕ_3 are related with ψ_1, ψ_2, ψ_3 by the three equations

$$\frac{\partial^2}{\partial y\partial z}\left[\frac{\partial\psi_1}{\partial x}-\frac{1}{2}(\phi_2+\phi_3)\right],\ \ldots,\text{ etc.}\quad .\ .\ .\ .\ .\ .\quad (8)$$

Equations (8) will be satisfied if

$$\phi_2+\phi_3=2\frac{\partial\psi_1}{\partial x},\quad \phi_3+\phi_1=2\frac{\partial\psi_2}{\partial y},\quad \phi_1+\phi_2=2\frac{\partial\psi_3}{\partial z},\quad .\ .\ .\quad (9)$$

i. e., if

$$\phi_1=-\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}+\frac{\partial\psi_3}{\partial z},\quad \phi_2=\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial y}+\frac{\partial\psi_3}{\partial z},\quad \phi_3=\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_3}{\partial z},$$
$$.\ .\ .\quad (10)$$

and when these expressions are substituted for ϕ_1, ϕ_2, ϕ_3 in (7) we have all of the six stress-components expressed in terms of ψ_1, ψ_2, ψ_3. The latter are known as *Morera's stress functions* *.

3. In a recent paper † I showed that the six equations

$$\left.\begin{aligned}
&\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=\frac{\partial^2 e_{yz}}{\partial y\,\partial z},\quad 2\frac{\partial^2 e_{xx}}{\partial y\,\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right),\\
&\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=\frac{\partial^2 e_{zx}}{\partial z\,\partial x},\quad 2\frac{\partial^2 e_{yy}}{\partial z\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right),\\
&\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2 e_{xy}}{\partial x\,\partial y},\quad 2\frac{\partial^2 e_{zz}}{\partial x\,\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right),
\end{aligned}\right\}\quad (11)$$

which are the accepted conditions for the existence of single-valued displacements u, v, w, can be regarded as a consequence of Castigliano's "Principle of Minimum Strain-Energy" (more commonly termed Castigliano's "Theorem of Least Work"). According to that principle the stress distribution resulting from given forces, applied to a body initially in a state of ease, can be deduced from the conditions of equilibrium combined with the conditions for a minimum value of the total strain-energy U. The conditions of equilibrium are automatically satisfied when the stresses are expressed in terms either of Maxwell's functions χ_1, χ_2, χ_3 or of Morera's functions ψ_1, ψ_2, ψ_3, and the conditions for a stationary

* *Ibid.* § 17.

† Southwell, R. V., Proc. Roy. Soc. A, cliv. pp. 4–21 (1936).

value of U can then be obtained by variational methods. I showed that in this way the first three of (11) can be deduced with the use of Maxwell's functions, the second three with the use of Morera's.

4. Commenting on these results, Professor G. I. Taylor remarked to me that by them three of the six conditions (11) would seem to be made redundant ; for if either the first or the second three of (11) are sufficient to ensure that U is stationary, then according to Castigliano's principle they should also ensure the existence of single-valued displacements, and no further conditions should be necessary. Once the paradox is revealed its resolution becomes a problem of some urgency, and this I attempt in the present note.

5. There would seem to be no doubt that in fact all six of (11) are necessary. As argued in a footnote to § 8 of my former paper, we can always find forms for u, v, w such that

$$\frac{\partial u}{\partial x}=e_{xx}, \quad \frac{\partial v}{\partial y}=e_{yy}, \quad \frac{\partial w}{\partial z}=e_{zz}, \qquad \ldots\ldots \quad (12)$$

and then, with suitable expressions for F_1, F_2, F_3, we may write

$$e_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}-F_1, \quad e_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}-F_2, \quad e_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-F_3. \quad . \quad (13)$$

If now the forms of $e_{xx}, \ldots, e_{yz}, \ldots$ are compatible with single-valued displacements, it must be possible to find supplementary functions u', v', w' which, added to u, v, w, will bring F_1, F_2, F_3 to zero without affecting the expressions for e_{xx}, e_{yy}, e_{zz}. That is to say, we must have

$$\left.\begin{aligned} F_1&=\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}, \quad F_2=\frac{\partial w'}{\partial x}+\frac{\partial u'}{\partial z}, \quad F_3=\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial z}, \\ \text{where} \qquad & \frac{\partial u'}{\partial x}=\frac{\partial v'}{\partial y}=\frac{\partial w'}{\partial z}=0. \end{aligned}\right\} \quad . \quad . \quad (14)$$

Eliminating u', v', w' from the first three of (14) we obtain three expressions of the type

$$2\frac{\partial^2 u'}{\partial y\,\partial z}=-\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z},$$

and from these, by the last three of (14), there result three relations which must be satisfied by F_1, F_2, F_3, of the type

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(-\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)=0. \\ \text{Three more conditions of the type} \qquad & \\ &\frac{\partial^2 F_1}{\partial y\,\partial z}=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

can be deduced from (14) ; and if now we substitute in (15) from (12) and (13), all six of (11) appear as necessary relations between the components of strain.

Conversely, if (11) are satisfied we can deduce (15), and from these that

$$F_1=f_1(x, y)+f_2(z, x), \quad F_2=f_3(y, z)+f_4(x, y), \quad F_3=f_5(z, x)+f_6(y, z),$$

where
$$\frac{\partial f_1}{\partial x}=\frac{\partial f_4}{\partial y}, \quad \frac{\partial f_2}{\partial x}=\frac{\partial f_5}{\partial z}, \quad \frac{\partial f_3}{\partial y}=\frac{\partial f_6}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The relations (16) imply that quantities u', v', w' exist such that

$$\left.\begin{aligned} f_1&=\frac{\partial w'}{\partial y}, \quad f_4=\frac{\partial w'}{\partial x}, \\ f_2&=\frac{\partial v'}{\partial z}, \quad f_5=\frac{\partial v'}{\partial x}, \\ f_3&=\frac{\partial u'}{\partial z}, \quad f_6=\frac{\partial u'}{\partial y}, \end{aligned}\right\} \quad \text{where} \quad \frac{\partial u'}{\partial x}=\frac{\partial v'}{\partial y}=\frac{\partial w'}{\partial z}=0,$$

and it follows that F_1, F_2, F_3 in (13) can be brought to zero, without affecting the expressions (12), by additions u', v', w' made to u, v, w respectively. Thus the six relations (11) are both necessary and sufficient.

6. This result is in accordance with accepted theory, but it contradicts the conclusions of § 4, and on that account the argument of my earlier paper (summarized in § 3) now calls for further scrutiny.

One possible explanation of the paradox would appear to be that neither Maxwell's nor Morera's stress functions represent the most general solution of the stress-equations (1). The expressions (4), although sufficient, are not necessary conditions for the satisfaction of equations (3), since we may write

$$X_x=\frac{\partial^2\chi_3}{\partial y^2}+\frac{\partial^2\chi_2}{\partial z^2}+\Phi_1(y, z), \ldots, \text{etc.}, \quad . \quad . \quad . \quad . \quad (17)$$

and still ensure satisfaction of the three stress-equations of equilibrium. In § 2, similarly, the expressions (9) are not necessary conditions for the satisfaction of (8), since we may substitute for them

$$\left.\begin{aligned} &\phi_2+\phi_3=2\frac{\partial\psi_1}{\partial x}+2\Psi_1, \ \ldots, \text{etc.}, \\ &\text{where} \\ &\Psi_1=f_1(x, y)+f_2(z, x), \quad \Psi_2=f_3(y, z)+f_4(x, y), \quad \Psi_3=f_5(z, x)+f_6(y, z), \end{aligned}\right\} \quad . \quad (18)$$

and then in place of (10) we have three equations of the type

$$\phi_1=-\left(\frac{\partial\psi_1}{\partial x}+\Psi_1\right)+\left(\frac{\partial\psi_2}{\partial y}+\Psi_2\right)+\left(\frac{\partial\psi_3}{\partial z}+\Psi_3\right). \quad . \quad . \quad (19)$$

On the other hand, without affecting (2) we can merge Ψ_1, . . ., etc. in

the stress functions χ_1, χ_2, χ_3 by adding expressions of the forms $\chi_1'(z, x)$, $\chi_2'(x, y)$, $\chi_3'(y, z)$ such that

$$\frac{\partial^2\chi_1'}{\partial z^2}=\Phi_2, \quad \frac{\partial^2\chi_2'}{\partial x^2}=\Phi_3, \quad \frac{\partial^2\chi_3'}{\partial y^2}=\Phi_1.$$

Therefore without loss of generality we may return from (17) to (4). Similarly, without affecting (5) we can merge Ψ_1, Ψ_2, Ψ_3 in the stress functions ψ_1, ψ_2, ψ_3 by adding expressions of the forms $\psi_1'=\mathrm{F}_1(x, y)+\mathrm{F}_2(z, x)$. . . ., etc., where

$$\frac{\partial}{\partial x}\mathrm{F}_1(x, y)=f_1(x, y), \quad \frac{\partial}{\partial x}\mathrm{F}_2(z, x)=f_2(z, x), \ldots, \text{ etc.}$$

Therefore without loss of generality we may return from (19) to (10).

7. Thus the stress functions both of Maxwell and of Morera do in fact represent the most general solution of (1), and the paradox of § 4 remains. We now state its correct explanation, which is, that *arguing from Castigliano's principle on the lines of* § 3 *we can deduce relations equivalent to all six of* (11).

In my earlier paper the variation of the total strain-energy, given by

$$\delta\mathrm{U}=\iiint[e_{xx}\delta\mathrm{X}_x+\ldots+e_{yz}\delta\mathrm{Y}_z+\ldots]\,dx\,dy\,dz, \quad \ldots \quad (20)$$

was expressed in terms *successively* of the variations $\delta\chi_1$, $\delta\chi_2$, $\delta\chi_3$ of Maxwell's stress functions and of the variations $\delta\psi_1$, $\delta\psi_2$, $\delta\psi_3$ of the stress functions of Morera ; then by Green's transformation each expression was converted into the sum of a volume and of a surface integral. Since $\delta\mathrm{U}$ must vanish for all permissible variations, and since the variations $\delta\chi_1$, $\delta\chi_2$, $\delta\chi_3$ of Maxwell's stress functions can take any values at any point within the boundary, it was deduced that the first three of (11) must be satisfied. Similar treatment, using Morera's stress functions, led in the same way to the last three of (11). Accordingly it was argued that all six of (11) must be satisfied, which means that the strain-components can all be expressed in terms of single-valued displacements u, v, w ; and on that understanding it was shown that no further condition is imposed by the surface integrals in the expressions for $\delta\mathrm{U}$. But in this argument both types of stress function were employed to establish the six equations (11) : in the argument which follows, *by taking full account of the restrictions which are imposed upon the variations at the boundary*, all six of (11) are shown to be required by Castigliano's principle.

8. If in the expression (20) we substitute for the strain-components from (12) and (13), then on applying Green's transformation to the surface integral we obtain

$$\delta\mathrm{U}=-\iiint(\mathrm{F}_1.\delta\mathrm{Y}_z+\mathrm{F}_2.\delta\mathrm{Z}_x+\mathrm{F}_3.\delta\mathrm{X}_y)\,dx\,dy\,dz, \quad \ldots \quad (21)$$

since the terms in u, v, w vanish for the volume integral in virtue of the stress equations of equilibrium, and for the surface integral in virtue of the three conditions

$$\left.\begin{aligned} 0=\delta X_\nu &= l\delta X_x+m\delta X_y+n\delta X_z,\\ 0=\delta Y_\nu &= l\delta Y_x+m\delta Y_y+n\delta Y_z,\\ 0=\delta Z_\nu &= l\delta Z_x+m\delta Z_y+n\delta Z_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which, for equilibrium, must be satisfied at every point of the boundary.

From (22) it follows that

$$\iint[\alpha(l\delta X_x+m\delta X_y+n\delta X_z)+\beta(l\delta Y_x+m\delta Y_y+n\delta Y_z) +\gamma(l\delta Z_x+m\delta Z_y+n\delta Z_z)]\,dS=0 \quad . \quad (23)$$

when α, β, γ stand for any functions of x, y, z. Applying Green's transformation, and restricting α, β, γ by the conditions

$$\frac{\partial\alpha}{\partial x}=\frac{\partial\beta}{\partial y}=\frac{\partial\gamma}{\partial z}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

we deduce that the volume integral

$$\iiint\left[\delta Y_z\left(\frac{\partial\beta}{\partial z}+\frac{\partial\gamma}{\partial y}\right)+\delta Z_x\left(\frac{\partial\gamma}{\partial x}+\frac{\partial\alpha}{\partial z}\right)+\delta X_y\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)\right]dx\,dy\,dz=0 \quad . \quad (25)$$

for all variations δY_z, δZ_x, δX_y, *when* α, β, γ *are functions restricted by* (24).

Now the variations δY_z, δZ_x, δX_y in (21) are not restricted by the conditions of equilibrium, since we can always find expressions for δX_x, δY_y, δZ_z which will satisfy these both at the surface and throughout the volume of the elastic solid. Therefore, using (25) in accordance with the rules of the Calculus of Variations, we deduce from (21) that since U must be stationary for all variations δY_z, δZ_x, δX_y, therefore

$$F_1=k\left(\frac{\partial\beta}{\partial z}+\frac{\partial\gamma}{\partial y}\right), \quad F_2=k\left(\frac{\partial\gamma}{\partial x}+\frac{\partial\alpha}{\partial z}\right), \quad F_3=k\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right), \quad . \quad (26)$$

where k is an arbitrary constant and α, β, γ are functions unrestricted except by (24). From (26) combined with (24) it follows that F_1, F_2, F_3 satisfy the six relations of types (15); and if we substitute in those relations from (12) and (13) all six of (11) appear as necessary relations between the components of strain.

Summary.

In an earlier paper, of the six equations (11) of compatibility three were deduced with the use of Maxwell's stress functions (§ 1), the other three with the use of those of Morera (§ 2), appeal being made in each instance to Castigliano's principle of minimum total strain-energy (§ 3). A paradox is thus presented, in that it would appear that three of the six equations are redundant (§ 4).

This paradox is examined in the present paper. In § 5 it is shown that all six equations are in fact necessary conditions of compatibility, and in § 6 that the stress functions either of Maxwell or Morera do in fact constitute the most general solution of the stress-equations of equilibrium. In § 7 the paradox is resolved by showing that more might have been proved in the earlier paper: when account is taken of the restrictions imposed by the conditions of equilibrium at the boundary, all six of the equations of compatibility can be deduced as a consequence of Castigliano's principle.

XXIV. *Notices respecting New Books.*

The Photographic Process. By Prof. J. E. Mack and Prof. M. J. Martin. [Pp. xvii+586.] (London: McGraw-Hill Publishing Co. Ltd., 1939. Price 33*s.*)

The authors of this treatise have produced a work which is intended to cover the studies in photography usually undertaken in a course given in a technical college or university. At the same time they endeavour to treat the subject in such a way as to appeal to the general student. Such a method of approach has its dangers, but the authors must be judged to be successful in what is a very difficult task. As the authors remark, there is in existence a very large number of elementary instruction books and, at the other end of the scale, a number of special treatises and encyclopædic references, but there was a distinct need for a comprehensive textbook which should survey the whole field in a manner suited to the needs of a student of university grade.

This survey the authors have provided.

The principal sections of the book deal with photographic optics, lenses, and cameras; the photographic emulsion and the latent image; exposure and development; colour and natural colour photography; positive prints; projection printing; scientific and technological photography; photomechanical reproduction; and pictorial photography.

There is a useful short bibliography and two appendices, one of which deals with explanations of the simple mathematics involved, the other with a chemical formulary and an account of some thirty experiments.

The authors are to be congratulated on having produced a much-needed addition to the literature of their subject.

Science since 1500. *A short History of Mathematics, Physics, Chemistry, Biology.* By H. T. Pledge. [Pp. 357.] (London: Science Museum. Published by H.M. Stationery Office, 1939. Price 7*s.* 6*d.*)

The author of this book has attempted a most difficult task—that of presenting a survey, in condensed form, of the development, during four centuries, of the sciences mentioned in the title. His book is intended to give a background to the various manuals on the history of special scientific subjects that have, from time to time, been published under the auspices of the Science Museum. But it is not popular in character; on the contrary, it is designed to appeal to the "specialist in other subjects, the scholarship candidate, the university student, the research worker."

As such it must be judged; and, if we judge it from this standpoint, the author must be deemed to have succeeded. True, he may be criticised for attempting, in limited space, to cover the ground in too great detail. But the details are of absorbing interest to the class of reader to whom he appeals, and if, on rare occasions, a detail here or there is wrong, that is surely no great matter.

Every student of the history of science will be grateful to the author for having brought together such a large mass of valuable material, and for having presented it in such a thoroughly interesting way.

The book merits, and should achieve, success. A. F.

Electronic Structure and Chemical Binding. By Prof. O. K. Rice. [Pp. xiv+511.] (London: McGraw-Hill Publishing Co., Ltd., 1940. Price 33*s.*)

The outlook on the fundamentals of chemistry has changed almost beyond recognition during the last twenty years. Nevertheless there exists a large population of students of chemistry the members of which are well acquainted—and it is all to the good that they should be—with the properties of matter as expounded in the classic treatises on physical chemistry; but not so familiar with the changed point of view due to the development of the modern theory of the atom. It is for such students that the book has been written, and the author begins his task by a detailed, clear, but quite elementary treatment of atomic and quantum theory, wave and corpuscular properties and allied topics, afterwards developing on this foundation a careful account of the hydrogen molecule, theories of valency, ionic crystals, complex compounds and crystals, concluding with a discussion of the structure of water, hydrates and aqueous solutions.

The book fills a niche in the literature of modern chemistry, and deserves success.

A Text-book of Heat.—Part II. By H. S. Allen and R. S. Maxwell. [Pp. ix+531–848+xi.] (Macmillan & Co. Price 10*s.* 6*d.*)

In this volume the authors have produced a compact but adequate and well-written account of the more theoretical and advanced portions of the subject of heat, such as are required by students taking physics at the pass standard. The first half is devoted to the laws of thermodynamics and their applications, followed by conduction and radiation.

Modern work receives due mention, and the inclusion of chapters on specific heats, probability, statistical methods, and quantum theory provides a useful foundation for the General and Special honours student.

There is a good selection of questions and numerical examples, quite a number being worked out in the text. The authors succeed in their aim of filling a place between an elementary text-book and a comprehensive treatise. A book of this standard, obtainable at a reasonable price and convenient in size, finds a ready place among works on heat. F. A. L.

It is with very deep sorrow that the Editors of the 'Philosophical Magazine' announce the death of their old and valued colleague, Sir Oliver Lodge. An obituary notice will appear in the October number of the Magazine.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

CONTENTS OF No. 201.

Page

XXV. Quaternion Analogy of Wave-tensor Calculus. By W. H. McCrea 261

XXVI. The Propagation of Electromagnetic Waves in an Ionized Medium and the Calculation of the True Heights of the Ionized Layers of the Atmosphere. By Olof Rydbeck, Cruft Laboratory, Harvard University, and the Chalmers Institute of Technology, Gothenburg 282

XXVII. On the Nature of Negative Resistances and Negative Resistance Sections. By S. P. Chakravarti, M.Sc. (London), D.I.C., A.M.I.E.E., Mem.I.R.E., University Lecturer, University of Calcutta 294

XXVIII. Notes on Mathieu Functions.—I. A Class of Hyperbolic Mathieu Functions. By W. G. Bickley, D.Sc., Imperial College of Science and Technology 312

XXIX. The Thermal and Electrical Conductivities of Metals and Alloys. By W. F. Chubb, Ph.D., B.Sc. 323

XXX. The Idea of Minimum Proper Time, and some Consequences of it. By M. Fahmy, The Egyptian University, Cairo 331

XXXI. The Mutual Inductance of Two Circles : a Problem in Approximation. By E. H. Neville 340

XXXII. Obituary :—Sir Oliver Joseph Lodge, D.Sc., Sc.D., LL.D., F.R.S. 341

XXXIII. Notices respecting New Books :—

Mr. D. Laugharne Thornton's Mechanics Applied to Vibrations and Balancing 344

Messrs. J. V. Uspensky and M. A. Heaslet's Elementary Number Theory 345

Mr. W. W. Rouse Ball's Mathematical Recreations and Essays 345

Prof. L. E. Dickson's Modern Elementary Theory of Numbers 346

Prof. I. S. Sokolnikoff's Advanced Calculus 347

Dr. Harris Hancock's Development of the Minkowski Geometry of Numbers 347

Mr. Blamey Stevens's The Psychology of Physics 347

Reports on Progress in Physics.—Vol. VI 348

*** It is requested that all Communications for this Work may be addressed, post-paid, to the care of Messrs. Taylor and Francis, Ltd., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

XXV. *Quaternion Analogy of Wave-tensor Calculus.*

By W. H. McCrea †.

[Received August 17, 1939.]

1. *Introduction.*

The relativistic quantum theory developed by Eddington ‡ employs the mathematical technique which he has called wave-tensor calculus. This is based upon the algebra of his E-numbers § and certain geometries derived from it. In order to follow the details of the work the present author has found it instructive to write down the analogous results for the simpler case in which *quaternion algebra* is used in place of E-number algebra. It seems worth while placing some of these results on record in case they may be of similar service to others. It might be added that for the purposes of this paper it is not necessary to assume any prior knowledge of quaternion theory.

The analogy necessarily cannot be complete. But most of the broader features of Eddington's calculus, such as the correspondence between wave tensors and space vectors, the matrix representation and corresponding reality conditions, the existence of the " simple " wave equation and the differential wave equation, the existence of a phase space, and the existence of double wave vectors, have their counterparts in the present work. Some of these counterparts are given by rather trivial results, which nevertheless have the advantage of being so obvious that they make it easy to anticipate the existence of Eddington's corresponding results. The main lack of completeness of the analogy will be seen to arise from the fact that in quaternion algebra there are no pairs of elements (not including the unity element) which commute, in addition to pairs which anticommute.

It must be emphasized that there is no suggestion of the present work having any *physical* significance analogous to that of Eddington's theory. Well-known physical applications of quaternions might be related to the work, but they would not be useful analogues. Our concern is only with the mathematical apparatus.

---

† Communicated by the Author.

‡ A. S. Eddington, 'Relativity Theory of Protons and Electrons' (Cambridge, 1936).

§ *Op. cit.* chapter II.

References to Eddington's book will be made, for example, thus: Chapter II, E II; paragraph 2.2, E 2.2; page 21, E 21; equation (2.25), E (2.25). Such references will for the most part indicate results in his work of which the present ones are analogues.

Certain of our results are adumbrated in E 3.8, where it is also shown, in effect, that E-number algebra is practically the product of two quaternion algebras.

2. *Tensors.*

In regard to the tensor character of sets of quantities obeying various rules of linear transformation we follow precisely the definitions in E 1.4, 1.5, since these are independent of the calculus to be developed. However, so as to avoid the suggestion of physical analogy while recalling the mathematical similarity, we shall speak of "*w*-vectors" and *w*-tensors" instead of "wave vectors" and "wave tensors." Some such qualification of the simple terms vector and tensor is needed, since it is essential to distinguish between their use here in regard to matrix transformations and their later use in connexion with geometrical representations. The definitions of *w*-vectors and *w*-tensors involve no geometrical conceptions.

These definitions are:

A set of n quantities $\psi \equiv (\psi_{(\alpha)})(\alpha = 1, \ldots, n)$ which transforms according to the rule expressed by the matrix equation

$$\psi' = q\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where q is a non-singular $n \times n$ matrix, and ψ, ψ' are treated as single-column matrices, is called a *covariant w-vector*. More properly, the whole aggregate of sets like ψ, ψ', ..., related by equations like (1) should be called a covariant *w*-vector, and each set ψ, or ψ', or ..., is a particular representation of it.

A set of n quantities $\chi^* \equiv \chi_{(\alpha)}(\alpha = 1, \ldots, n)$ which transforms according to the rule

$$\chi^{*\prime} = \chi^* q^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $qq^{-1} = \mathbf{1}$, the unit matrix of order n, and χ^*, $\chi^{*\prime}$ are treated as single-row matrices, is then called a *contravariant w-vector*.

A set of n^2 quantities $\mathbf{T} = (\mathrm{T}_{(\alpha\beta)})(\alpha, \beta = 1, \ldots, n)$ which transforms according to the rule

$$\mathbf{T}' = q\mathbf{T}q^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\mathbf{T}$, $\mathbf{T}'$ are treated as square matrices, is then called a *mixed w-tensor*.

A set of n^2 quantities $\mathbf{S} \equiv (\mathrm{S}_{(\alpha\beta)})(\alpha, \beta = 1, \ldots, n)$ which transforms according to the rule

$$\mathbf{S}' = q\mathbf{S}\tilde{q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\tilde{q}$ is the transpose of the matrix q, and $\mathbf{S}$, $\mathbf{S}'$ are treated as square matrices, is called a *covariant w-tensor.*

It will be noted that the product $\psi\chi^*$ is a particular case of a mixed w-tensor, while the product $\chi^*\psi$ is invariant.

3. *Quaternions.*

Instead of starting with four symbols E_1, E_2, E_3, E_4 as Eddington does (E (2.21)), we shall start with two symbols i_1, i_2, in addition to a unity symbol $\mathbf{1}$, satisfying the same conditions, *i. e.*,

$$i_1\mathbf{1}=\mathbf{1}i_1,\ i_2\mathbf{1}=\mathbf{1}i_2,\ i_1i_2=-i_2i_1,\ i_1^2=-\mathbf{1},\ i_2^2=-\mathbf{1}. \quad . \quad . \quad . \quad (5)$$

Then, if we define i_3 by

$$i_3=i_1i_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we find

$$i_3\mathbf{1}=\mathbf{1}i_3,\ i_2i_3=-i_3i_2=i_1,\ i_3i_1=-i_1i_3=i_2,\ i_3^2=-\mathbf{1}. \quad . \quad . \quad (7)$$

It follows that the four quantities $\mathbf{1}$, i_1, i_2, i_3 form a complete set, so that the operations of addition, subtraction, and multiplication applied to any finite number of expressions of the form

$$\mathrm{A}=a_0\mathbf{1}+a_1i_1+a_2i_2+a_3i_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where a_0, a_1, a_2, a_3 are scalar numbers, always yield a single expression of this form. Actually A is seen to be a quaternion, and we have merely recovered *quaternion algebra.*

Here the product of two symbols gives a third which completes the fundamental set, apart from the unity symbol. In Eddington's case the product of four anticommuting symbols gives a fifth which anticommutes with each of the four and completes what he calls a *pentad*, of which our single triad i_1, i_2, i_3 forms the analogue. But such a pentad does not complete the fundamental set of E-symbols. The latter consists of fifteen symbols, other than the unity symbol, each of which anticommutes with eight others and commutes with the rest. As already noted, this feature of the existence of *both* anticommuting and commuting pairs finds no analogue in quaternion algebra. Consequently we have also no analogues of Eddington's *triads* and *antitriads* (E (2.33), (2.36)), nor of any features of his work which depend essentially on their existence.

It is to be noted that Eddington uses $i\mathbf{1}$ in place of $\mathbf{1}$, where i is the ordinary scalar symbol used in complex numbers (E (2.33)). To do the same here would slightly improve our analogy, but the gain would not be sufficient to warrant a departure from the normal procedure of quaternion algebra. We recall also the familiar result that algebraic manipulations can be performed without inserting the $\mathbf{1}$ explicitly in terms like $a_0\mathbf{1}$, so we shall usually omit writing it.

In a quaternion like A the separate terms a_0, a_1i_1, a_2i_2, a_3i_3 are called

the *components* of A; a_0 is called the *scalar part* of A, denoted by $\mathscr{S}$A; $a_1i_1+a_2i_2+a_3i_3$ is called the *vector part* of A, denoted by $\mathscr{V}$A. $\mathscr{S}$A is the analogue of Eddington's qs T(E(2·395)), and to complete the analogy we say that A is *degenerate* if $\mathscr{S}$A=0. We shall write $a_\mu i_\mu$, or $a_\nu i_\nu$, or . . ., for the general component of $\mathscr{V}$A, it being always implied, unless otherwise stated, that μ, ν, . . ., can take any of the values 1, 2, 3; summation over μ, or ν, or . . ., is understood to be over these values.

Quaternion algebra contains the zero symbol 0 which is such that A=0 if, and only if, $a_0=a_1=a_2=a_3=0$. Therefore kA=0, if and only if $k=0$ or A=0, k being a scalar.

Elementary properties.—It is a trivial matter to prove the following properties of a quaternion A from the definitions just given; they are the analogues of Eddington's results as indicated.

If A commutes with i_μ, then each non-vanishing component of A commutes with i_μ (E 2.5(c)). In this case A is then of the form $a_0+a_\mu i_\mu$.

If A anticommutes with i_μ, then each non-vanishing component of A anticommutes with i_μ (E 2.5(d)). In this case A is then of the form

$$a_\nu i_\nu+a_\sigma i_\sigma \ (\mu\neq\nu\neq\sigma).$$

For any A we have (E 2.5(f))

$$\mathrm{A}-\Sigma_\mu i_\mu \mathrm{A} i_\mu=4\mathscr{S}\mathrm{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$a_\mu=-\mathscr{S}(\mathrm{A}i_\mu)=-\mathscr{S}(i_\mu\mathrm{A}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Combining (9), (10), we have (E 2.5(g))

$$4a_\mu=-\mathrm{A}i_\mu+\Sigma_\nu i_\nu \mathrm{A} i_\mu i_\nu=-i_\mu\mathrm{A}+\Sigma_\nu i_\nu i_\mu \mathrm{A} i_\nu. \quad . \quad . \quad . \quad . \quad (11)$$

For any two quaternions A, B, we have (E 2.5(h))

$$\mathscr{S}(\mathrm{AB})=a_0b_0-\Sigma_\mu a_\mu b_\mu=\mathscr{S}(\mathrm{BA}).$$

Conjugate and Reciprocal.—If $\mathrm{A}=a_0+\Sigma_\mu a_\mu i_\mu$, $\bar{\mathrm{A}}=a_0-\Sigma_\mu a_\mu i_\mu$, then the quaternion $\bar{\mathrm{A}}$ is called the *conjugate* of A (sometimes written $\mathscr{K}$A), and consequently A is the conjugate of $\bar{\mathrm{A}}$. We find

$$\mathrm{A}\bar{\mathrm{A}}=\bar{\mathrm{A}}\mathrm{A}=a_0^2+a_1^2+a_2^2+a_3^2.$$

This quantity is called the *norm* of A and is written $\mathscr{N}$A. We have $\mathscr{N}\mathrm{A}=\mathscr{N}\bar{\mathrm{A}}$. If $\mathscr{N}$A=0 we say that A is *singular*; otherwise it is non-singular.

It is easy to show that if A is non-singular it has a unique *reciprocal* A^{-1} such that

$$\mathrm{A}\mathrm{A}^{-1}=\mathrm{A}^{-1}\mathrm{A}=1,$$

and, in fact, that $\mathrm{A}^{-1}=\bar{\mathrm{A}}/\mathscr{N}\mathrm{A}$.

If A is singular, then

$$\mathrm{A}\bar{\mathrm{A}}=\bar{\mathrm{A}}\mathrm{A}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Then there is no quaternion B such that

$$AB=1 \quad \text{or} \quad BA=1.$$

For premultiplying and postmultiplying these respectively by $\bar{A}$ would give, using (12), $0=\bar{A}$, and, consequently, $A=0$, giving $AB=BA=0$ for all B, which yields a contradiction. Therefore a singular quaternion has no reciprocal (E 26).

However, if B is an arbitrary quaternion, then from (12), $A(\bar{A}B)=0$, $(B\bar{A})A=0$. So there are in this case infinitely many quaternions C, D (say) such that $AC=0$, $DA=0$, and these may be called *pseudo-reciprocals* of the singular quaternion A (E(2.64)). [It does not, of course, follow immediately that C, D are necessarily of the forms $\bar{A}B$, $B\bar{A}$, though this can be proved.]

Consider now the equation

$$AB=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It follows that $\bar{A}AB=0$, *i. e.* $(\mathcal{N}A)\,B=0$, so that if $\mathcal{N}A\neq 0$, we have $B=0$, since $\mathcal{N}A$ is a scalar. So if in (13) A is non-singular, then B is zero. But if A is singular, we have seen that there are infinitely many non-zero quaternions B which satisfy (13). Now if A is non-zero, then every such B is singular. For (13) gives $AB\bar{B}=0$, *i. e.*, $A(\mathcal{N}B)=0$, so that if $A\neq 0$, $\mathcal{N}B=0$ since $\mathcal{N}B$ is a scalar (E 26).

Transformation of quaternions (E 2.7).—If t be any non-singular quaternion and t^{-1} be its reciprocal, let

$$\underset{\sim}{i}_\mu=ti_\mu t^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Then the quantities $\underset{\sim}{i}_\mu$ have the same properties as the original set i_μ. For example,

$$\underset{\sim}{i}_1\underset{\sim}{i}_2=ti_1t^{-1}ti_2t^{-1}=ti_1i_2t^{-1}=ti_3t^{-1}=-\underset{\sim}{i}_3,$$
$$\underset{\sim}{i}_1{}^2=ti_1t^{-1}ti_1t^{-1}=ti_1{}^2t^{-1}=-tt^{-1}=-1.$$

To consider the converse, let the $\underset{\sim}{i}_\mu$ be any set of three symbols having the properties (5), (6), (7) of the i_μ, so that

$$\underset{\sim}{i}_\mu 1=1\underset{\sim}{i}_\mu,\ \underset{\sim}{i}_2\underset{\sim}{i}_3=\underset{\sim}{i}_1=-\underset{\sim}{i}_3\underset{\sim}{i}_2,\ \underset{\sim}{i}_3\underset{\sim}{i}_1=\underset{\sim}{i}_2=-\underset{\sim}{i}_1\underset{\sim}{i}_3,\ \underset{\sim}{i}_1\underset{\sim}{i}_2=\underset{\sim}{i}_3=-\underset{\sim}{i}_2\underset{\sim}{i}_1,\ \underset{\sim}{i}_\mu{}^2=-1. \quad (15)$$

Define

$$Q=\alpha(1-\Sigma\underset{\sim}{i}_\mu i_\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where α is a scalar number ($\neq 0$). By direct multiplication we verify that

$$\underset{\sim}{i}_\mu Qi_\mu=-Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Suppose now that the $\underset{\sim}{i}_\mu$ are quaternions in the algebra defined by the i_μ (i-quaternions, say) (E 27 Case (a)). Then Q is a quaternion and we have, if Q is non-singular, by postmultiplying (17) by $i_\mu Q^{-1}$,

$$\underset{\sim}{i}_\mu=Qi_\mu Q^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In order to allow for the possibility of Q being singular, define

$$Q^{(\mu)}=\alpha(i_\mu-\Sigma_\nu \underset{\sim}{i}_\nu i_\mu i_\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It is a straightforward matter to verify that

$$Q-\Sigma_\mu Q^{(\mu)}\underset{\sim}{i}_\mu=4\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\underset{\sim}{i}_\mu Q^{(\mu)} i_\mu=-Q^{(\mu)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From (20) at least one of the quaternions Q, $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$ is certainly not zero ; let R be such a one, and let $R=r_0+\Sigma_\mu r_\mu i_\mu$. Then we can show that R is also non-singular. For suppose R is singular ; then there exists a pseudo-reciprocal S such that

$$RS=0, \quad (S\neq 0).$$

From (17), (21)

$$R=-\underset{\sim}{i}_\mu R i_\mu.$$

Postmultiplying by $i_\mu S$ this gives

$$Ri_\mu S=0.$$

From (9), (11)

$$4r_0=R-\Sigma_\mu i_\mu R i_\mu, \quad 4r_\nu=-i_\nu R+\Sigma_\mu i_\mu i_\nu R i_\mu,$$

yielding then

$$4r_0 S=RS-\Sigma_\mu i_\mu(Ri_\mu S)=0,$$

$$4r_\nu S=-i_\nu RS-\Sigma_\mu i_\mu i_\nu(Ri_\mu S)=0.$$

But since $R\neq 0$, at least one of its coefficients r_0, r_1, r_2, r_3 is not zero. Hence we should have $S=0$, which is a contradiction. Hence R cannot be singular.

Proceeding then as in (18), we find

$$\underset{\sim}{i}_\mu=Ri_\mu R^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

We have therefore proved : *A necessary and sufficient condition for* i_μ, $\underset{\sim}{i}_\mu$ ($\mu=1, 2, 3$) *to be two sets of symbols having the fundamental properties* (15) *of quaternion elements is that they satisfy relations of the form* $\underset{\sim}{i}_\mu=ti_\mu t^{-1}$, *where t is a quaternion, provided the one set be quaternions in the algebra generated by the other set.*

If the last provision is not satisfied, we can get the analogue of E 28 Case (*b*) by supposing instead that the $\underset{\sim}{i}_\mu$ are symbols which commute with all the i_μ. Direct calculation then gives in this case

$$Q^2=4\alpha^2.$$

Take $4\alpha^2=1$, and postmultiply (17) by $i_\mu Q$, which gives

$$\underset{\sim}{i}_\mu=Qi_\mu Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

as the law of transformation.

No further reference to this case will be made before section 9 ; in the meantime we assume the conditions of the above theorem to apply.

Properties of the Transformation.—The transformation is,

$$\underline{i}_\mu = t i_\mu t^{-1}. \quad \ldots \ldots \ldots \quad (24,$$

Suppose another quaternion u exists such that

$$\underline{i}_\mu = u i_\mu u^{-1}.$$

Then $t i_\mu t^{-1} = u i_\mu u^{-1}$ gives $u^{-1} t i_\mu = i_\mu u^{-1} t$. That is $(u^{-1}t)$ commutes with every i_μ, and so, by one of our elementary properties, it must be a scalar k (say). Therefore $t = ku$, so that *t is unique except for an arbitrary scalar factor.*

This has important consequences. In the first place it might appear (E 29) possible to obtain a more general transformation than (22) by using t in place of R, where t is a linear combination of Q, $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$, *i. e.* (say)

$$\begin{aligned} t &= \tau_0 Q + \Sigma_\mu \tau_\mu Q^{(\mu)} \\ &= \alpha(\tau_0 + \Sigma \tau_\mu i_\mu - \Sigma_\nu \underline{i}_\nu (\tau_0 + \Sigma_\mu \tau_\mu i_\mu) i_\nu) \\ &= \alpha(T - \Sigma_\nu \underline{i}_\nu T i_\nu), \end{aligned}$$

where T is an arbitrary quaternion. But this t must now be merely a scalar multiple of R, and so provides only a more symmetrical way of writing R.

In the second place, had we interchanged the rôles of i_μ and $\underline{i}_\mu$ in deriving (18) we should have

$$i_\mu = \underline{Q} \underline{i}_\mu \underline{Q}^{-1},$$

where $Q = \alpha(1 - \Sigma i_\mu \underline{i}_\mu)$. This can be written

$$\underline{i}_\mu = \underline{Q}^{-1} i_\mu \underline{Q},$$

and so comparing with (18) we see that $\underline{Q}$, Q^{-1} differ only by a scalar factor, and hence that α could be chosen to give $Q^{-1} = \underline{Q}$(E 28).

This applies, of course, only provided Q is non-singular. If it is singular we have seen that some $Q^{(\mu)}$ must be non-singular, and by a similar argument for this $Q^{(\mu)}$ we find that α can be chosen to give

$$\begin{aligned} Q^{(\mu)-1} &= \alpha(\underline{i}_\mu - \Sigma i_\nu \underline{i}_\mu \underline{i}_\nu) \\ &\equiv \alpha(\underline{i}_\mu - \Sigma i_\nu i_\mu \underline{i}_\nu). \end{aligned}$$

The latter identity is verified for any particular value of μ by multiplying out and rearranging the terms ; the final form corresponds to E (2.81).

Another elementary result is that all the transformations like (24) form a *group.*

Finally, we may note that $\underline{i}_\mu$ is necessarily a *degenerate* i-quaternion. This can be seen in various ways ; probably the simplest is to prove that $\bar{\underline{i}}_\mu = -\underline{i}_\mu$, from which it immediately follows that $\mathscr{S} \underline{i}_\mu = 0$.

Rotations.—In virtue of the relations (24) any quaternion A may be written in either of the forms

$$A=a_0+\Sigma_\mu a_\mu i_\mu \quad \text{or} \quad A=\underline{a}_0+\Sigma_\mu \underline{a}_\mu \underline{i}_\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

According to Eddington's terminology we express this by saying that A can be referred to either of the *symbolic frames* $\mathbf{1}, i_1, i_2, i_3$ or $\mathbf{1}, \underline{i}_1, \underline{i}_2, \underline{i}_3$. Since the two sets of symbols have the same mutual relations, we can say that the frames have the same structure, and can describe the transformation from one to the other as a *rotation* of the frames (E 2.9). Such a rotation we shall call an r-rotation in order to suggest the analogy to relativity rotations (E 31).

Suppose now we have a quaternion

$$A=a_0+\Sigma_\mu a_\mu i_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and we form another quaternion

$$A'=a_0+\Sigma_\mu a_\mu \underline{i}_\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

such that the components of A′ in the second frame are the same as the components of A in the first. Then we say that A′ is got from A by rotating it with the frame.

We see from (24), (27) that

$$A'=tAt^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and since this is formally like (3) we extend the definition used in (3) by calling a quaternion transformed in this manner a mixed w-tensor (E 2.9). The results given in the next section justify this.

Expressing the $\underline{i}_\mu$ explicitly in terms of the i_μ, we could put A′ into the form

$$A'=a_0'+\Sigma_\mu a_\mu' i_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and so express the rotation of A with respect to the original frame. We return to this in section 5. But there are two simple properties we note here. First, since the $\underline{i}_\mu$ are degenerate, we have at once

$$a_0'=a_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Second, the transformation (28) gives

$$\mathcal{N}A'=\mathcal{N}A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

or, using (30),

$$a_1'^2+a_2'^2+a_3'^2=a_1^2+a_2^2+a_3^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

4. *Matrix Representation.*

It is easily seen that the matrices

$$\mathbf{1}=\begin{bmatrix}1 & .\\ . & 1\end{bmatrix}, \quad \mathbf{i}_1=\begin{bmatrix}i & .\\ . & -i\end{bmatrix}, \quad \mathbf{i}_2=\begin{bmatrix}. & 1\\ -1 & .\end{bmatrix}, \quad \mathbf{i}_3=\begin{bmatrix}. & i\\ i & .\end{bmatrix}, \quad . \quad (33)$$

combine according to the same rules as **1**, i_1, i_2, i_3 in (5), (6), (7). Consequently, a quaternion

$$\mathrm{A}=a_0+\Sigma_\mu a_\mu i_\mu \quad \ldots\ldots\ldots \quad (34)$$

can be *represented* by the matrix

$$\mathbf{A}=\begin{bmatrix} a_0+ia_1 & a_2+ia_3 \\ -a_2+ia_3 & a_0-ia_1 \end{bmatrix}, \quad \ldots\ldots\ldots \quad (35)$$

i. e., every theorem on quaternions remains true when the quaternions are replaced by corresponding matrices of this form, and every theorem on matrices of this form remains true when the matrices are replaced by the corresponding quaternions. Here, and subsequently except in section 9, we consider only matrices whose elements are scalar numbers (E III.)

Now, however, an *arbitrary* 2×2 matrix

$$\begin{bmatrix} x & y \\ t & w \end{bmatrix}$$

can be expressed in the form (35). We have merely to put

$$a_0=\tfrac{1}{2}(x+w),\ a_1=-\tfrac{1}{2}i(x-w),\ a_2=\tfrac{1}{2}(y-z),\ a_3=-\tfrac{1}{2}i(y+z). \quad (36)$$

Therefore the general result just described can be stated briefly: Any quaternion can be regarded as a 2×2 matrix, and any 2×2 matrix can be regarded as a quaternion (E 3.3).

The matrix representation (35) is not unique. It is easy to see that it can be replaced by $t\mathbf{A}t^{-1}$, where t is any non-singular 2×2 matrix. In fact, this follows from section 3 in virtue of the preceding paragraph. But we see then that any quaternion relation proved by using one matrix representation must hold good for all matrix representations, which is merely saying the same thing as that a quaternion relation holding with regard to one symbolic frame must hold with regard to all such frames.

Properties of the Representation.—The *spur* (or *trace*) of a matrix is the sum of the elements in its principal diagonal. We see from (35)

$$\text{spur } \mathbf{A}=2a_0=2\mathscr{S}\mathrm{A}. \quad \ldots\ldots\ldots \quad (37)$$

It is well known that spur **A** is invariant under the transformation $t\mathbf{A}t^{-1}$, and this reproduces (30).

From (35) $$\det \mathbf{A}=a_0^2+a_1^2+a_2^2+a_3^2=\mathscr{N}\mathrm{A}, \quad \ldots\ldots \quad (38)$$

which is also invariant under the same transformation, giving again (31), (32). A square matrix **A** is singular or non-singular according as det **A** is zero or non-zero. Hence from (38) a singular quaternion has a singular matrix representation, and conversely.

The *characteristic equation* of the matrix **A** in (35) is

$$\begin{vmatrix} a_0+ia_1-\lambda & a_2+ia_3 \\ -a_2+ia_3 & a_0-ia_1-\lambda \end{vmatrix} = 0,$$

which is

$$(a_0-\lambda)^2+a_1^2+a_2^2+a_3^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

By the Cayley-Hamilton theorem **A** must satisfy this equation, giving for the corresponding quaternion, as can easily be verified directly,

$$\mathrm{A}^2-2a_0\mathrm{A}+\mathscr{N}\mathrm{A}=0. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The roots of (40) are the *eigenvalues* of A. The general quaternion can clearly not satisfy an equation of lower degree than (40), *i. e.*, an equation of degree 1, since such an equation would reduce to A=a scalar. So (40) is the *minimum equation* of the general quaternion (E 3.7). Hence we see that the present representation by 2×2 matrices is *irreducible*, which follows also from the fact that any 2×2 matrix represents a quaternion.

We may remark that the *regular* representation of A is given by the matrix

$$\mathfrak{A}=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.$$

The characteristic equation of $\mathfrak{A}$ is found to be

$$\{(a_0-\lambda)^2+a_1^2+a_2^2+a_3^2\}^2=0,$$

showing, as we expect, that the representation $\mathfrak{A}$ is *reducible* to the representation **A**.

The analogous result for E-numbers is not discussed by Eddington. But it can be seen that his representation by 4×4 matrices is, in fact, irreducible. This follows either from his remark that any 4×4 matrix can be expressed in the form of his matrix of a general E-number (E 3.3), or from the present author's calculation†, which shows directly that the polynomial equation of lowest degree satisfied by a general E-number is of degree 4. The present author has given ‡ also the regular representation of an E-number, which requires a 16×16 matrix, and the result just stated shows that this representation is reducible. It is clear that the irreducible representation used by Eddington is the most convenient for his purposes.

---

† W. H. McCrea, Journ. London Math. Soc. xiii. pp. 283–288 (1938).
‡ W. H. McCrea, Proc. Cambridge Phil. Soc. xxxv. pp. 123–125 (1939).

5. *Mixed w-tensors.*

The work of E 4.1 can be imitated very closely in our case. As in E (4.11) we have

$$e^{i\bar{\mu}\theta}=\cos\theta+i_\mu\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

which gives $e^{i\mu\theta}i_\mu=i_\mu e^{i\mu\theta}$, $e^{i\mu\theta}i_\nu=i_\nu e^{-i\mu\theta}(\mu\neq\nu)$. . . . . (42)

Therefore if in (28) we take $t=e^{ki_1\theta}$, so that $t^{-1}=e^{-ki_1\theta}$, we have

$$\begin{aligned}A'&=a_0'+\Sigma_\mu a_\mu' i_\mu=a_0+a_1i_1+a_2i_2e^{-i_1\theta}+a_3i_3e^{-i_1\theta}\\&=a_0+a_1i_1+(a_2\cos\theta-a_3\sin\theta)i_2+(a_2\sin\theta+a_3\cos\theta)i_3 \quad . \quad (43)\end{aligned}$$

Now we may certainly "map" A as a vector in four-dimensional euclidean space, with components a_0, a_1, a_2, a_3 referred to rectangular axes and similarly for any other quaternion. But then (43) shows that what we conventionally define according to section 3 as the "rotation" of A corresponding to the transformation from A to A′ transforms the vector (a_0, a_1, a_2, a_3) into the vector $(a_0, a_1, a_2\cos\theta-a_3\sin\theta, a_2\sin\theta+a_3\cos\theta)$, and so corresponds to an ordinary euclidean rotation through angle θ in the i_2, i_3-plane. Similarly, we can get rotations in the planes (i_3, i_1), (i_1, i_2), but *not* in the planes $(\mathbf{1}, i_1)$, $(\mathbf{1}, i_2)$, $(\mathbf{1}, i_3)$.

Further, every transformation of the form $t\,(\ )\,t^{-1}$ can be interpreted as the result of such euclidean rotations. For from (30), (32) it leaves a_0 invariant, and moves (a_1, a_2, a_3) to another point on the sphere $a_1^2+a_2^2+a_3^2=$const. in the subspace $a_0=0$. But this latter result is well known to be obtainable by successive rotations in the planes (i_2, i_3), (i_3, i_1), (i_1, i_2).

These conclusions may be expressed by saying that a quaternion A regarded as a mixed w-tensor can be resolved into an *invariant* a_0, and a *space vector* (a_1, a_2, a_3) in three-dimensional euclidean space. The term space vector serves to distinguish the latter from a w-vector. This is the analogy of Eddington's resolution of a wave-tensor (E 54) and his isolation of a five-dimensional euclidean space. But his resolution provides, in addition, a part of the wave tensor which is identified as a space tensor, and for this we have no analogue.

We have just seen that the mixed w-tensor transformation transforms the point represented by the space vector (a_1, a_2, a_3) into another point on the *sphere* $a_1^2+a_2^2+a_3^2=$const. in the three-dimensional space. The two-dimensional spherical space thus obtained is the analogue of Eddington's four-dimensional spherical space (E 4.4).

If we keep say the i_1-axis fixed we restrict ourselves to rotations with t of the form $e^{ki_1\theta}$, and we have at once that a_0, a_1 are then both invariant, and (a_2, a_3) behaves as a space vector in a *two*-dimensional euclidean space. This gives the analogue of E (4.42), but again there are parts of Eddington's resolution (E (4.42) (*b*), (*c*)) which are not represented here.

We may note, further, that if we consider an element of this two-dimensional space with sides $i_2 da_2$, $i_3 da_3$ its scalar area is $d\alpha = da_2 da_3$. Alternatively, we may represent the area by the product $i_2 da_2 i_3 da_3 = i_1 d\alpha$, which is a vector normal to its plane in the three-space. The device of representing an element of area by a vector perpendicular to its plane is familiar. This slight result appears to exhaust our analogy to E 4.6.

Naturally, the results in this section are well known in elementary quaternion theory, and, in fact, account for the definitions of $\mathscr{S}$A and $\mathscr{V}$A. But they are not usually reached by the present route, which is followed only in order to produce the desired analogies.

Another well-known result is found useful. Suppose we ask, what quaternion T remains fixed under the transformation (28)? We must have

$$\mathrm{T}' = t\mathrm{T}t^{-1} = \mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

so that therefore $t\mathrm{T} = \mathrm{T}t$.

Now it can be proved that a necessary and sufficient condition for T to commute with t is that T should be an algebraic function of t, which in virtue of (40) reduces to a linear function of t. Therefore (44) requires

$$\mathrm{T} = \alpha + \beta t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where α, β are scalars. This is easy to understand geometrically from the above discussion. For the transformation has been seen to be representable as a rotation of the euclidean three-space, keeping the 1-direction fixed. Now the rotation has an invariant axis represented by a space-vector V (say), and the only space vectors which are not rotated lie along V and so are scalar multiples of V. To get the general fixed quaternion we can add any arbitrary component parallel to the fixed 1-direction. But obviously

$$t' = ttt^{-1} = t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

So we may take $\mathrm{V} = \mathscr{V}t$, and putting these results together we recover (45). It is important to notice that the vector part of t gives the axis of the corresponding rotation.

6. *w-vectors.*

In section 5 we explored the correspondence between mixed w-tensors and quaternions; we now consider the relation between w-vectors and quaternions. To do so we revert to the matrix representation (E 40 (1)).

We have seen that any 2×2 matrix can be regarded as a quaternion. Consider then the matrix † constructed as follows:

$$\mathrm{C} = \psi\chi^* = \begin{bmatrix} \psi_{(1)} \\ \psi_{(2)} \end{bmatrix} [\chi_{(1)} \quad \chi_{(2)}] = \begin{bmatrix} \psi_{(1)}\chi_{(1)} & \psi_{(1)}\chi_{(2)} \\ \psi_{(2)}\chi_{(1)} & \psi_{(2)}(\chi_2) \end{bmatrix}. \quad . \quad (47).$$

† Matrix suffixes are placed in brackets to distinguish them from suffixes of quaternion coefficients.

Then **C** represents a quaternion C (say). But the general quaternion cannot be represented by a factorizable matrix of this form. If a quaternion does have such a representation it will be said to be *pure* (E 40, 65).

Let now a quaternion A have the representation

$$\mathrm{A}=\begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

This is of the form (47) if

$$x=\psi_{(1)}\,\chi_{(1)},\quad y=\psi_{(1)}\,\chi_{(2)},\quad z=\psi_{(2)}\,\chi_{(1)},\quad w=\psi_{(2)}\,\chi_{(2)}.$$

These give as equations for $\psi_{(1)}, \ldots, \chi_{(2)}$,

$$\left.\begin{matrix} z\psi_{(1)}-x\psi_{(2)}=0 \\ w\psi_{(1)}-y\psi_{(2)}=0 \end{matrix}\right\} \quad \left.\begin{matrix} \chi_{(1)}y-\chi_{(2)}x=0 \\ \chi_{(1)}w-\chi_{(2)}z=0 \end{matrix}\right\},$$

which may be written in matrix form as

$$\begin{bmatrix} z & -x \\ w & -y \end{bmatrix}\begin{bmatrix} \psi_{(1)} \\ \psi_{(2)} \end{bmatrix}=0, \qquad [\chi_{(1)}\ \ \chi_{(2)}]\begin{bmatrix} y & w \\ -x & -z \end{bmatrix}=0. \quad . \quad (48)$$

A necessary and sufficient condition for these to have non-zero solutions is that the determinants of the coefficients should vanish. But this is seen to give in both cases $\det\mathbf{A}=0$. *Hence a necessary and sufficient condition for a quaternion to be pure is that it should be singular.*

Further, using (36), (48) may be written

$$\begin{bmatrix} -a_2+ia_3 & -a_0-ia_1 \\ a_0-ia_1 & -a_2-ia_3 \end{bmatrix}\begin{bmatrix} \psi_{(1)} \\ \psi_{(2)} \end{bmatrix}=0, \quad [\chi_{(1)}\ \ \chi_{(2)}]\begin{bmatrix} a_2+ia_3 & a_0-ia_1 \\ -a_0-ia_1 & a_2-ia_3 \end{bmatrix}=0.$$

Using (33) these are

$$(-a_2\mathbf{1}+a_3i_1-a_0i_2-a_1i_3)\psi=0, \quad \chi^*(a_2\mathbf{1}+a_3i_1+a_0i_2-a_1i_3)=0,$$

or, premultiplying the first and postmultiplying the second by i_2,

$$(a_0\mathbf{1}-a_1i_1-a_2i_2-a_3i_3)\psi=0, \quad \chi^*(a_0\mathbf{1}-a_1i_1-a_2i_2-a_3i_3)=0, \quad . \quad (49)$$

which are simply $$\bar{\mathrm{A}}\psi=0, \quad \chi^*\bar{\mathrm{A}}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

These are the analogues of Eddington's "simple wave equations" (E (5.43), (5.44)) and exhibit clearly the rôle of these equations as being merely a way of writing the equations which determine the factors of a pure matrix. The form (50) shows them to be tensor equations and therefore independent in this form of the particular frame employed. We refer to them as simple w-equations.

The general theorem involved here is that a necessary and sufficient condition for a non-vanishing $n\times n$ matrix to be pure is that it should be of rank 1. In our case of $n=2$ this gives at once the single condition that the determinant of the matrix should vanish; the above working was necessary only in order to derive the w-equations. In Eddington's

case the set of equations like E (5.45), (5.46) must express the conditions for the matrix (E (3.61) to be of rank 1.

Now, if A is any quaternion, it is clear from section 5 that A could be regarded as being obtained by a suitable rotation of a suitably chosen quaternion B whose space vector lies initially along the t_1-axis (say). Hence there exists a transformation giving

$$A=tBt^{-1}=t(b_0+b_1i_1)t^{-1} \text{ (say)}.$$

If, then, A is singular, $\mathcal{N}A=\mathcal{N}B=0$, and hence

$$B=b_0(1\pm ii_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

giving standard forms of a non-degenerate pure quaternion, i. e., $b_0=a_0\neq 0$. If A is singular and degenerate we could similarly transform it to standard forms (say)

$$C=c(i_2\pm ii_3) \quad (c\neq 0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

In this case A and C are null space vectors. The forms (51), (52) are analogous to E (5.54), (5.56).

We can *normalize* a non-degenerate quaternion A by taking a scalar multiple of it, A_n, whose spur is 1, that is $\mathcal{S}A_n=\frac{1}{2}$. Hence a normalized pure quaternion has standard form, from (51),

$$B_n=\tfrac{1}{2}(1\pm ii_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

This gives at once $B_n^2=B_n$. But such an equation is invariant under rotations, and therefore

$$A_n^2=A_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

is a necessary condition for A_n to be a normalized non-degenerate pure quaternion. It is easy to show that it is also sufficient. But (54) is also the condition for A_n to be *idempotent* (E 5.6).

Consider now the pair of idempotent quaternions given by the standard forms (53)

$$B_a=\tfrac{1}{2}(1+ii_1), \quad B_b=\tfrac{1}{2}(1-ii_1).$$

They satisfy the set of relations

$$B_a^2=B_a, \quad B_b^2=B_b, \quad B_aB_b=0, \quad B_a+B_b=1,$$

and thus form an elementary example of a *spectral set* (E 5.7).

Before leaving the subject of pure quaternions we deduce a simple property from the theorem on matrices which gives: If X, Y, Z are matrices then

$$\text{spur (XY)}=\text{spur (YX)}, \quad \text{spur (XYZ)}=\text{spur (YZX)}=\text{spur (ZXY)},$$

provided the products indicated exist and are square.

Hence if in (10) we take $\mathbf{A}=\psi\chi^*$ we obtain

$$2a_0=2\mathscr{S}\mathbf{A}=\text{spur }\mathbf{A}=\text{spur }\psi\chi^*=\text{spur }\chi^*\psi, \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$-2a_\mu=2\mathscr{S}(\mathbf{A}i_\mu)=\text{spur }(\mathbf{A}\mathbf{i}_\mu)=\text{spur }(\psi\chi^*\mathbf{i}_\mu)=\text{spur }(\chi^*\mathbf{i}_\mu\psi). \quad . \quad (56)$$

But in (55), (56) the last matrices reduce to a single element. So we have

$$a_0=\tfrac{1}{2}\chi^*\psi, \quad a_\mu=-\tfrac{1}{2}\chi^*\mathbf{i}_\mu\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

which are compact formulæ for the coefficients of a pure quaternion (E (3.37)).

7. *Covariant w-tensors.*

Since any 2×2 matrix can be treated as a quaternion, equation (4) in the case $n=2$ can be interpreted as a quaternion equation, and we may call a quaternion S which transforms in this way a covariant w-tensor.

In the matrix representation (33) all the individual matrices are symmetrical, except $\mathbf{i}_2$, which is antisymmetrical. Therefore if

$$q=q_0+q_1i_1+q_2i_2+q_3i_3,$$

then

$$\tilde{q}=q_0+q_1i_1-q_2i_2+q_3i_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

This particular correspondence between q, $\tilde{q}$ is, however, *not* independent of the particular matrix representation. We may associate this representation with an arbitrarily selected symbolic frame, and then the representation associated with any equivalent frame is uniquely determined. There is then a different correspondence between q, $\tilde{q}$ for each different frame. For instance, we could take a new frame 1, $\underline{i}_1$, $\underline{i}_2$, $\underline{i}_3$, such that $\underline{i}_1=i_1$, $\underline{i}_2=(i_2-i_3)/\sqrt{2}$, $\underline{i}_3=(i_2+i_3)/\sqrt{2}$; then if $q=\underline{q}_0+\underline{q}_1\underline{i}_1+\underline{q}_2\underline{i}_2+\underline{q}_3\underline{i}_3$, we have $\tilde{q}=\underline{q}_0+\underline{q}_1\underline{i}_1-\underline{q}_3\underline{i}_2-\underline{q}_2\underline{i}_3$. Nevertheless, this very lack of invariance in regard to rotations of the frame plays an essential part in Eddington's application of his analogous result (E 96).

In this section we shall give results appropriate to the representation (33), and so to the correspondence (58). In any other representation there would be corresponding, but not identical, results.

Consider first the special case where q is of the form

$$q=e^{\frac{1}{2}i_\mu\theta}=\cos\tfrac{1}{2}\theta+i_\mu\sin\tfrac{1}{2}\theta, \text{ so that } q^{-1}=\cos\tfrac{1}{2}\theta-i_\mu\sin\tfrac{1}{2}\theta. \quad . \quad (59)$$

Then from (58)

$$\tilde{q}=q\ (\mu=1,\ 3), \quad \tilde{q}=q^{-1}\ (\mu=2). \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Therefore the case $\mu=2$ gives the same transformation for a covariant w-tensor S as for a mixed w-tensor. The cases $\mu=1$, 3 give for the transformation

$$\mathrm{S}'=q\mathrm{S}q,$$

which Eddington calls an antiperpendicular rotation (E 4.16). To illustrate it take $\mu=1$, and let

$$S=s_0+s_1i_1+s_2i_2+s_3i_3.$$

Then
$$S'=e^{\frac{1}{2}i_1\theta}Se^{\frac{1}{2}i_1\theta}$$
$$=s_0e^{i_1\theta}+s_1i_1e^{i_1\theta}+s_2i_2+s_3i_3, \text{ using (42)}$$
$$=(s_0\cos\theta-s_1\sin\theta)+(s_0\sin\theta+s_1\cos\theta)\,i_1+s_2i_2+s_3i_3.$$

Hence this is seen to produce a rotation in the plane $(1, i_1)$. The general result is that antiperpendicular rotations take place in planes in which r-rotations are forbidden, and conversely (E 52); but, of course, an antiperpendicular rotation in the plane $(1, i_2)$ is not a covariant tensor transformation.

We proceed now to a more general treatment. We can "map" a covariant w-tensor S as a vector in four-dimensional euclidean space, with components s_0, s_1, s_2, s_3 referred to rectangular axes, just as we mapped a mixed w-tensor. But when we apply a covariant tensor transformation we shall in general get a different new vector from any we could get by applying a mixed tensor transformation.

To see this in detail, let S, A be a covariant and a mixed w-tensor respectively, and let them be initially represented by the same vector in the four-space (E 95). Applying the appropriate transformations with transformation matrix q, assuming this is non-singular, we get new vectors representing S′, A′, where

$$S'=qS\tilde{q}, \quad A'=qAq^{-1},$$

and so, since S=A, we have $S'=A'q\tilde{q}$.

The relative displacement of the points represented by the vectors S′, A′ is therefore given by the vector

$$S'-A'=A'(q\tilde{q}-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

As q varies these relative displacements trace out a certain space, which we shall call the p-space of S. It is the analogue of Eddington's *phase*-space (E 7.2).

However, this statement does not completely define the p-space. We might define it as the region traced out by the relative displacement in the original frame, or as that traced out in a frame rotating with A, or we might use some other specified frame †. This appears to be an aspect of the difficulty stated by Eddington (E 99). The analogue of the definition adopted by him would appear to be given by referring

† Compare, for example, the fact that the space traced out by the radius vector from the Earth to the Moon is different when referred to "fixed" axes through the Sun from what it is when referred to axes through the Sun which revolve with the Earth.

the p-space to axes rotating with A. To do this we must then refer (61) to the frame **1**, $\underline{i}_1$, $\underline{i}_2$, $\underline{i}_3$, where

$$\underline{i}_\mu = q i_\mu q^{-1} \quad \ldots\ldots\ldots\ldots \quad (62)$$

Now if $q = q_0 + \Sigma_\mu q_\mu i_\mu$, $q' = q_0 + \Sigma_\mu q_\mu \underline{i}_\mu$,

then, as in (46),

$$q' = q \quad \ldots\ldots\ldots\ldots \quad (63)$$

Also, from (62), (63),

$$i_\mu = q^{-1}\underline{i}_\mu q = q'^{-1}\underline{i}_\mu q' \quad \ldots\ldots\ldots\ldots \quad (64)$$

Therefore

$$\begin{aligned}\tilde{q} &= q_0 + q_1 i_1 - q_2 i_2 + q_3 i_3 \\ &= q'^{-1}(q_0 + q_1\underline{i}_1 - q_2\underline{i}_2 + q_3\underline{i}_3)q' \\ &= q'^{-1}(\tilde{q})'q', \text{ say,} \quad \ldots\ldots\ldots\ldots \quad (65)\end{aligned}$$

where, for the reason already explained, $(\tilde{q})'$ is not the same as $\tilde{q}'$, that is, the transpose of q does not transform into the transpose $\tilde{q}'$ of q'. We have then from (65)

$$q\tilde{q} = q'q'^{-1}(\tilde{q})'q' = (\tilde{q})'q' \quad \ldots\ldots\ldots\ldots \quad (67)$$

Naturally, this simple form of relation does not apply to any quaternion other than that used to generate the transformation (or a linear function of it).

We have, further, by direct calculation

$$\begin{aligned}(\tilde{q})'q' &= (q_0^2 - q_1^2 + q_2^2 - q_3^2) + 2(q_0q_1 - q_2q_3)\underline{i}_1 + 2(q_0q_3 + q_1q_2)\underline{i}_3 \\ &= p_0 + p_1\underline{i}_1 + p_3\underline{i}_3 \text{ (say).} \quad \ldots\ldots\ldots\ldots \quad (68)\end{aligned}$$

Also $$A' = qAq^{-1} = qSq^{-1} = s_0 + s_1\underline{i}_1 + s_2\underline{i}_2 + s_3\underline{i}_3. \quad \ldots\ldots\ldots\ldots \quad (69)$$

Combining (68), (69) with (61) we get an explicit expression for the displacement S′—A′ referred to the desired frame. For given S it is seen to depend only on the three parameters p_0, p_1, p_3, which are clearly independent, and so it traces out a *three*-dimensional p-space in this frame. Also, since the components of S′—A′ involve the three parameters linearly, the p-space is a hyperplane. It should be noted that we have not restricted q as Eddington does in E 7.3 in order to secure a closed space.

It has to be noted that the whole discussion depends upon the arbitrary setting up of a (1-1) correspondence between a transform of S and a transform of A, namely, by taking the transforms generated by the same transformation quaternion q.

8. *Differential w-equations.*

Let $C = \psi\chi^* = c_0 + \Sigma_\mu c_\mu i_\mu$ be a pure quaternion, and let c_0, c_1, c_2, c_3, and consequently $\psi_{(1)}$, $\psi_{(2)}$, $\chi_{(1)}$, $\chi_{(2)}$ be functions of the coefficients of a quaternion

$$X = x_0 + \Sigma_\mu x_\mu i_\mu.$$

Then we may regard C as defining a "vector field" in the four-space in which X is the "position vector." Our object is to discuss the *divergence* of this field.

Following E 8.2 we define operators

$$\nabla=\frac{\partial}{\partial x_0}-\Sigma i_\mu\frac{\partial}{\partial x_\mu},\quad \nabla^*=\frac{\delta}{\delta x_0}-\Sigma\frac{\delta}{\delta x_\mu}i_\mu,$$

where an operator $\frac{\partial}{\partial x}$ operates on functions written *after* it, and $\frac{\delta}{\delta x}$ denotes the same operation performed on functions written *before* it. A distinction of this sort is obviously required in order to be able to adhere to quaternion and matrix notation.

Using (57) we now have

$$\frac{\partial c_0}{\partial x_0}=\frac{1}{2}\frac{\partial}{\partial x_0}(\chi^*\psi)=\frac{1}{2}\chi^*\frac{\delta}{\delta x_0}\psi+\frac{1}{2}\chi^*\frac{\partial}{\partial x_0}\psi,$$

$$\frac{\partial c_\mu}{\partial x_\mu}=-\frac{1}{2}\frac{\partial}{\partial x_\mu}(\chi^*i_\mu\psi)=-\frac{1}{2}\chi^*\frac{\delta}{\delta x_\mu}i_\mu\psi-\frac{1}{2}\chi^*i_\mu\frac{\partial}{\partial x_\mu}\psi.$$

Therefore, defining the divergence of C by

$$\text{div C}=\frac{\partial c_0}{\partial x_0}+\frac{\partial c_1}{\partial x_1}+\frac{\partial c_2}{\partial x_2}+\frac{\partial c_3}{\partial x_3},$$

we have

$$\text{div C}=\tfrac{1}{2}\chi^*(\nabla^*+\nabla)\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Any given 2×1 matrix can be expressed as $\mathrm{M}\psi$, where ψ is another given 2×1 matrix and M is a 2×2 matrix, which is, in fact, not uniquely determined by this requirement. Now $\nabla\psi$ is a 2×1 matrix, so we shall let M be any 2×2 matrix such that

$$\nabla\psi=\mathrm{M}\psi.$$

Similarly, let M* be any 2×2 matrix such that

$$\chi^*\nabla^*=\chi^*\mathrm{M}^*.$$

Then from (70)

$$\text{div C}=\tfrac{1}{2}\chi^*(\mathrm{M}^*+\mathrm{M})\psi.$$

A *sufficient* condition for div C$=0$ is then $\mathrm{M}^*+\mathrm{M}=0$, giving†

$$(\nabla-\mathrm{M})\psi=0,\quad \chi^*(\nabla^*+\mathrm{M})=0. \quad . \quad . \quad . \quad . \quad . \quad (71)$$

These we call *differential w-equations*. Their invariance under rotations of the frame to which X is referred follows as in Eddington's case (E 119).

We shall not go further with the analogy to Eddington's differential wave equations (E 8.2). The object of this section is to show the contrast, emphasized by Eddington in his corresponding work, between the simple w-equations of section 6, which hold identically for any pure quaternion, and the present differential w-equations which depend upon a number of additional assumptions about the relevant pure quaternion.

† *Cf.* E. Cartan, 'Leçons sur la théorie des spineurs' (1938), § 71.

9. *Double Quaternions.*

Let $\mathbf{1}, i_1, i_2, i_3$ and $\mathbf{1}, j_1, j_2, j_3$ be two sets of quaternion elements, each having the fundamental properties (5), (6), (7), and such that every element of one set commutes with every element of the other set. Consider then the set of sixteen products

$$\begin{aligned}&\mathbf{1}, j_1, j_2, j_3, i_1, i_1j_1, i_1j_2, i_1j_3, i_2, i_2j_1, i_2j_2, i_2j_3, i_3, i_3j_1, i_3j_2, i_3j_3\\ \equiv&\mathbf{1}, k_{01}, k_{02}, k_{03}, k_{10}, k_{11}, \ldots, k_{33}\\ \equiv&\mathbf{1}, (k_\mu) \quad (\mu=1, \ldots, 15), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (72)\end{aligned}$$

where we can use a double suffix or single suffix notation for the k's to suit our convenience, without causing confusion.

The assumptions lead to relations of the types shown by the following examples :—

$$\mathbf{1}k_\mu=k_\mu\mathbf{1},$$

$$k_{01}k_{02}=k_{03}=-k_{02}k_{01}, \quad k_{01}k_{10}=k_{11}=k_{10}k_{01}, \quad k_{01}k_{11}=-k_{01}=k_{11}k_{01},$$

$$k_{01}k_{12}=k_{13}=-k_{12}k_{01}, \quad k_{12}k_{23}=k_{31}=k_{23}k_{12}, \quad {k_{01}}^2=-1, \quad {k_{12}}^2=1.$$

Such relations show that each k_μ anticommutes with eight others and commutes with the rest ; the squares of six of the k_μ's are -1 and the squares of the rest are 1: the product of any two k_μ's is, apart from sign, another k_μ or 1. The latter shows that the sixteen elements (72) form a complete set. Hence we obtain an algebra of quantities like

$$\mathrm{X}=x_0+\sum_{\mu=1}^{15} x_\mu k_\mu. \quad \ldots\ldots\ldots \quad (73)$$

This algebra is almost *equivalent* to E-number algebra. (The precise equivalence is obtained by inserting a factor i in certain of the k_μ's (E 3.8).) But this is not the feature which concerns us here. We are interested in the fact that its relations to quaternion algebra is *analogous* to the relation of Eddington's algebra of double wave tensors to that of his E-numbers (EX). It will therefore be appropriate to call a quantity like (73) a *double quaternion* †, and the algebra of such quantities is in the standard sense the *product* of two quaternion algebras.

It is the latter aspect which it is desirable to keep in evidence for the purpose of the analogy. So we shall not give, for instance, the irreducible representation of a double quaternion by a 4×4 matrix, but shall consider two other more appropriate representations. First, we could use a "mixed" representation by a 2×2 matrix, in which the elements are quaternions of, say, the i_μ-system, and in forming the product of i_μ with such a matrix we should adopt the rule indicated by

$$i_\mu\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}i_\mu a & i_\mu b\\ i_\mu c & i_\mu d\end{bmatrix}, \qquad \begin{bmatrix}a & b\\ c & d\end{bmatrix}i_\mu=\begin{bmatrix}ai_\mu & bi_\mu\\ ci_\mu & di_\mu\end{bmatrix}.$$

† This is, of course, different from the more familiar *bi-quaternion*.

We should then represent, for example, $i_2 j_3$ by

$$i_2 \begin{bmatrix} . & i \\ i & . \end{bmatrix} = \begin{bmatrix} . & ii_2 \\ ii_2 & . \end{bmatrix},$$

and so on. Second, we can proceed to replace the quaternions of the i_μ-system by 2×2 matrices, so that a double quaternion is represented by a 2×2 matrix, each of whose elements is itself a 2×2 matrix. The general rules for combining such composite matrices, and corresponding 2×1 and 1×2 matrices, will be sufficiently illustrated by the following development (where for clarity we drop the brackets on the matrix suffixes and write row suffixes below and column suffixes above) :

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} (\chi^1 \quad \chi^2) \, [\omega^1 \quad \omega^2] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (74)$$

$$= \begin{bmatrix} \begin{pmatrix} \psi_1\phi_1 \\ \psi_2\phi_1 \end{pmatrix} & \begin{pmatrix} \psi_1\phi_2 \\ \psi_2\phi_2 \end{pmatrix} \end{bmatrix} [(\chi^1\omega^1 \quad \chi^2\omega^1) \quad (\chi^1\omega^2 \quad \chi^2\omega^2)] \; . \; . \; . \; . \; . \qquad (75)$$

$$= \begin{bmatrix} \begin{pmatrix} \psi_1\chi^1\phi_1\omega^1 & \psi_1\chi^2\phi_1\omega^1 \\ \psi_2\chi^1\phi_1\omega^1 & \psi_2\chi^2\phi_1\omega^1 \end{pmatrix} & \begin{pmatrix} \psi_1\chi^1\phi_1\omega^2 & \psi_1\chi^2\phi_1\omega^2 \\ \psi_2\chi^1\phi_1\omega^2 & \psi_2\chi^2\phi_1\omega^2 \end{pmatrix} \\ \begin{pmatrix} \psi_1\chi^1\phi_2\omega^1 & \psi_1\chi^2\phi_2\omega^1 \\ \psi_2\chi^1\phi_2\omega^1 & \psi_2\chi^2\phi_2\omega^1 \end{pmatrix} & \begin{pmatrix} \psi_1\chi^1\phi_2\omega^2 & \psi_1\chi^2\phi_2\omega^2 \\ \chi_2\chi^1\phi_2\omega^2 & \psi_2\chi^2\phi_2\omega^2 \end{pmatrix} \end{bmatrix} . \; . \; . \qquad (76)$$

$$= \begin{pmatrix} \psi_1\chi^1 & \psi_1\chi^2 \\ \psi_2\chi^1 & \chi_2\chi^2 \end{pmatrix} \begin{bmatrix} \phi_1\omega^1 & \phi_1\omega^2 \\ \phi_2\omega^1 & \phi_2\omega^2 \end{bmatrix} . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (77)$$

This is given in order to illustrate also some points analogous to those in E 10.4. (74) is the product of four w-vectors (E (10.411)) ; (76) is the resulting double quaternion ; (75) expresses it as the product of two *double w-vectors* ; (77) expresses it as the product of two w-tensors (E (10.413)). It will be obvious that the general double quaternion is not resolvable into such factors. We remark also that Q in (23) is the analogue of Eddington's interchange operator (E 10.3). To pursue the whole analogy further than this would unduly prolong our discussion.

10. *Reality Conditions and Right- and Left-handed Frames.*

We have nowhere put any restriction upon the scalar numbers appearing in the work and have therefore in particular not discussed the analogues of Eddington's reality conditions (E VI and 7.3). The analogy could be worked through, but we shall not consider the details, since in

Eddington's work these conditions are closely bound up with the physical applications for which they are intended, and which do not concern us here.

We shall merely remark, in the first place, that the possibility of representing i_1, i_2, i_3 by the matrices in (33), of which one contains only real elements and two contain only imaginary elements, is the simplest possible illustration of Newman's general theorem (E 42). It is given by taking p, q (E 42) to be both unity. The correspondence between reality and symmetry in these matrices is also like that in E 7.3.

In the second place, the following up of the analogy would require us to replace the unity element **1** in the quaternions by i_0 (say) where $i_0 = \pm i\mathbf{1}$, and i_0 is then the analogue of Eddington's E_{16} (E (2.38)). The symbolic frames $(i\mathbf{1}, i_1, i_2, i_3)$ and $(-i\mathbf{1}, i_1, i_2, i_3)$ provide a trivial analogy of right- and left-handed frames (E 3.9). For since $ti\mathbf{1}t^{-1} = i\mathbf{1}$, it is not possible to transform one of these frames into the other by a "rotation." The frame $(1, i_1, i_2, i_3)$ would then actually be a little more closely analogous to Eddington's *neutral set* (E (6·412)) than to his other cases.

XXVI. *The Propagation of Electromagnetic Waves in an Ionized Medium and the Calculation of the True Heights of the Ionized Layers of the Atmosphere.*

By Olof Rydbeck, Cruft Laboratory, Harvard University, and the Chalmers Institute of Technology, Gothenburg *.

[Received January 10, 1940.]

Let us consider the wave-equation for the propagation of electromagnetic waves in an ionized medium if there be no magnetic field.

$$\nabla^2\Pi+\left(\frac{2\pi f}{c}\right)^2\left(1-\frac{f_0^2}{f^2}\right)\Pi=0\ \dagger. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Π=the wave-function from which any of the potentials may be deduced. f=the frequency of the wave, and $f_0^2=\dfrac{Ne^2}{m_e\pi}$, where N is the number of electrons per cubic cm., e is the charge of the electron, and m_e its mass.

(1) is very similar to the Schrödinger equation, and it is clearly seen that the equivalent potential energy of the wave-particles or wave-packets is proportional to N.

Making the energy transformation

$$\frac{hf}{2}=\frac{mc^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $hf/2$ is the zero-point energy of the oscillator, (1) becomes

$$\nabla^2\Pi+\frac{8\pi^2m}{h^2}\left(\frac{hf}{2}-\frac{hNe^2}{2\pi m_e f}\right)\Pi=0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The equivalent potential energy of the wave-packet thus seems to be

$$V=\frac{hNe^2}{2\pi m_e f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the force acting on it is

$$F_s'=-\frac{he^2}{2\pi m_e . f}\cdot\frac{\partial N}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4\,a)$$

The mechanical interpretation is only valid when eq. (3) is reducible to the Hamilton-Jacobi equation for the action function of a particle of

---

* Communicated by the Author.

† The influence of heavier ions and of collisional friction is not considered in this connexion.

energy $hf/2$ travelling in a force field with a potential energy V, as is well known. Whenever the relative change in phase velocity in a wave-length is not very small, the reduction of eq. (3) to the Hamilton-Jacobi equation is no longer a permissible approximation. In the regions where this is the case, geometrical optics is no longer valid. In the case of the ionized layers of the ionosphere the geometrical optics or the mechanical interpretation is not valid for the long commercial radio waves, whereas it holds very well for the short broadcasting waves, except near the level where V approaches $hf/2$.

If the angle of incidence upon the uniformed ionized layer is ϕ, then the vertical component of the initial kinetic energy is $\frac{hf}{2}\cos^2\phi$ and the constant horizontal kinetic energy is $\frac{hf}{2}\sin^2\phi$. At the level of potential energy V, the vertical component of the transmission velocity must be

$$v_v = \left(\frac{2}{m}\left(\frac{hf}{2}\cos^2\phi - \mathrm{V}\right)\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For $v_v = 0$, $\mathrm{V} = \mathrm{V}_c = \frac{hf}{2}\cos^2\phi$, where V_c is the critical potential energy. This is the equivalent of the ordinary cosine-law for the determination of the critical frequency. If $\cos\phi$ is sufficiently small, *i. e.*, sufficiently oblique incidence, the wave packet never penetrates into the vicinity of the level, where V is equal to $hf/2$ and where the mechanical interpretation is not valid. Usually the energies of the exploring signals used in the ionospheric measurements are so high, however, and the wave lengths become so small, that one is quite justified in assuming the mechanical interpretation to hold up to the classical level of reflexion at vertical incidence, *i. e.*, where $\mathrm{V} = hf/2$. In the following we shall base our calculations on that assumption, and we shall later on discuss the error thus made.

If we assume the electronic density to be a function of the height only, *i. e.*, a uniform ionosphere, the mechanical interpretation thus means that the horizontal component of the transmission velocity is constant. As it is the same for an unrefracted ray travelling along the side of the equivalent triangle, formed by the ray path, the unrefracted ray will reach the top of the triangle at the same time as the refracted ray reaches the apex of its path. This was shown already in 1926 by Breit and Tuve *. The height to the top of the triangle is defined as the virtual height. Strictly speaking, this definition only has a meaning when the influence of the terrestrial magnetic field is neglected and when the mechanical interpretation is valid over the path.

---

* G. Breit and M. A. Tuve, Phys. Rev. xxviii. p. 554 (1926).

In the case of a spherical, uniform ionosphere the transmission is a case of central motion, when the mechanical interpretation is valid. Thus the wave-packet moves with constant angular momentum, *i. e.*, constant surface velocity. This is also expressed by the optical invariant. Thus the unrefracted ray will reach the top of the equivalent triangle after the refracted ray has passed the apex. Therefore, the virtual path-length is less than the path-length of the equivalent triangle. The difference is very small, however, when the height of the layer is small compared to the radius, as is the case of the ionosphere.

The Determination of the True Height of Reflexion.

The total time of travel for a reflected signal at vertical incidence is

$$T=2\int_0^{h_T}\frac{dh}{V_v}=2\int_0^{h_T}\frac{dh}{\left(\frac{2}{m}\left(\frac{hf}{2}-V\right)\right)^{\frac{1}{2}}}, \quad \ldots\ldots \quad (6)$$

where h_T is the true height of reflexion. The so-called virtual height is h_v, where $h_v=\frac{T.c}{2}$.

Thus

$$h_v=\int_0^{h_T}\frac{c\,dh}{\left(\frac{2}{m}\left(\frac{hf}{2}-V\right)\right)^{\frac{1}{2}}}. \quad \ldots\ldots \quad (6\,a)$$

Now, our measurements give h_v as a function-frequency, *i. e.*, as a function of the energy of the ray-particles which enter the ionized region.

If we desire to find out how the gravitational force varies with the distance from the earth, for example, we should have to make a similar experiment by throwing projectiles upwards at different speeds (we do not consider the air resistance and the rotation of the earth) and noticing their time of return. For every initial energy of the projectile we get the corresponding time of travel and the virtual height corresponding to constant velocity and a perfect, elastic reflexion. It seems natural that the times of travel as functions of initial energy should give us information regarding the gravitational force as a function of height. This can be shown to be the case. The time of travel as a function of initial energy determines the gravitational force as a function of height uniquely (if no other influences are considered). It should be the same for the exploring signal in the case where the potential energy is a function of the electronic density only.

Now, if the potential energy is a single valued function of the height, we can write the height as a function of the corresponding potential energy, viz. :

$$h_T=h_T(V). \quad \ldots\ldots \quad (7)$$

This changes (6 *a*) into

$$h_v=\int_0^{\frac{hf}{2}}\frac{h_T'(V)\,dV}{\left(1-\frac{2V}{hf}\right)^{\frac{1}{2}}}+h(0), \quad \ldots\ldots\ldots \quad (8)$$

where $$h_T'(V)=\frac{dh_T}{dV}.$$

Let us now take the case of the ordinary component of the ray when the magnetic field is perpendicular to the direction of propagation, as is practically the case at Huancayo, Peru.

Inserting the value of V (6 *a*) in (8) we get, if we change to the variable ω_0^2 instead of V, that

$$(\omega_0=2\pi f_0) \qquad h_v(\omega^2)=\omega\int_0^{\omega}\frac{h_T'(\omega_0^2)\,d(\omega_0^2)}{(\omega^2-\omega_0^2)^{\frac{1}{2}}}+h(0). \quad \ldots\ldots \quad (9)$$

This is an integral equation of the Abelian type *.

$h_v(\omega^2)$ is known and it is $h_T(\omega^2)$ we desire to obtain.

A similar integral equation occurs in a problem of seismology when the depth of a refracted seismic wave is to be determined. It was also used by Abel in an application of a similar kind.

Now $$h_T(\omega)=h(0)+\int_0^{\omega}h_T'(\omega_0^2)\,d(\omega_0^2),$$

where $h(0)$ is the height of zero ionic density (the lower boundary of the ionized layer). The expression for the beta function, $\beta(\frac{1}{2},\frac{1}{2})$, is easily transformed into

$$\beta(\tfrac{1}{2},\tfrac{1}{2})=\pi=\int_{\omega_0}^{\omega}\frac{2\omega_1\,d\omega_1}{(\omega^2-\omega_1^2)^{\frac{1}{2}}(\omega_1^2-\omega_0^2)^{\frac{1}{2}}}. \quad \ldots\ldots \quad (10)$$

by a simple transformation of the variable. On multiplying h_T by $\beta(\frac{1}{2},\frac{1}{2})$ we get

$$h_T(\omega^2)=\frac{1}{\pi}\int_0^{\omega}2\omega_0\,d\omega_0\int_{\omega_0}^{\omega}\frac{h_T'(\omega_0^2)2\omega_1\,d\omega_1}{(\omega^2-\omega_1^2)^{\frac{1}{2}}(\omega_1^2-\omega_0^2)^{\frac{1}{2}}}+h(0).$$

Now, on using Dirichlet's formula †, we get the necessary inversion of the order of integration, if we assume that h_T' is continuous throughout the range.

Doing so, we get

$$h_T(\omega^2)=\frac{2}{\pi}\int_0^{\omega}\frac{h_v(\omega_1^2)\,d\omega_1}{(\omega^2-\omega_1^2)^{\frac{1}{2}}}, \quad \ldots\ldots \quad (11)$$

---

* N. H. Abel, 'Solution de quelques problèmes à l'aide d'intégrales definies' (1823), Œuvres, p. 11, 1.

† Whittaker and Watson, 'Modern Analysis,' p. 77.

which can be written

$$h_T(\omega)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} h_v(\omega \sin \phi)\, d\phi. \quad . \quad . \quad . \quad . \quad . \quad (11\,a)$$

The true height is finally obtained in the simple form of the Schlömilch integral, the only thing we need know is the virtual height as a function of the frequency.

The true height is easily obtained by planimetration from the virtual height curve projected on an angular frequency scale. To begin with we take a few simple examples (assumed to be valid up to the frequency ω_c).

$$\left.\begin{aligned}
&(a)\ \text{if } h_v(\omega)=h(0)+k_1\left(\frac{\omega}{\omega_c}\right), \text{ we get } h_T(\omega)=h(0)+\frac{2}{\pi}k_1\left(\frac{\omega}{\omega_c}\right).\\
&(b)\ \text{if } h_v(\omega)=h(0)+k_2\left(\frac{\omega}{\omega_c}\right)^2, \text{ we get } h_T(\omega)=h(0)+\frac{1}{2}k_2\left(\frac{\omega}{\omega_c}\right)^2.\\
&(c)\ \text{if } h_v(\omega)=h(0)+k_n\left(\frac{\omega}{\omega_c}\right)^n, \text{ we get } h_T(\omega)\\
&\qquad =h(0)+\frac{1}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\,.k_n\left(\frac{\omega}{\omega_c}\right)^n.
\end{aligned}\right\}$$

In all these cases the constant k means the increase in virtual height at the frequency ω_c. Although the cases a, b, and d give an exaggerated increase in virtual height for lower frequencies, it still gives a maximum true height that is only 63·5, 50, and 47·3 per cent. respectively of the maximum virtual height. Case c gives a closer approximation to actual conditions. When n is large, it gives a rapid rise in virtual height first at frequencies near the critical frequency. As is seen the true height is very much smaller.

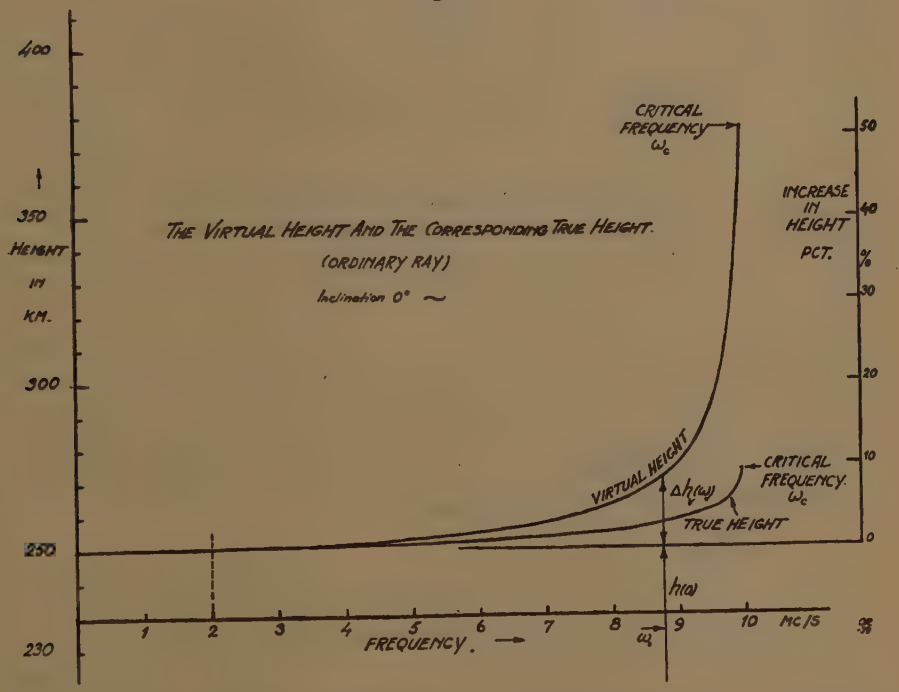
Fig. 1 shows a typical case, the virtual height of the ordinary ray as a function of frequency for the F-region in the evening. Measured values only extend down to about 2 mc./s., but it is easily seen that the curve can be extended by its tangent down to the frequency zero. The maximum increase in virtual height is about 50 per cent., a value that is frequently obtained in the F-region after sunset. The true height computed from this curve is also shown. The increase in true height is hardly 10 per cent. This is as should be expected from (11 a) and examples a, b, etc. *The rapid increase near the critical frequency contributes very little to the integral.*

Sometimes the F-region is diffuse and the virtual height recorded varies all the time, so that the virtual height line is a broad band. The true variation in height is then obtained by integration along the upper

and lower lines of this black region. The virtual height may vary within some 50 or 60 per cent. of $h(0)$, but the integration usually gives a variation in true height of only some 8 to 15 per cent.

The above method does not give a dependable idea of the very low electronic densities of the layers. This could never be expected. These regions cannot be measured anyway, because at such low frequencies the exploring signals would always be reflected from ionic regions of low intensity below the main layer.

Fig. 1.



When lower layers are recorded at the same time, it is very often possible to extend the tangent of the virtual height line of the higher layers at the lowest frequency at which reflexion is registered to the point where integration should begin. Usually the slope is so low that this is permissible.

In the present application the angle between the direction of propagation and the terrestrial magnetic field is 90°. However, the solution can be used with an error less than the experimental error at angles down to about 75°. The electron distribution is first calculated under the assumption that the angle is 90°, *i. e.*, transverse transmission. The virtual

height is then calculated from this distribution with the correct value of group-velocity at the angle in question, assuming that the ray path does not practically depart from the vertical. At an angle of 75° the virtual height thus calculated certainly is less than 3 per cent. smaller than the measured virtual height. Now, if the increase in virtual height is say 100 km., then the uncertainty is 3·0 km. However, the exploring pulse, which cannot easily be made as small as one desires, takes up a space of, say 7–15 km. on the ionosphere record.

The error due to the mechanical interpretation can be estimated as follows. As has been shown in the wave-mechanics, the wave penetrates slightly into the region where V is larger than $hf/2$. In the case where the electron density increases with the square of the height, the actual penetration corresponds to an eighth of the wave-length, as this case exactly corresponds to the linear oscillator in the quantum mechanics. For other distributions one may assume that an eighth of a wave-length still is a good approximation, *i. e.*, one assumes the Bohr-Sommerfeld phase integral condition to be valid. It is clear that the correction $\lambda/8$ can be neglected in comparison with h_v for the wave-lengths, ranging from say 200 metres and downwards, which are used for the exploration of the ionosphere.

Many ionospheric observatories are located at places where the earth's magnetic field vector is almost vertical. Thus it should be of special interest to examine if a solution similar to the one just discussed can be obtained in such a case. It is easy to show that the phase integral for the extraordinary ray at longitudinal transmission and vertical incidence can be written

$$\frac{\Delta(\omega)}{2} = \frac{\omega}{c} h(0) + \frac{\omega}{c} \int_{\omega_0=0}^{\omega_0^2=\omega(\omega-\omega_H)} \left(1-\frac{\omega_0^2}{\omega(\omega-\omega_H)}\right)^{\frac{1}{2}} dz. \quad . \quad . \quad (12)$$

$\Delta(\omega)$ is the change in phase between the transmitter and the receiver in radians and ω_H is the angular gyro-frequency of the terrestrial magnetic field. For the sake of simplicity we call $\omega(\omega-\omega_H)=\omega_r^2$: Thus $\omega_r=0$ for $\omega=\omega_H$. Then

$$\frac{\omega_r}{\omega}\frac{\Delta(\omega)}{2} = \frac{\omega_r h(0)}{c} + \frac{\omega_r}{c}\int_{\omega_0=0}^{\omega_0^2=\omega_r^2}\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}} dz$$

or

$$\frac{d}{d\omega_r}\cdot\frac{\omega_r}{\omega}\cdot\frac{\Delta(\omega)}{2} = \frac{h(0)}{c} + \frac{1}{c}\int_{\omega_0=0}^{\omega_0^2=\omega_r^2} \frac{dz}{\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}}} \cdot \quad . \quad . \quad . \quad (13)$$

But

$$\frac{d}{d\omega_r} = \frac{\omega_r}{\omega-\frac{\omega_H}{2}}\cdot\frac{d}{d\omega},$$

and
$$\frac{d}{d\omega}\frac{A(\omega)}{2}=\frac{h(0)}{c}+\frac{1}{c}\int_{\omega_0=0}^{\omega_0^2=\omega(\omega-\omega_H)}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega(\omega-\omega_H)}\right)^{\frac{1}{2}}}\times\left[1+\frac{1}{2}\frac{\omega_0^2}{\omega(\omega-\omega_H)}\cdot\frac{\omega_H}{\omega-\omega_H}\right]. \quad (14)$$

But
$$\int_{h_0}^{h_T}\frac{dz}{v_v}=\int_{\omega_0=0}^{\omega_0^2=\omega(\omega-\omega_H)}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega(\omega-\omega_H)}\right)^{\frac{1}{2}}}\left[1+\frac{1}{2}\frac{\omega_0^2}{\omega(\omega-\omega_H)}\cdot\frac{\omega_H}{\omega-\omega_H}\right]$$
$$=h_v(\omega)-h(0)=\Delta h_v(\omega), \quad (15)$$

as can easily be verified.

Thus
$$\frac{1}{2}\frac{d\Delta(\omega)}{d\omega}=\frac{h(0)}{c}+\frac{\Delta h_v(\omega)}{c}=\frac{h_v(\omega)}{c}, \quad (16)$$

and with (13) this gives
$$\frac{\Delta(\omega)}{2\omega}+\frac{\omega_r^2}{\omega-\frac{\omega_H}{2}}\left[\frac{1}{\omega}\left(\frac{h(0)}{c}+\frac{\Delta h_v(\omega)}{c}\right)-\frac{1}{2}\frac{\Delta(\omega)}{\omega^2}\right]$$
$$=\frac{h(0)}{c}+\frac{1}{c}\int_{\omega_0=0}^{\omega_0^2=\omega_r^2}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}}}.$$

By a simple rearrangement of terms, this comes out to be
$$\frac{\omega_H}{2\omega-\omega_H}\left[\frac{\Delta(\omega)}{2\omega}-\frac{h_v(\omega)}{c}\right]+\frac{h_v(\omega)}{c}=\frac{h(0)}{c}+\frac{1}{c}\int_{\omega_0=0}^{\omega_0^2=\omega_r^2}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}}}. \quad (17)$$

Now, $\Delta(\omega)-\Delta(\omega_H)=\frac{2}{c}\int_{\omega_H}^{\omega}h_v(\omega')d\omega'$, by (18), but $\Delta(\omega_H)=\frac{2}{c}\omega_H h(0)$,

because at a frequency ω_H, $\omega_r=0$, and the signal is reflected at the lowest boundary of the layer, *i. e.*, at the height $h(0)$. Thus
$$\Delta(\omega)=\frac{2}{c}\omega h(0)+\frac{2}{c}\int_{\omega_H}^{\omega}\Delta h_v(\omega')\,d\omega',$$

where $\Delta h_v(\omega')=h'(\omega')-h_v(0)$=the increase in virtual height at the frequency ω'. Putting this in (17) we finally get
$$\Delta h_v(\omega)-\frac{\omega_\Delta}{2\omega-\omega_H}\left[\Delta h_v(\omega)-\frac{1}{\omega}\int_{\omega_H}^{\omega}\Delta h_v(\omega')\,d\omega'\right]=\int_{\omega_0=0}^{\omega_0^2=\omega_r^2}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}}},$$

or
$$\Delta H_v(\omega)=\Delta h_v(\omega)-\Delta_2 H(\omega)=\int_{\omega_0=0}^{\omega_0^2=\omega_r^2}\frac{dz}{\left(1-\frac{\omega_0^2}{\omega_r^2}\right)^{\frac{1}{2}}}, \quad (18)$$

where $$\Delta_2 H(\omega) = \frac{\omega}{2\omega - \omega_H} [\Delta h_v(\omega) - \overline{\Delta h_v}(\omega)]$$

and $$\overline{\Delta h_v}(\omega) = \frac{1}{\omega} \int_{\omega_H}^{\omega} \Delta h_v(\omega')\, d\omega'.$$

$\Delta_2 H(\omega)$ thus is the correction we must apply to $\Delta h_v(\omega)$ before we can translate it into ω_r, for the application of the solution in the form of the Schlömilch integral. At Cambridge, Mass., $\frac{\omega_H}{2\omega - \omega_H}$ is about 0·68 for a frequency of 2 mc./s., for 4 mc./s. it is about 0·26, and for 7 mc./s. it is 0·13, which gives an idea of the magnitude of the correction.

The solution of (18) then is

$$\Delta h_T(\omega) = \frac{2}{\pi} \int_0^{\pi/2} \Delta H_v \left(\frac{\omega_H}{2} + \sqrt{\omega(\omega - \omega_H) \sin^2 \phi + \frac{\omega_H^2}{4}} \right) d\phi.$$

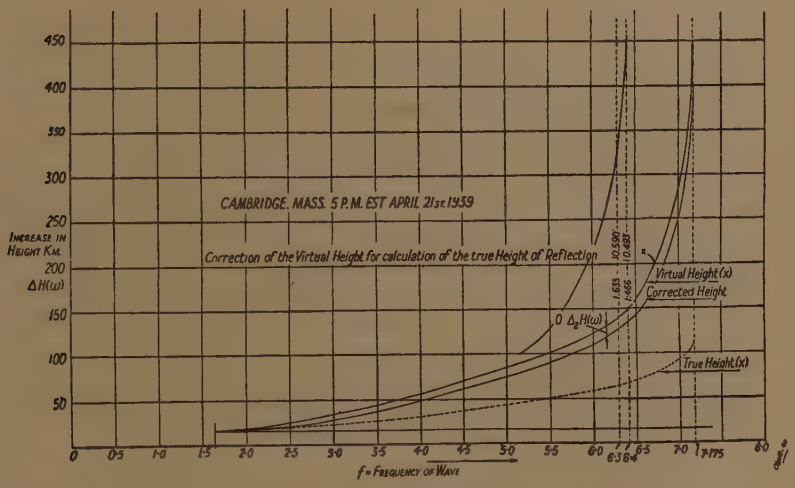
As ΔH_v always can be constructed from the measured uncorrected height, h_v, the true height, h_T, can always be obtained. Now, the above method can be used at angles of propagation (angle between the direction of propagation of the wave and the direction of the magnetic field) up to 20°. The error is smaller than the present experimental error. The error can be calculated in the same manner as before. At an angle of 15° the average error should be less than about 1·5 per cent., or around one-tenth of the correction, $\Delta_2 H(\omega)$. For 20° the correction should be less than 2·5 per cent., which still is less than the experimental error with our present technique.

Thus we can apply this solution, for example, to ionosphere records obtained at Cambridge, Mass., because the angle of propagation at Cambridge is only about 16° (at vertical incidence of course). We shall then be able to determine quite accurately the electron distribution from the measurements. Fig. 2 shows an application of this method to a Cambridge, Mass., recording obtained on April 21st, 1939, at 5 P.M. E.S.T. Only steady, clean F_2 signals were recorded all over the range, even down to a frequency of 2 mc./s. indicating negligible loss and retardation in the E region. The extraordinary ray is reflected from a virtual height of 700 km. at the approximate critical frequency of 7·18 mc./s. The ordinary ray was returned from just about the same virtual height at its approximate critical frequency of 6·40 mc./s. If the magnetic field had the same strength at the point of reflexion as at the earth, this frequency would have been 6·30 mc./s. instead of 6·40 mc./s. This indicates that the magnetic field only is about 0·49 oersted at the true height. In computing these values and the true height, it was possible to obtain rather high accuracy because the record was very clean and sharp. This showed once more the value of the high resolving power of the recording equipment used.

First, the correction $\Delta_2 H(\omega)$ was constructed. $\Delta h_v(\omega)$ was obtained by using a planimeter. The correction is shown on fig. 2. At a frequency of 6 mc./s. it is 13 km., which is just about the width of the pulse on the record. As the correction due to the fact that the angle is 16° and not 0° would amount to somewhere around a tenth of the pulse width, it is clearly seen that the present method must be regarded as very accurate for use with the Cambridge recordings.

Next the corrected height, $\Delta H_v(\omega)$, is plotted against $\omega_r = (\omega(\omega - \omega_H))^{\frac{1}{2}}$. This is shown on fig. 3. It is indicated as the Corrected Virtual Height. The figure further shows the calculation of the true height for a number of points. Take point (1) for example. The curve is now replotted

Fig. 2.


CAMBRIDGE, MASS. 5 P.M. EST APRIL 21st 1939
Correction of the Virtual Height for calculation of the true Height of Reflection
Virtual Height (x)
Corrected Height
True Height (x)
Increase in Height Km.
ΔH(ω)
f = Frequency of Wave

The correction of the virtual height for the calculation of the true height of reflexion.

against the angular variable, ϕ. ϕ runs from 0° to 90°, while the frequency ω_r runs from 0 (*i. e.*, the point where the actual frequency is $=\omega_H$) to the value it has at point 1. Line (1) is the plot of this transcribed virtual height. The average value of the height of this curve is obtained by a planimeter. This value is the *true* height. In such a way point after point on the true height curve is constructed. Once one is used to the procedure it works quite fast.

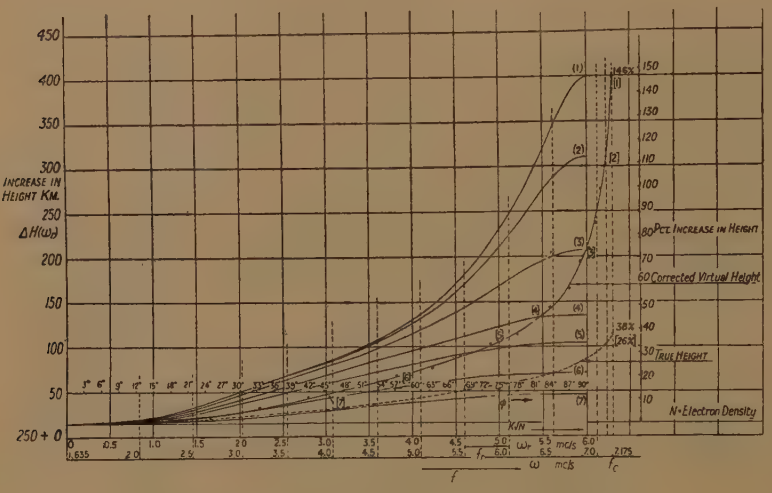
The dashed curve shows the increase in true height thus constructed. While the maximum increase in virtual height is 146 per cent. of $(h(0))$, the true height only increases to a maximum of 38 per cent., or about one-fourth of the increase in virtual height. In spite of the tremendous rise in virtual height the maximum increase in true height is only 105 km.

This means that the thickness of the lower side of the F_2 layer is 105 km. The true height curve is now transferred to the original recording, fig. 2, for comparison. It is rather instructive.

As we now know the true height of reflexion, we also know that the terrestrial magnetic field at this point is about 0·49 oersted. Thus the magnetic field has decreased to about 0·49 oersted at around 370 km. above the surface of the earth at this particular location.

From fig. 3 we can plot the true electron density at the time in question. Fig. 4 shows what it looks like. For comparison a dashed parabola is drawn with its apex at the point of maximum electron density. It is

Fig. 3.



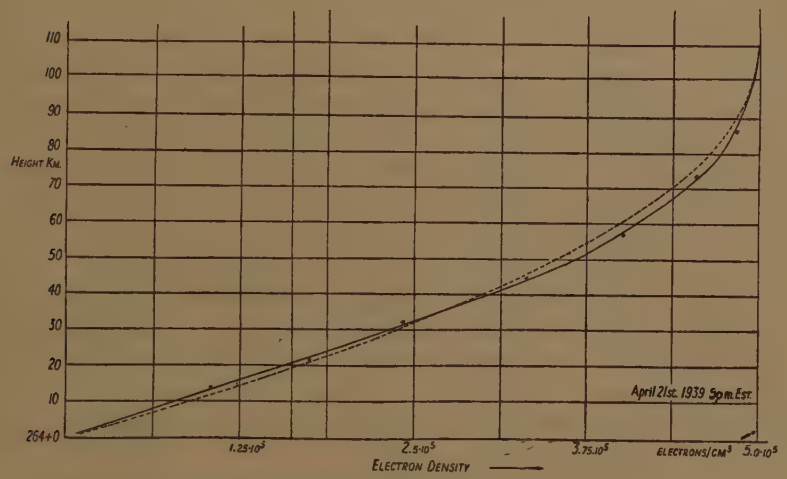
The calculation of the true height of reflexion.

made to fit at the lowest densities. It is clearly seen what a good approximation this is over the whole range. It has been believed for some time that the F_2 layer generally is parabolic in nature. This confirms that belief. The smaller electron densities at the lower side of the layer cannot be determined. At the corresponding critical frequencies for the extraordinary ray, which are slightly higher than ω_H, the extraordinary ray is either completely absorbed or reflected from a lower layer, if it is reflected at all. Thus we have to continue the virtual height curve with a tangent to the gyro-frequency and assume that the electron density goes to zero at a certain height. It is clear from the nature of the solution that the omission of the small portion of low density ionization at the lower edge of the layer will not appreciably affect the shape of the main distribution as it is obtained from the integral equation. It is not

the fault of the method that we do not get any data of reflexion from these low electron densities.

The very highest values of virtual height near the critical frequency may be uncertain, if the rise in height is very rapid. That only means that the corresponding true heights near the critical frequency are uncertain, as the value of the true height depends only upon the magnitude of the virtual heights at frequencies equal to or lower than the frequency at which the true height is to be calculated.

Fig. 4.



The true distribution of electrons in the F_2-layer.

The methods just outlined should be of value at equatorial ionospheric observatories and at ionospheric observatories at the higher latitudes such as in New England, Canada and Northern Europe. It is quite likely that the accuracy of the method at other geographical locations will be sufficient to give at least the character of the distribution accurately enough. It should not be too difficult to estimate the errors in such cases. It is further felt that the application of this method at northern latitudes might lead to a definite knowledge of the true variation with height of the terrestrial magnetic field in the upper ionosphere.

The author's thanks are due to Professor H. R. Mimmo, of Harvard University, who has been kind enough to let him carry on this research at his institution.

XXVII. *On the Nature of Negative Resistances and Negative Resistance Sections.*

By S. P. Chakravarti, M.Sc.(London), D.I.C., A.M.I.E.E., Mem.I.R.E., University Lecturer, University of Calcutta *.

[Received June 19, 1939.]

Summary.

The paper is in two parts. Part I. relates to comparative studies of " stability," " value and nature of impedance," " linearity," and " phase-distortion " made on dynatron, transitron, and feed-back types of negative resistance element for the frequency range 0·5 kc/s to 1·0 Mc/s. "Amplification Effect," " Band-pass Effect," and negative resistance equalizers have been explained on the basis of the above studies.

Part II. shows the (*a*) negative resistance (impedance) sections as two-way amplifiers, and (*b*) negative resistance (impedance) sections as two-way ampli-filters. It has been found that to obtain amplification, the impedance in the shunt must essentially be negative, and, in addition, to obtain filter action, either series or shunt impedance must be of the square type. Properties and applications of the sections have been discussed.

Introduction.

Negative resistances formed from vacuum tubes have been treated from various aspects during last twenty years. They originated with the " dynatron effect " discovered by Hull, who discussed the effect of a magnetic field on the secondary emission and further developed the dynatron oscillator. Van der Pol developed " negative inductance " and " negative capacitance," and evolved circuits in which the magnitude of impedance was independent of frequency, but its angle depended on it. Crisson developed the " feed-back type " of negative resistances by connecting the input and output terminals of one-way amplifier in series or parallel and used them for the twin 21-type of repeater. Black developed the stabilized negative feed-back amplifier which effected an extraordinary improvement in constancy of amplification and freedom from non-linearity and consequent cross-modulation. Cabot discussed the " resistance tuning " which could give maximum of selectivity without undue

* Communicated by the Author.

sacrifice in the audio quality of reception and maintain the same optimum compromise over the entire tuning range. Herold supplied the third type of negative resistance through the use of a double-grid tube employing negative transconductance which has been named "Transitron" by Brunetti.

None of the authors appears to deal with (1) those fundamental properties of the negative resistance element—"stability," "value and character of impedance," "linearity," and "phase-shift," and their variation with frequency, knowledge of which is so desirable for the elucidation of its operation in a.c. circuits; and (2) the properties of symmetrical sections formed both from positive and negative resistances or wholly from negative resistances. The present paper deals with these two aspects.

Part I.—Nature of Negative Resistance.

Negative resistances formed from vacuum tubes have been of three types, namely :—

(1) *Dynatron type*, in which the internal resistance of a triode or screen-grid tube under secondary emission condition has been used to obtain negative resistance (in the present treatment, negative resistance from a screen-grid tube has been considered).

(2) *Transitron type*, in which a five electrode or double-grid tube employing negative transconductance has been used. The falling portion of E_2–I_2 characteristic has been used for obtaining the negative resistance.

(3) *Feed-back or regenerative type*, in which the input and the output terminals of a one-way amplifier are connected in series or parallel

They are shown in fig. 1 in suitable connexion for use in an a.c. system. In the dynatron type, C blocks the d.c. from circulating in the main circuit to which the element may be connected, and is of such a value that at a.c. frequencies involved its impedance can be neglected; and L presents large impedance at frequencies involved to prevent a.c. from taking a parallel path through the H.T. battery. The impedance between terminals T_1 and T_2 is almost non-reactive of value—R_a+R. In the transitron type, similar condenser and chokes have been inserted from the same considerations. In the feed-back type, none of these precautions is necessary. The dynatron type depends upon the secondary emission for its negative resistance effect, whereas the other types do not. The transitron type gives much smaller resistance value than the other two types without using an external positive resistance.

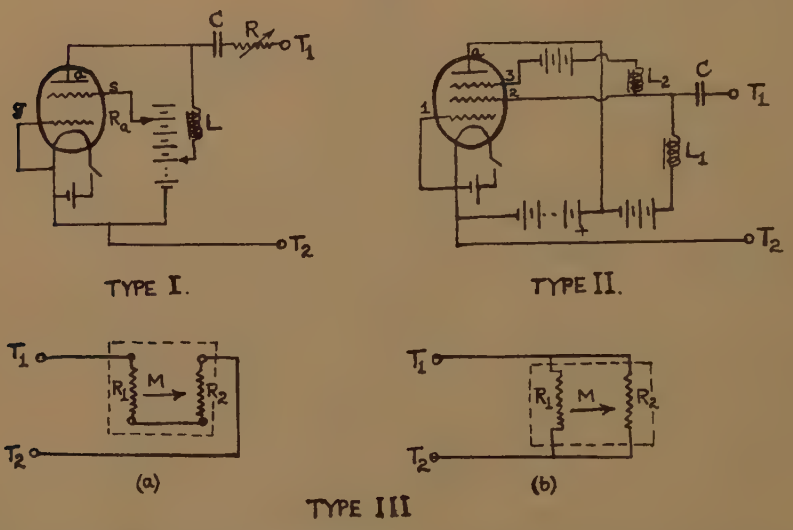
The following comparative studies have been undertaken for the frequency range 0·5 kc./s. to 1·0 Mc./s.:—(1) The stability of the negative

resistance element in a.c. circuit ; (2) the value and character of the element impedance ; (3) the linearity of the element ; and (4) the phase-shift caused by the element.

1. *Stability of the Element.*

Instability of the dynatron type of negative resistance results from (*a*) variation of filament current, plate voltage, and screen-grid voltage ; (*b*) variation of the degree of contamination and mechanical nature of the surface emitting secondary electrons ; and (*c*) oscillations set up in the negative resistance circuit on switching. From stability studies on Mazda SG. 215 described in one of the author's papers, it can be said

Fig. 1.



that the variation of a.c. resistance of a new tube at a given frequency is negligible up to 4 per cent. change in plate voltage, 4·6 per cent. change in filament current, and 3·3 per cent. change in screen-grid voltage ; and that the variation for an old tube (in continuous use for about two years) is similar, though the actual value alters slightly with age. It has also been mentioned therein that oscillations set up in the negative resistance circuit designed for frequencies up to 15 kc./s. cease on closing the terminals through a filter network or merely an impedance. Recent experiments on circuits designed for higher frequencies up to 1·0 Mc./s. have shown that the above condition still holds good for the cessation of oscillations provided suitable values of blocking condenser, choke, and coupling between main a.c. circuit and H.F. source are used and proper shielding is done for these frequencies.

The transitron type of negative resistance formed from type 58 tube has been found to be comparatively more stable than the first type, since it does not depend on the secondary emission. The variation of a.c. resistance at a given frequency has been found to be negligible up to 6 per cent. change in E_p, E_2, and E_{32}, and up to 5 per cent. change in filament current. The actual value also remains constant for a longer period. Oscillations are also set up in this type on switching, but cease as before on connecting it to the main circuit.

The instability of the feed-back type of negative resistance depends upon the adjustment of amplifier performance, and results in " singing." The series and the shunt negative resistance values are given by R_1+R_2-M and R_1R_2/R_1+R_2-M respectively, where R_1 and R_2 are impedances between input and output terminals respectively, and M is mutual impedance given by e.m.f. generated in the output divided by current in the input. The difficulty lies in adjusting M without phase-shift, but once this type has been set in it remains fairly stable for a long time. From the stability point of view, the transitron type appears to be the best of the three.

A fairly close agreement between the measured and calculated values of gain, phase-shift, and impedance in the author's experiments has been a sufficient proof of the stability of the element in a.c. circuits.

2. *Value and Character of the Impedance.*

Voltage—current characteristics of the circuits have been obtained with alternating current of frequencies from 0·5 kc./s. to 1·0 Mc./s. A.C. voltages have been obtained from Standard Telephones and Weston Companies' oscillators of good stability and negligible harmonic percentage ; and voltages and currents have been measured by Moullin's thermionic voltmeter and a Cambridge thermo-galvanometer respectively. Fig. 2 shows the typical audio and high frequency characteristics.

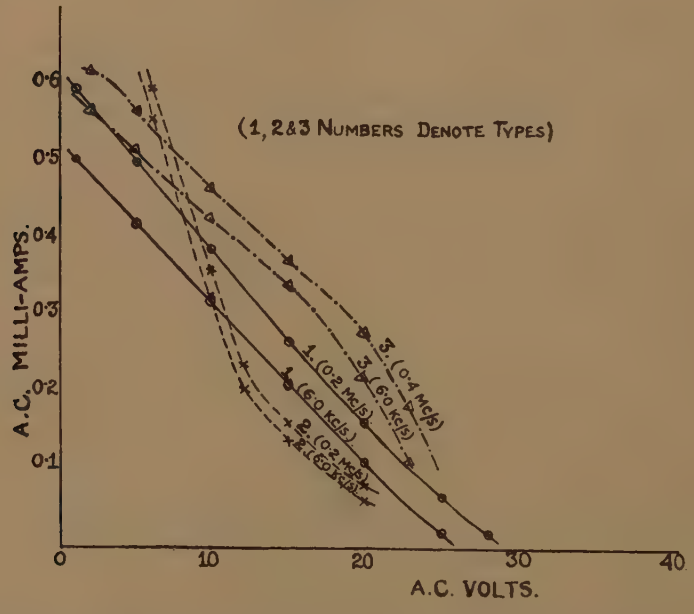
(a) *Dynatron type.*—It has been observed that in the V_a–I_a static characteristic for S.G. 215 ($V_s=60_v$, $V_g=0$) the negative resistance portion lies between limiting values of V_a, 8, and 54 volts ; whereas in the a.c. characteristics with working point corresponding to $V_a=34_v$, it lies between a.c. voltage limits, 1 and 30 volts. It is possible to increase the a.c. voltage range by altering V_s and V_g. With $V_s=60_v$, $V_g=0$, and $V_a=34_v$, the internal impedance$=-46\times10^3$ ohms. Fig. 3 shows the values of a.c. negative impedance at frequencies from 1·0 kc./s. to 1·0 Mc./s. with 12 volts a.c.

(b) *Transitron type.*—It has been observed that in the E_2-I_2 characteristic for type 58 tube (with $E_p=26_v$, $E_1=0$, $E_{32}=-98_v$) the negative resistance portion lies between limiting values of E_2, 80 and 95 volts ;

whereas in a.c. characteristics with working-point corresponding to $E_2=84$ volts, it lies between a.c. voltage limits, 1·5 and 19 volts. The length of the negative portion of the characteristic increases with both E_p and E_2. The slope can be varied by introducing a small bias on grid No. 1. With $E_p=26_v$, $E_1=0$, $E_g=84_v$, and $E_{32}=-92_v$, the impedance $=-3823$ ohms. Fig. 3 shows the values of a.c. negative impedance at frequencies from 1·0 kc./s. to 1·0 Mc./s. with 12 a.c. volts.

(c) *Feed-back type.*—A series type of negative resistance formed by connecting in series the input and the output terminals of a one-stage

Fig. 2.



resistance coupled amplifier using Osram HL2 tube has $R_1=50{,}000$ ohms, $R_2=20{,}000$ ohms, and M varying from 140×10^3 to 80×10^3 ohms with frequency. Fig. 3 shows the values of a.c. negative impedance at frequencies from 1 kc./s. to 1·0 Mc./s. with 12 a.c. volts.

Fig. 3 indicates a general resemblance in the variation of negative impedance with frequency for all the three types, and shows close similarity for types 1 and 2. It will be observed that for types 1 and 2 the variation is similar to a multiple resonance curve—the impedance value at 0·1 to 0·5 kc./s. is high, then decreases slowly with increase of frequency up to 4 kc./s., then increases to high values at 10 to 12 kc./s., then decreases, again increases and so on, and subsequently falls down above 400 kc./s. If the

negative impedance of type 1 be represented as shown in fig. 3 *a*, where C_{af}=the anode-filament (and choke self) capacitance, and L=the inductance of the choke inserted in the battery path, and further C_{af} be considered to be less at higher frequencies as discussed by Hartshorn, the

Fig. 3.

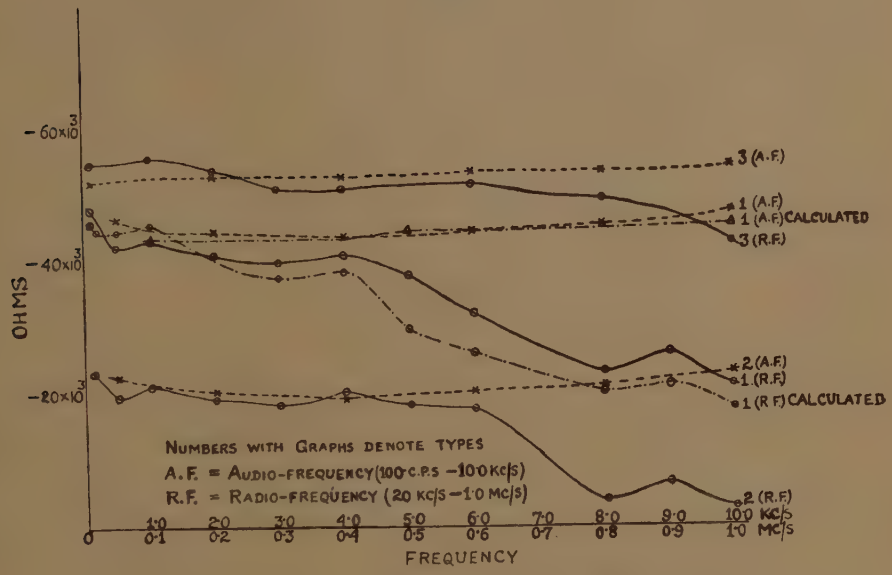
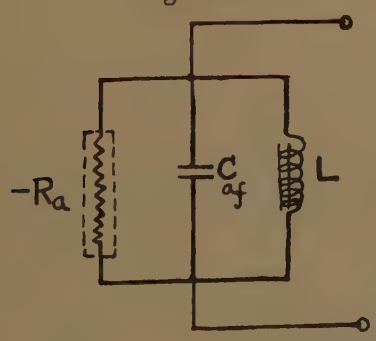


Fig. 3 *a*.



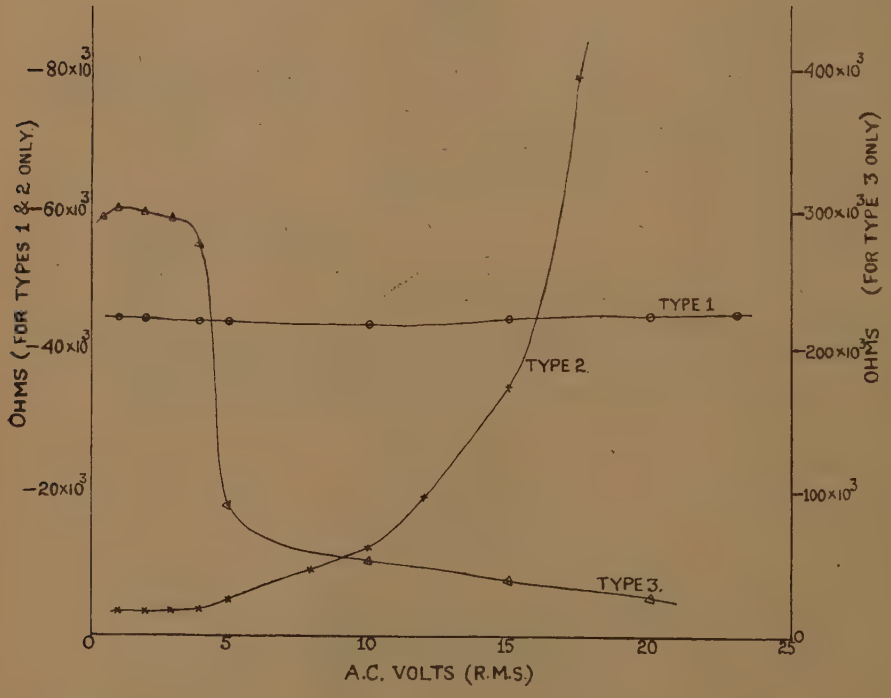
peculiar variation of negative impedance with frequency can be accounted for. Fig. 3 shows *for comparison* the variation of negative impedance with frequency calculated from the expression $Z=\frac{R_a\omega L}{-\omega L+jR_a(\omega^2 LC_{af}-1)}$, with $R_a=-46\times10^3$ ohms, L=25 H, and C_{af} varying from 10 to 8 micro-micro farads. It will be observed that there is a close similarity between

the experimental and theoretical curves. The negative element cannot strictly be regarded as non-reactive except at very low frequencies.

3. *Linearity of the Element.*

The normal type of positive resistance is invariably linear, and a tube amplifier system is linear within certain limits. For the dynatron type, the negative impedance value has been found to be the same between 1 and 30 volts at a given frequency for the frequency range 0·5 kc./s. to

Fig. 4.



1·0 Mc./s. under stated working conditions, and the permissible volume variation without non-linear distortion would be about 30 decibels. It has been found possible to have volume variation up to 40 db. with the same tube by altering the working conditions. With other tubes, still larger volume variation can be expected.

For the transitron type, the negative impedance has been found to increase with the increase of voltage for the frequency range 0·5 kc./s. to 1·0 Mc./s. The negative impedance value varies slowly between 1 and 4 volts and rapidly after 4 volts. The type may be considered as a non-linear impedance.

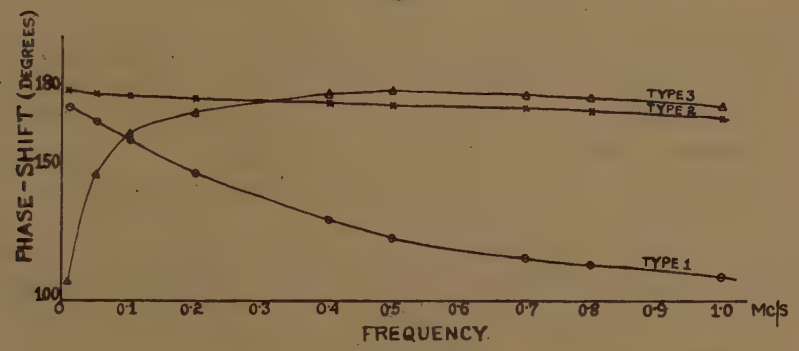
For the feed-back type, the linearity depends upon that of the amplifier. The amplifier described above has been linear for input voltages from 0·3 to 3·0 volts, and the negative impedance has been found to be the same for voltages between 1 and 4 volts, a range comparatively smaller than that of the dynatron type.

Fig. 4 shows the impedance variation with voltage for the three types. From the linearity point of view, the dynatron type appears to be well suited for use in a.c. circuits.

4. *Phase-shift caused by the Element.*

The normal type of positive resistance does not introduce any phase difference between the current and voltage vectors. Measurement of the phase difference for three types of negative impedance has been made by the method described by the author in one of the previous papers. [1]

Fig. 5.



For dynatron and transitron types, the angle of phase-shift is proportional to $1/\omega$ up to 0·5 Mc./s., but varies in a non-linear way with $1/\omega$ beyond 0·5 Mc./s. To obtain a pure negative resistance from a one-way amplifier, it is necessary that there shall be no phase-shift in amplification. Even in a resistance coupled amplifier, small inductance and capacitance in wiring and tube and capacitances of blocking condenser and input and output impedances are involved, and hence phase-shifts are produced at high frequencies. Fig. 5 shows the measured values of phase-shift at frequencies from 10 kc./s. to 1 Mc./s. It will be noted that the phase distortion will be inappreciable for types 1 and 2, at least for the higher frequency range.

5. *Explanation of some Negative Impedance Effects.*

The investigations suggest that the dynatron type is the most suitable one to use in a.c. circuits from all points of view. A single negative

resistance element (of the type) or a combination of one or more such elements forming a network connected in an a.c. circuit (1) can remain sufficiently stable ; (2) will permit volume variation up to 40 db. or more without non-linear distortion ; (3) will not introduce appreciable phase-distortion ; and (4) will introduce gain varying with frequency depending upon the variation of the negative impedance with frequency for the frequency range involved.

(a) "*Amplification and Band-pass Effects.*"—When a filter section of the usual type is terminated by a negative resistance (impedance) of magnitude equal to the characteristic impedance in the transmission band, gain is obtained in the transmission band. The transmission loss due to reflexion at the termination becomes gain in this case, the reflected wave being reversed in sign to that in the case of a positive resistance termination. The reflexion gain adds up with network attenuation to give a total insertion gain.

When a low-pass or high-pass filter section is terminated by a similar negative impedance, it transforms into a band-pass filter. This has been realized due to the peculiar variation of reflexion gain with frequency, resulting from the combination of two variations, viz. (1) that of negative impedance with frequency as shown in fig. 3, and (2) that of the characteristic impedance of the filter section with frequency.

(b) "*Equalizer*"—The realization of two types of equalizer—one in which loss increases with frequency and another in which gain increases with frequency has been possible due to (1) decrease or increase of negative impedance value with frequency over portions of frequency ranges in fig. 3, and (2) magnification of the rate of decrease or increase by positive and negative impedance of the square type which can only be derived from ordinary negative resistance.

Part II.—Nature of Negative Resistance Sections.

It was pointed out by Campbell and Mouradian about 1928 that three negative impedances could be grouped into a T or a π network which would amplify waves traversing a line without causing echoes if the impedance values are suitably chosen, but no work on the subject has appeared so far. It has been found by the author that to obtain amplification without causing appreciable echo effects all the three impedances need not be negative, and it will suffice if only the impedance in the shunt is negative, and that, in addition, to obtain filter action either series or shunt impedance must also be of the square type. It must be noted that the negative resistance element cannot strictly be considered non-reactive above 10 kc./s., though it has been mentioned as "negative resistance" in the following treatment. For building sections, negative resistance of type 1 has been used by the author.

1. *Negative Resistance Sections as Two-way Amplifiers.*

A normal type of tube amplifier is a one-way device, introduces gain and phase-shift (to frequency) components which vary with frequency in a manner depending upon the type of amplifier used and give frequency and phase distortion, is linear within certain limiting values of the input, and has different input and output impedances.

The present section deals with more or less pure resistance sections, containing negative resistance of the normal type either in the shunt arm or in both series and shunt arms and possessing the following distinguishing properties :—(1) They are symmetrical and two-way devices ; (2) they introduce almost the same gain over the frequency range in which the negative impedance does not vary rapidly with frequency ; (3) they introduce no phase distortion over much larger frequency range ; (4) they are linear for much greater range of input ; and (5) they have the same input and output impedances.

It will be observed that in each of the cases discussed the gain will be the same provided the $|R_1/R_2|$ ratio is the same, but the characteristic resistance will vary for different values of R_1 and R_2.

Case I.—A section in which total series resistance $= R_1$, and total shunt resistance $= -R_2$.

For T-section, characteristic resistance

$$Z_0 = \sqrt{\frac{R_1^2}{4} - R_1R_2}, \quad \ldots\ldots\ldots \quad (1)$$

and for π-section,

$$Z_0' = \sqrt{\frac{R_1R_2}{\frac{R_1}{4R_2} - 1}} \quad \ldots\ldots\ldots \quad (2)$$

Since Z_0 and Z_0' have to be *non-reactive* within the transmission band, R_1 must be greater than $4R_2$ in magnitude.

The ratio of sending-end current to receiving-end current for both types of section when the section is terminated in its characteristic resistance is given by

$$k = 1 - \frac{R_1}{2R_2} - \frac{\sqrt{\frac{R_1^2}{4} - R_1R_2}}{R_2}, \quad \ldots\ldots \quad (3)$$

which will give gain, since k is negative. The variation of gain with frequency will depend upon the variation of $-R_2$ with frequency, and there will be no appreciable phase distortion.

Case II.—A section in which total series resistance $= -R_1$ and total shunt resistance $= -R_2$.

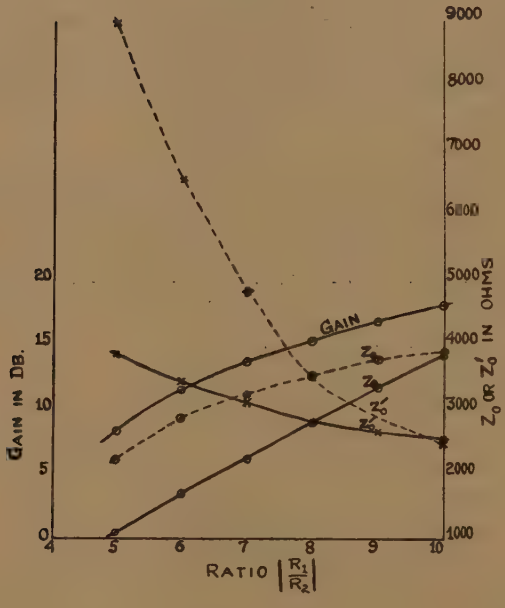
For T-section, characteristic resistance

$$Z_0=\sqrt{R_1R_2+\frac{R_1^2}{4}} \quad \ldots\ldots\ldots\ldots (4)$$

and for π-section,

$$Z_0'=\sqrt{\frac{R_1R_2}{1+\frac{R_1}{4R_2}}} \quad \ldots\ldots\ldots\ldots (5)$$

Fig. 6.



The ratio

$$k=1+\frac{R_1}{2R_2}-\frac{\sqrt{R_1R_2+\frac{R_1^2}{4}}}{R_2}, \quad \ldots\ldots (6)$$

which can give gain by suitably adjusting the absolute magnitudes of R_1 and R_2. The variation of gain with frequency will depend upon the variation of $-R_1$ and $-R_2$ with frequency. If both R_1 and R_2 vary in the same way the gain will almost be the same at all frequencies. There will be no appreciable phase distortion.

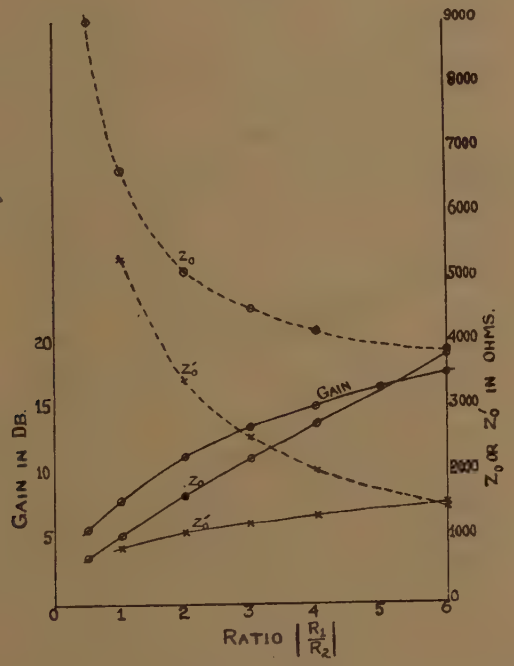
Fig. 6 shows the variation of gain and characteristic resistance of sections in Case I. with the ratio $|R_1/R_2|$. The ratio $|R_1/R_2|$ has been varied (as shown by the full curve) by keeping R_2 constant and changing R_1,

and (as shown by the dotted curve) by keeping R_1 constant and changing R_2. The gain is the same in both cases, but the characteristic resistance in the latter is higher and varies slowly and rapidly with $|R_1/R_2|$ for T and π sections respectively. Fig. 7 shows similar curves for sections in Case II.

2. *Negative Resistance Sections as Two-way Ampli-filters.*

An ampli-filter may be called a device which transmits a certain specified band of frequencies and at the same time introduces gain to

Fig. 7.



frequency components in the transmission band. The present section deals with sections containing negative resistances of the ordinary type and negative resistances proportional to the square of the frequency and possessing the following properties :—(1) They are symmetrical and two-way devices ; (2) they act as wave-filters transmitting certain specified band or bands ; (3) they introduce gain to the frequency components in the transmitting band or bands which varies with frequency ; (4) they introduce no phase-distortion over the frequency band transmitted ; (5) they are linear for much greater range of input ; and (6) they have the same input and output impedances.

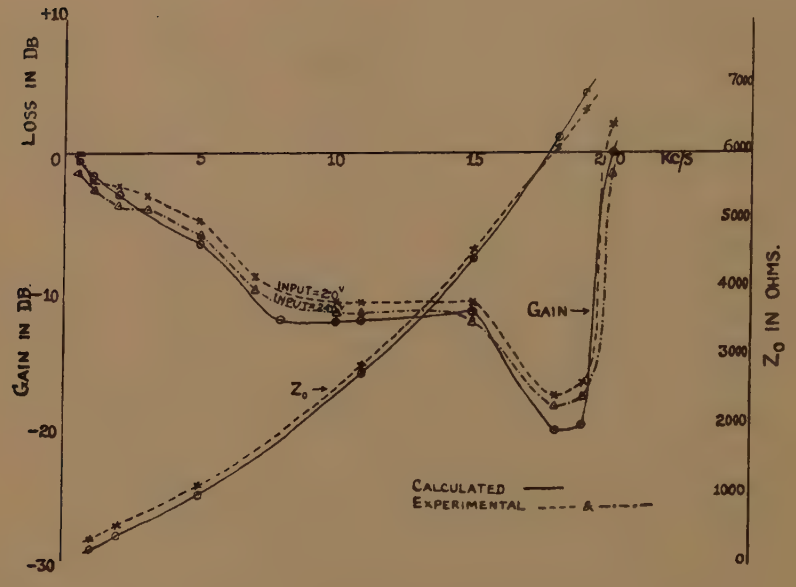
It will be observed that in each of the cases discussed the gain will be the same provided the R_1/R_2 ratio is the same, but the characteristic resistance will be different for different values of R_1 and R_2.

Case I.—A T-section in which total series resistance$=-\omega^2R_1$ and total shunt resistance$=-R_2$.

Characteristic resistance

$$Z_0=\frac{\omega^2R_1}{2}\sqrt{1+\frac{4R^2}{\omega^2R_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Fig. 8.



The ratio

$$k=1+\frac{\omega^2R_1}{2R_2}-\frac{\omega^2R_1}{2R_2}\sqrt{\left(1+\frac{4R_2}{\omega^2R_1}\right)} \quad . \quad . \quad . \quad . \quad (8)$$

In the section built for experiment, total series resistance

$$=-\omega^2\cdot\frac{1}{12\times10^5}\text{ ohms}$$

(which corresponds to L$=$10 mH., $-$L$=-$10 mH., and $-r=-$120 ohms in the negative resistance circuit), and total shunt resistance $=-$1000 ohms. The variation of measured and calculated gain and characteristic resistance with frequency is shown in fig. 8. The measured values

of gain at two inputs plotted in fig. 8 show that the section acts almost as a linear amplifier.

It will be seen that the section behaves as a low-pass filter of cut-off frequency of about 19·5kc./s., and sharpness of cut-off of 12·5 db. in 400 c.p.s. introduces gain to frequency components in the transmitted band increasing with frequency, and has its characteristic resistance increasing with frequency—a property opposite to that of a T-section normal low-pass filter or a loaded telephone circuit.

This type of section should prove of advantage as (*a*) a *repeater* where it would equalize the gain and the characteristic impedance in addition to offering two-way communication without the use of hybrid coils; and (*b*) a *directional filter* in carrier telephone repeater where it would act as equalizer and auxiliary amplifier in addition to its normal work. The low-pass directional filter in Standard Telephone Co.'s repeater for 3-channel carrier telephone system has the cut-off of about 16·5 kc./s. and the sharpness of cut-off about 40 db. in 1 kc./s.

Case II.—A T-section in which total series resistance $= \omega^2 R_1$ and total shunt resistance $= -R_2$.

Characteristic resistance

$$Z_0 = \sqrt{\frac{\omega^4 R_1^2}{4} - \omega^2 R_1 R_2} \quad \ldots\ldots\ldots \quad (9)$$

The frequencies at which Z_0 is zero are 0 and $\frac{1}{\pi}\sqrt{\frac{R_2}{R_1}}$.

The ratio

$$k = 1 - \frac{\omega^2 R_1}{2R_2} - \frac{\sqrt{\frac{\omega^4 R_1^2}{4} - \omega^2 R_1 R_2}}{R_2} \quad \ldots \quad (10)$$

In the section built for experiment, total series resistance

$$= \omega^2 \cdot \frac{1}{12 \times 10^5} \text{ ohms},$$

and total shunt resistance $= -1000$ ohms. The variation of measured and calculated gain and characteristic resistance with frequency is shown in fig. 9. The measured values of gain at two inputs plotted in fig. 9 show that the section acts as a linear amplifier.

It will be observed that the section behaves as a high-pass filter of cut-off frequency of about 16·5 kc./s. (*i. e.* $\pi^{-1}\sqrt{(R_2/R_1)}$), and sharpness of cut-off 16·5 db. in 5 kc./s., introduces gain to frequency components in the transmitted band increasing with frequency and has Z_0 non-reactive within the transmission band increasing with frequency and reactive outside this band. It can be used as a high-pass *directional filter* in carrier

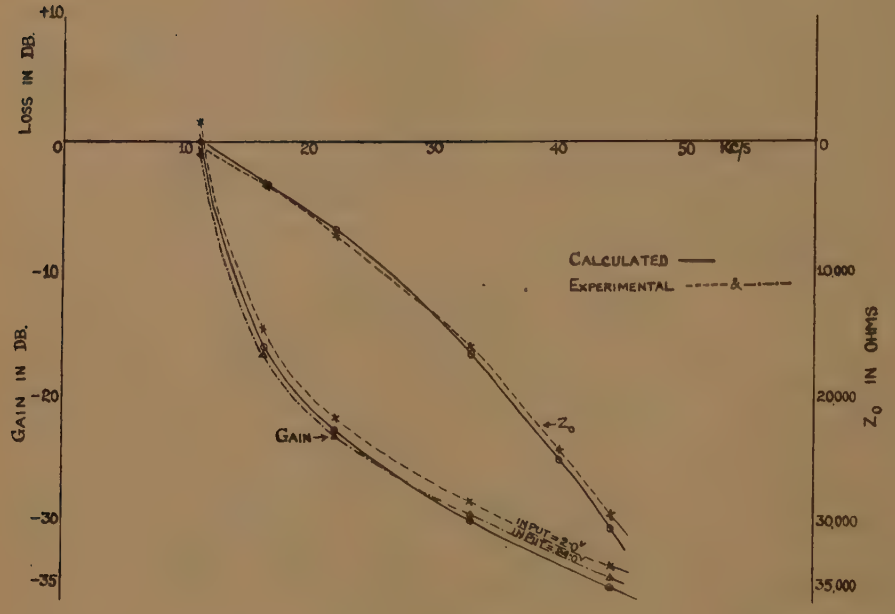
telephone repeaters which in Standard Telephone Co.'s 3-channel carrier telephone system has the cut-off of about 18 kc./s. and the sharpness of cut-off about 40 db. in 1 kc./s.

Case III.—A T-section in which total series resistance $= -R_1$ and total shunt resistance $= -\omega^2 R_2$

Characteristic resistance

$$Z_0 = \sqrt{\omega^2 R_1 R_2 + \frac{R_1^2}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Fig. 9.



The ratio

$$k = 1 + \frac{R_1}{2\omega^2 R_2} - \frac{\sqrt{\omega^2 R_1 R_2 + \frac{R_1^2}{4}}}{\omega_2 R^2} \quad . \quad . \quad . \quad (12)$$

In the section built for experiment, $-R_1 = -1000$ ohms and

$$-\omega^2 R_2 = -\frac{\omega^2}{12 \times 10^5} \text{ohms.}$$

The variation of measured and calculated gain and characteristic resistance with frequency is shown in fig. 10. Measurements at two inputs show that the section acts as a linear amplifier.

It will be seen that the section behaves as a high-pass filter of cut-off frequency of about 1·0 kc./s. and sharpness of cut-off of the order of 20 db. in 500 c.p.s., introduces gain to frequency components in the transmission band decreasing with frequency, and has Z_0 increasing with frequency—a property similar to that of a normal high-pass filter of the T type.

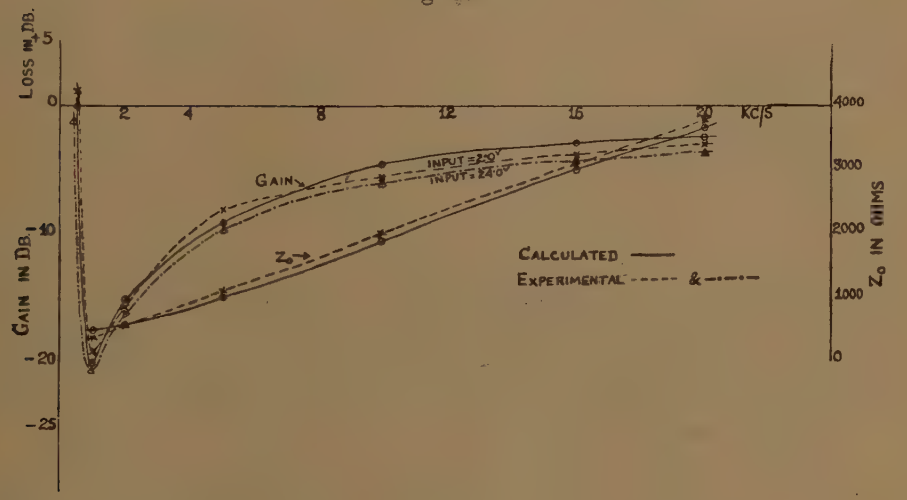
Case IV.—A T-section in which total series resistance$=R_1$ and total shunt resistance $-\omega^2R_2$.

Characteristic resistance

$$Z_0=\frac{R_1}{2}\sqrt{\left(1-\frac{4\omega^2R_2}{R_1}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The frequency at which Z_0 is zero is given by $f_0=\frac{1}{4\pi}\sqrt{\frac{R_1}{R_2}}$.

Fig. 10.



The ratio

$$k=1-\frac{R_1}{2\omega^2R_2}-\frac{R_1}{2\omega^2R_2}\sqrt{1-\frac{4\omega^2R_2}{R_1}} \quad . \quad . \quad . \quad (14)$$

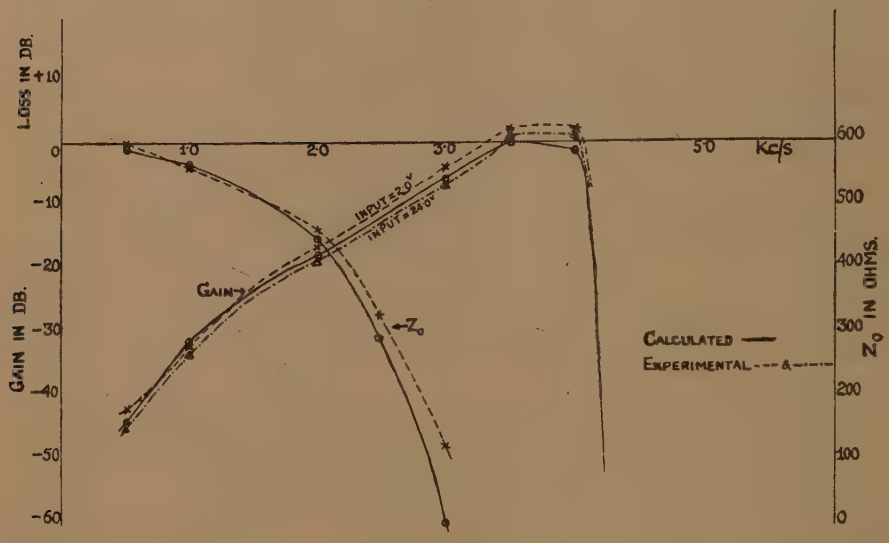
In the section built for experiment, $R_1=1200$ ohms and

$$-\omega^2R_2=-\frac{\omega^2}{12\times10^5}\text{ ohms.}$$

The variation of measured and calculated gain and characteristic resistance is shown in fig. 11. Measurements at two inputs show that the section acts as a linear amplifier.

It will be seen from the gain characteristic that from 3·0 to 4·1 kc./s. (f_0=3·0 kc./s.) the gain is comparatively small, but beyond these limits it increases considerably, and the section may be considered as a band-elimination ampli-filter.

Fig. 11.



Conclusion.

The whole of the experimental work was carried out at the Kanodia Electrical Communication Engineering Laboratories of the Calcutta University. The author desires to thank Prof. P. N. Ghosh of the Department of Applied Physics for his continued interest in the work and for giving him all facilities.

References.

(1) A. W. Hull, "Dynatron—A Tube possessing Negative Resistance," Proc. Inst. R. E. vi. (February 1918).
(2) Van der Pol, "A new Transformation in Alternating Current Theory and an Application to the Theory of Audition," Proc. I. R. E. xviii. p. 221 (February 1930).
(3) H. Mouradian, "Long Distance Transmission Problems," 'Journal of the Franklin Institute,' ccvii. (February 1929).
(4) George Crisson, "Negative Impedance and the Twin 21-Type Repeater," B. S. T. J. (July 1931).
(5) L. Hartshorn, "Variation of the Resistances and Inter-electrode Capacities of Thermionic Valves with Frequency," 'Experimental Wireless,' pp. 413–421 (August 1931).
(6) S. Cabot, "Resistance Tuning," Proc. I. R. E. (June 1934).

(7) S. P. Chakravarti, "Voltage Amplification by Electric Wave Filters terminated in Negative Resistances," Proc. Indian Academy of Science (1934).
(8) H. S. Black, "Stabilised Feed-back Amplifiers," B. S. T. J. (January 1934).
(9) E. W. Herold, "Negative Resistance and Devices for obtaining it," Proc. I. R. E. pp. 1201–1223 (October 1935).
(10) S. P. Chakravarti, "Sur les Filters se terminant par des resistances negatives," *L'Onde Electrique* (March 1936).
(11) S. P. Chakravarti, "On Negative Resistance Equalizers for Telephone Transmission Systems," Phil. Mag. xxiii. (May 1937).
(12) S. P. Chakravarti, "Band-pass Effect in Wave-filters terminated in Negative Resistances," Phil. Mag. xxvi. p. 173 (July 1938).

XXVIII. *Notes on Mathieu Functions.*—I. *A Class of Hyperbolic Mathieu Functions.*

By W. G. Bickley, D.Sc.,
Imperial College of Science and Technology *.

[Received May 30, 1940.]

It was pointed out by the present writer some years ago [1] that the solution of the problem of forced convection in two dimensions for any cylinder involves the use of Mathieu functions. The periodic solutions of Mathieu's equation, and the associated characteristic numbers, have been studied in detail [2], and even tabulated [3]. Some attention has also been devoted to the associated functions of hyperbolic type for the equation

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\lambda\phi=0,$$

when λ is positive [4], [5]. None had apparently been given to the corresponding functions in the case of negative λ, which arise in the problem of forced convection, and also in the Oseen approximation to the flow of a viscous fluid past an elliptic cylinder—or indeed, with Southwell and Squire's [6] modification, past any cylinder. The last problem has recently received some attention [7], [8].

Some results concerning the functions for negative λ are given in the following pages. Mostly they were obtained some years ago, and, on account of certain difficulties not yet completely overcome, left unpublished, although Meksyn had access to the author's notes. Attention is now being elsewhere given to these functions, and it has been suggested that results obtained should be published, to prevent overlap and the possible waste of duplicated effort. Hence, although incomplete at several points, the present summary is given.

1. *Fundamentals and Notation.*

The equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda u=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

when transformed by the use of elliptic coordinates, connected with the rectangular coordinates (x, y) by the relations

$$x=c\cosh\xi\cos\eta, \quad y=c\sinh\xi\sin\eta, \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

* Communicated by the Author.

can be broken up into the ordinary differential equations

$$\frac{d^2X}{d\xi^2}-(A-\lambda c^2\cosh^2\xi)X=0, \quad . \quad . \quad . \quad . \quad . \quad (1.31)$$

$$\frac{d^2Y}{d\eta^2}+(A-\lambda c^2\cos^2\eta)Y=0, \quad . \quad . \quad . \quad . \quad . \quad (1.32)$$

by the substitution $u=XY$, where X and Y are functions of ξ and η respectively.

Ince [3] has standardized the equation (1.32) in the form

$$\frac{d^2y}{d\eta^2}+(a-2\theta\cos 2\eta)y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.33)$$

and has tabulated the characteristic numbers (values of a) corresponding to periodic solutions for certain values of θ, as well as values of the periodic solutions themselves. These solutions, and the associated characteristic numbers, are defined as follows :—

$$ce_{2n}(\eta,\theta)=\Sigma A_{2r}\cos 2r\eta, \qquad \text{char. no., } a_{2n}, \quad . \quad . \quad (1.41)$$

$$se_{2n+1}(\eta,\theta)=\Sigma B_{2r+1}\sin(2r+1)\eta, \qquad \text{,, } b_{2n+1}, \quad . \quad . \quad (1.42)$$

$$ce_{2n+1}(\eta,\theta)=\Sigma A_{2r+1}\cos(2r+1)\eta, \qquad \text{,, } a_{2n+1}, \quad . \quad . \quad (1.43)$$

$$se_{2n+2}(\eta,\theta)=\Sigma B_{2r+2}\sin(2r+2)\eta, \qquad \text{,, } b_{2n+2}, \quad . \quad . \quad (1.44)$$

where n takes all integral values from 0 to ∞, as does r for any value of n; the ratios of successive coefficients, A or B, are given by recurrence relations, and the values are made definite by normalizing the functions, so that if y stands for any one of them,

$$\int_0^{2\pi} y^2\,d\eta=\pi.$$

For the equation of forced convection, or for Oseen's equation, the sign of λ in (1.1) is negative, which means that the sign of θ in (1.33) is changed. Since, however, this change of sign is also equivalent to replacing η by $(\frac{1}{2}\pi-\eta)$, it is clear that the characteristic numbers are unchanged and the fundamental set of periodic solutions is as follows :—

$$ce_{2n}(\eta,-\theta)=(-)^n ce_{2n}(\tfrac{1}{2}\pi-\eta,\theta) \quad =\Sigma(-)^{n+r}A_{2r}\cos 2r\eta, \quad . \quad . \quad (1.51)$$

$$ce_{2n+1}(\eta,-\theta)=(-)^n se_{2n+1}(\tfrac{1}{2}\pi-\eta,\theta) \quad =\Sigma(-)^{n+r}B_{2r+1}\cos(2r+1)\eta, \quad (1.52)$$

$$se_{2n+1}(\eta,-\theta)=(-)^n ce_{2n+1}(\tfrac{1}{2}\pi-\eta,\theta) \quad =\Sigma(-)^{n+r}A_{2r+1}\sin(2r+1)\eta, \quad (1.53)$$

$$se_{2n+2}(\eta,-\theta)=(-)^{n+1}se_{2n+2}(\tfrac{1}{2}\pi-\eta,\theta)=\Sigma(-)^{n+r+1}B_{2r+2}\sin(2r+2)\eta. \quad . \quad . \quad . \quad (1.54)$$

Corresponding to each characteristic number, there is also a second solution, but these are not periodic, and are therefore seldom relevant to problems in mathematical physics.

Corresponding to each member of the set (1.4) there are functions of hyperbolic type, solutions of the equation

$$\frac{d^2y}{d\xi^2}-(a-2\theta\cosh 2\xi)y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$

and it is evident that if $f(\eta)$ is a solution of (1.33) for any value of a, then $f(i\xi)$ is a solution of (1.6) for the same value of a. We may thus define

$$Ce_{2n}(\xi,\theta)=\Sigma A_{2r}\cosh 2r\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.71)$$

$$Se_{2n+1}(\xi,\theta)=\Sigma B_{2r+1}\sinh(2r+1)\xi, \quad . \quad . \quad . \quad . \quad (1.72)$$

$$Ce_{2n+1}(\xi,\theta)=\Sigma A_{2r+1}\cosh(2r+1)\xi, \quad . \quad . \quad . \quad . \quad (1.73)$$

$$Se_{2n+2}(\xi,\theta)=\Sigma B_{2r+2}\sinh(2r+2)\xi. \quad . \quad . \quad . \quad . \quad (1.74)$$

In the same way, corresponding to each function of the set (1.5), there is a solution of the equation

$$\frac{d^2y}{d\xi^2}-(a+2\theta\cosh 2\xi)y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.8)$$

and we may define

$$Ceh_{2n}(\xi,\theta)=\Sigma(-)^rA_{2r}\cosh 2r\xi, \quad . \quad . \quad . \quad . \quad . \quad (1.91)$$

$$Ceh_{2n+1}(\xi,\theta)=\Sigma(-)^rB_{2r+1}\cosh(2r+1)\xi, \quad . \quad . \quad . \quad (1.92)$$

$$Seh_{2n+1}(\xi,\theta)=\Sigma(-)^rA_{2r+1}\sinh(2r+1)\xi, \quad . \quad . \quad . \quad (1.93)$$

$$Seh_{2n+2}(\xi,\theta)=\Sigma(-)^rB_{2r+2}\sinh(2r+2)\xi. \quad . \quad . \quad . \quad (1.94)$$

It appears that ultimately, for sufficiently large r, the coefficients of any of the set (1.4) oscillate in sign, and decrease rapidly with increasing r. The series of groups (1.7) and (1.9) are absolutely and uniformly convergent for any finite value of ξ; for large ξ the functions of group (1.7) oscillate and tend to zero, but those of group (1.9)—where the terms are ultimately all of the same sign—tend monotonically to infinity with ξ.

To solve the forced convection and the Oseen problems, however, we need solutions of (1.8) which tend to *zero* as $\xi\to\infty$; the search for such solutions, and the investigation of their properties and those of the set (1.9), were the motives of the research whose results are given in the ensuing sections of this paper.

2. *Series of Bessel Functions.*

It is known [2] that, in addition to the Fourier series (1.4), the periodic Mathieu functions are expansible in series of Bessel functions, of which the formula

$$A_0ce_{2n}(\eta)=ce_{2n}(\tfrac{1}{2}\pi)\Sigma(-)^rA_{2r}J_{2r}(k\cos\eta),$$

where $k^2=4\theta(=\lambda c^2)$, is typical. Similar expansions for functions of group (1.7) have $k\cosh\xi$ as the argument of the Bessel functions. Changing the

sign of θ means replacing k by ik, so that it seems probable that solutions of (1.8) exist which are expansible in terms of I—and also K—type of Bessel function. It is indeed easy to verify that such expansions, with coefficients proportional to those of the fundamental Fourier series (1.4), in which the argument of the Bessel function may be either $k\cosh\xi$ or $k\sinh\xi$, do formally satisfy the equation (1.8).

Dealing first with the argument $k\cosh\xi$, we find the series

$$\left.\begin{aligned} Ceic_{2n}(\xi)&=\Sigma(-)^r A_{2r}I_{2r}(k\cosh\xi),\\ Cekc_{2n}(\xi)&=\Sigma(-)^r A_{2r}K_{2r}(k\cosh\xi),\end{aligned}\right\}a_{2n} \quad . \quad . \quad (2.11),\ (2.12)$$

$$\left.\begin{aligned} Ceic_{2n+1}(\xi)&=\Sigma(-)^r B_{2r+1}I_{2r+1}(k\cosh\xi),\\ Cekc_{2n+1}(\xi)&=\Sigma(-)^r B_{2r+1}K_{2r+1}(k\cosh\xi).\end{aligned}\right\}b_{2n+1} \quad . \quad . \quad (2.13),\ (2.14)$$

The notation indicates by Ce (or Se) the type of the corresponding periodic trigonometrical function of the set (1.5), the third letter (i or k) indicates the type of Bessel Function, while the fourth letter (c or s) shows whether the argument of the Bessel functions is $k\cosh\xi$ or $k\sinh\xi$. Assuming for the moment that these series do converge, it is evident that the two functions *Ceic* and *Cekc* are not numerical multiples one of the other, since the terms of one increase, and of the other decrease, with increasing ξ. They therefore provide a fundamental set of solutions, and, moreover, *Cekc* is a solution which tends to zero at infinity.

The convergence of the series must, however, be examined. We recall that [3]

$$\frac{A_{2n}}{A_{2n-2}}=\frac{-\theta}{4n^2\left\{1-\dfrac{a}{4n^2}+\dfrac{\theta}{4n^2}\dfrac{A_{2n+2}}{A_{2n}}\right\}}, \quad . \quad . \quad . \quad . \quad (2.21)$$

and that

$$I_{2r}(x)=\frac{(x/2)^{2r}}{2r\,!}\left(1+\frac{x^2}{4(2r+1)}+\ldots\right) \quad . \quad . \quad . \quad (2.31)$$

$$K_{2r}(x)=\frac{(2r-1)\,!}{2(x/2)^{2r}}\left(1-\frac{x^2}{4(2r-1)}+\ldots\right), \quad . \quad . \quad (2.32)$$

whence it follows that

$$\frac{A_{2r+2}I_{2r+2}(k\cosh\xi)}{A_{2r}I_{2r}(k\cosh\xi)}=\frac{k^4\cosh^2\xi}{256r^4}(1+\ldots). \quad . \quad . \quad . \quad (2.41)$$

$$\frac{A_{2r+2}K_{2r+2}(k\cosh\xi)}{A_{2r}K_{2r}(k\cosh\xi)}=\frac{1}{\cosh^2\xi}\left(1-\frac{3}{2r}+\ldots\right). \quad . \quad . \quad (2.42)$$

It is clear from (2.41) that $Ceic_{2n}$ converges for all finite values of ξ; it also follows from (2.42) that *Cekc* converges for all values of ξ, but the convergence becomes slower as $\xi\to0$, and ceases to be uniform in a range including $\xi=0$. This slow and non-uniform convergence has at least two serious consequences, the first analytic, that the series for *Cekc* cannot be

differentiated term by term, the second numerical, that computation is difficult *, in the neighbourhood of $\xi=0$. It is, for instance, despite the argument $k\cosh\xi$, not true that $Cekc$ has a zero derivative when $\xi=0$. This last fact also follows from the consideration that $Ceic$ *has* a zero derivative when $\xi=0$, so that if $Cekc$ also had a zero derivative there, these functions would be solutions of the same differential equation satisfying the same boundary conditions (y finite, y' zero) at $\xi=0$, and would therefore be multiples one of the other; their behaviour for large ξ shows this to be untrue.

Next, considering the Bessel functions with argument $k\sinh\xi$ we find two more sets of solutions †

$$Ceis_{2n}(\xi)=\Sigma A_{2r}I_{2r}(k\sinh\xi), \qquad (a_{2n}) \quad . \quad . \quad (2.51)$$

$$Seis_{2n+1}(\xi)=\Sigma A_{2r+1}I_{2r+1}(k\sinh\xi), \quad (a_{2n+1}). \quad . \quad (2.52)$$

This exhausts the possibilities of the simple series, and that without providing any solution at all corresponding to the set b_{2n+2} of characteristic numbers, or any solutions corresponding to a_{2n+1} which vanish at infinity. On the other hand, we have now three solutions,

$$Ceh_{2n}(\xi), \quad Ceic_{2n}(\xi), \quad Ceis_{2n}(\xi),$$

corresponding to a_{2n}, all of which have a vanishing derivative at $\xi=0$, and tend to infinity with ξ; they must be multiples one of another. Their ratios will be obtained later.

3. *Integral Formulæ.*

If, in the formulae of the preceding section, we insert the definite integral representation of the Bessel functions,

$$I_n(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\phi}\cos n\phi\, d\phi,$$

$$K_n(x)=\int_0^{\infty} e^{-x\cosh\phi}\cosh n\phi\, d\phi,$$

we obtain formulæ which are intimately related to Whittaker's integral equation [9] for the periodic Mathieu functions. Thus, by (2.11) and the first of the above,

$$Ceic_{2n}(\xi)=\frac{1}{2\pi}\int_0^{2\pi} e^{k\cosh\xi\cos\phi}\Sigma(-)^r A_{2r}\cos 2r\phi\, d\phi,$$

$$=\frac{1}{2\pi}\int_0^{2\pi} e^{k\cosh\xi\cos\phi}ce_{2n}(\phi,-\theta)\, d\phi, \quad . \quad . \quad . \quad . \quad (3.11)$$

* It was to overcome this difficulty that the investigations in "The Numerical Summation of Slowly Convergent Series . . .," Phil. Mag. xxii. p. 754 (1936), were undertaken.

† The corresponding K series would seem not to converge when $\sinh\xi<1$; there is an enigma here, so far unresolved!

and, similarly from (2.12),

$$Cekc_{2n}(\xi)=\int_0^\infty e^{-k\cosh\xi\cosh\phi}Ceh_{2n}(\phi)\,d\phi. \quad . \quad . \quad . \quad . \quad . \quad (3.12)$$

(2.13) and (2.14) yield similar formulæ in which $(2n+1)$ replaces $2n$, while from (2.51) and (2.52) we obtain

$$Ceis_{2n}(\xi)=\frac{1}{2\pi}\int_0^{2\pi} e^{k\sinh\xi\cos\phi}ce_{2n}(\phi,\theta)d\phi, \quad . \quad . \quad . \quad (3.13)$$

$$Seis_{2n+1}(\xi)=\frac{1}{2\pi}\int_0^{2\pi} e^{k\sinh\xi\cos\phi}ce_{2n+1}(\phi,\theta)\,d\phi. \quad . \quad . \quad (3.14)$$

These integral formulæ are all of the general form

$$F(\xi)=\int e^{\pm k\mathrm{X}\Phi}f(\phi)\,d\phi,$$

where X is $\cosh\xi$ or $\sinh\xi$, and Φ is a trigonometric or hyperbolic function of ϕ, while $F(\xi)$ and $f(\phi)$ are solutions of the same or of associated Mathieu equations; a systematic examination of all the possibilities, however, discloses nothing further.

There is, however, an integral due to Poole [10], of a similar form, which suggests the exploration of the possibilities of the formula

$$F(\xi)=\int \mathrm{X}'\Phi' e^{\pm k\mathrm{X}\Phi}f(\phi)\,d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

where $\mathrm{X}'=d\mathrm{X}/d\xi$ and $\Phi'=d\Phi/d\phi$. The following results emerge :—

(a) With $\mathrm{X}=\cosh\xi$, $\Phi=\cos\phi$,

$$\begin{aligned}
F(\xi)&=\frac{1}{2\pi}\int_0^{2\pi} k\sinh\xi\sin\phi e^{k\cosh\xi\cos\phi}se_{2n}(\phi,-\theta)\,d\phi\\
&=\frac{k\sinh\xi}{2\pi}\Sigma(-)^r\mathrm{B}_{2r}\int_0^{2\pi} e^{k\cosh\xi\cos\phi}\sin\phi\sin 2r\phi\,d\phi\\
&=\frac{k\sinh\xi}{4\pi}\Sigma(-)^r\mathrm{B}_{2r}\int_0^{2\pi} e^{k\cosh\xi\cos\phi}\{\cos(2r-1)\phi-\cos(2r+1)\phi\}\,d\phi\\
&=\tfrac{1}{2}k\sinh\xi\Sigma(-)^r\mathrm{B}_{2r}\{\mathrm{I}_{2r-1}(k\cosh\xi)-\mathrm{I}_{2r+1}(k\cosh\xi)\}\\
&=\tanh\xi\Sigma(-)_r\mathrm{B}_{2r}.2r\mathrm{I}_{2r}(k\cosh\xi).
\end{aligned}$$

This is evidently an odd function of ξ, associated with the characteristic namber b_{2n}, so that it may appropriately be denoted by $Seic_{2n}(\xi)$. In a similar way the following set of formulæ can be constructed.

$$Seic_{2n}(\xi)=\tanh\xi\Sigma(-)^r2r\mathrm{B}_{2r}\mathrm{I}_{2r}(k\cosh\xi) \quad . \quad . \quad . \quad . \quad (3.31)$$

$$Seic_{2n+1}(\xi)=\tanh\xi\Sigma(-)^r(2r+1)\mathrm{A}_{2r+1}\mathrm{I}_{2r+1}(k\cosh\xi). \quad . \quad (3.32)$$

The coefficients—$2r\mathrm{B}_{2r}$ and $(2r+1)\mathrm{A}_{2r+1}$—are those of $se'_{2n}(\eta,-\theta)$ and $se'_{2n+1}(\eta,-\theta)$ respectively.

(*b*) With $X=\sinh\xi$, $\Phi=\cos\phi$,

$$Sei_{2n}(\xi)=\coth\xi\Sigma 2rB_{2r}I_{2r}(k\sinh\xi), \quad . \quad . \quad . \quad . \quad . \quad (3.33)$$

$$Cei_{2n+1}(\xi)=\coth\xi\Sigma(2r+1)B_{2r+1}I_{2r+1}(k\sinh\xi). \quad . \quad . \quad (3.34)$$

(*c*) With $X=\cosh\xi$, $\Phi=\cosh\phi$,

$$Sekc_{2n}(\xi)=\tanh\xi\Sigma(-)^r 2rB_{2r}K_{2r}(k\cosh\xi). \quad . \quad . \quad . \quad . \quad (3.35)$$

$$Sekc_{2n+1}(\xi)=\tanh\xi\Sigma(-)^r(2r+1)A_{2r+1}K_{2r+1}(k\cosh\xi). \quad . \quad (3.36)$$

4. *Asymptotic Expansions for Large ξ.*

Asymptotic expansions for the "J" type of hyperbolic Mathieu function have been given by Maclaurin [11] and Jeffreys [5]. Analogous expansions for the "I" and "K" types exist, but for these latter, more convenient formulæ are obtained by using e^{ξ} instead of $\cosh\xi$ as the variable.

The differential equation is

$$d^2y/d\xi^2-(a+\tfrac{1}{4}k^2e^{2\xi}+\tfrac{1}{4}k^2e^{-2\xi})y=0.$$

For large ξ a first approximation is

$$d^2y/d\xi^2-\tfrac{1}{4}k^2e^{2\xi}y=0,$$

which suggests writing

$$y=v\exp(\pm\tfrac{1}{2}ke^{\xi}).$$

Actually it is better to write

$$y=v\exp(\pm\tfrac{1}{2}ke^{\xi}+\beta\xi), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

and the equation for v is then

$$v''+(\pm ke^{\xi}+2\beta)v'+\{\pm(\beta+\tfrac{1}{2})ke^{\xi}+\beta^2-a-\tfrac{1}{4}k^2e^{-2\xi}\}v=0. \quad . \quad (4.21)$$

The coefficient of v remains finite as $\xi\to\infty$ if we take $\beta=-\tfrac{1}{2}$, and with this value of β our equation for v is

$$v''+(ke^{\xi}-1)v'-(a-\tfrac{1}{4}+\tfrac{1}{4}k^2e^{-2\xi})v=0, \quad . \quad . \quad . \quad (4.22)$$

where the ambiguous sign—associated with k and therefore easily restored later—has been temporarily dropped.

Assuming that

$$v=c_0+c_1e^{-\xi}+c_2e^{-2\xi}+\ldots+c_ne^{-n\xi}+\ldots, \quad . \quad . \quad . \quad (4.3)$$

we find that the differential equation is formally satisfied if the coefficients satisfy the recurrence formulæ

$$kc_1+(a-1/4)c_0=0, \quad 2kc_2+(a-9/4)c_1=0,$$

$$3kc_3+(a-25/4)c_2+\tfrac{1}{4}k^2c_0=0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$nkc_n+\{a-(2n-1)^2/4\}c_{n-1}+\tfrac{1}{4}k^2c_{n-3}=0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

whence

$$c_1/c_0=-(4a-1)/4k, \quad c_2/c_0=(4a-1)(4a-9)/2\,!(4k)^2,$$
$$c_3/c_0=-\{(4a-1)(4a-9)(4a-25)/3\,!(4k)^3+k/12\},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

The leading term in c_n/c_0 tends to infinity with n in a manner which shows that the expansion (4.3) is asymptotic, and not convergent. The alternative sign for k gives two distinct expansions, one of which tends to infinity, and the other to zero, as ξ tends to infinity. Evidently $+k$ will relate to the " I " type function, and $-k$ to the " K " type.

It is now easy to determine the value of c_0 corresponding to any of the functions defined in the previous sections, by replacing the I and K functions by their asymptotic expansions, and since only the leading terms are necessary, we can at the same time replace $\cosh\xi$ by $\frac{1}{2}e^{\xi}$.

Thus, for example,

$$Ceic_{2n}(\xi)=\Sigma(-)^r A_{2r} I_{2r}(k\cosh\xi)$$
$$\sim(\pi k)^{-\frac{1}{2}}\Sigma(-)^r A_{2r}\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\ldots$$

i. e.,

$$Ceic_{2n}(\xi)\sim(\pi k)^{-\frac{1}{2}}ce_{2n}(\pi/2,\theta)\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\times\left(1-\frac{4a_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad (4.51)$$

Similarly, we have

$$Ceis_{2n}(\xi)\sim(\pi k)^{-\frac{1}{2}}ce_{2n}(0,\theta)\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\times\left(1-\frac{4a_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad (4.52)$$

By what has been said in section 2, $Ceh_{2n}(\xi)$ is a multiple of $Ceic$ and $Ceis$. The precise relation can be obtained by comparing the value at $\xi=0$.

$$Ceh_{2n}(0)=\Sigma(-)^r A_{2r}=ce_{2n}(\tfrac{1}{2}\pi,\theta),$$
$$Ceis_{2n}(0)=A_0,$$

so that

$$Ceh_{2n}(\xi)\sim\frac{ce_{2n}(0.\theta)ce_{2n}(\frac{1}{2}\pi,\theta)}{(\pi k)^{\frac{1}{2}}A_0}\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\left(1-\frac{4a_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad (4.53)$$

From these results we now see that, *for all values of* ξ,

$$ce_{2n}(0,\theta)\,Ceic_{2n}(\xi)=ce_{2n}(\tfrac{1}{2}\pi,\theta)\,Ceis_{2n}(\xi)$$
$$=A_0 Ceh_{2n}(\xi). \quad (4.54)$$

By using the asymptotic expansion of the K functions, we find that

$$Cekc_{2n}(\xi)\sim ce_{2n}(\tfrac{1}{2}\pi,\theta)(\pi/k)^{\frac{1}{2}}\exp(-\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\left(1+\frac{4a_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad \ldots \quad (4.55)$$

Corresponding to the characteristic number b_{2n+1} we derive, similarly, the set of results

$$\begin{aligned}\tfrac{1}{2}k\mathrm{B}_1Ceh_{2n+1}(\xi)&=se_{2n+1}(\tfrac{1}{2}\pi,\theta)Ceis_{2n+1}(\xi)\\&=se'_{2n+1}(0.\theta)\,Ceic_{2n+1}(\xi)\\&\sim(\pi k)^{-\frac{1}{2}}se'_{2n+1}(0,\theta)se_{2n+1}(\tfrac{1}{2}\pi,\theta)\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\\&\qquad\times\left(1-\frac{4b_{2n+1}-1}{4k}e^{-\xi}+\ldots\right)\ldots \quad (4.61)\end{aligned}$$

$$Cekc_{2n+1}(\xi)\sim(\pi/k)^{\frac{1}{2}}se_{2n+1}(\tfrac{1}{2}\pi,\theta)\exp(-\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\times\left(1+\frac{4b_{2n+1}-1}{4k}e^{-\xi}+\ldots\right). \quad . \quad (4.62)$$

Corresponding to the characteristic number a_{2n+1} we have

$$\begin{aligned}\tfrac{1}{2}k\mathrm{A}_1Seh_{2n+1}(\xi)&=-ce'_{2n+1}(\tfrac{1}{2}\pi,\theta)Seis_{2n+1}(\xi)\\&=ce_{2n+1}(0,\theta)Seic_{2n+1}(\xi)\\&\sim-(\pi k)^{-\frac{1}{2}}ce_{2n+1}(0,\theta)ce'_{2n+1}(\tfrac{1}{2}\pi,\theta)\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\\&\qquad\times\left(1-\frac{4a_{2n+1}-1}{4k}e^{-\xi}+\ldots\right). \quad . \quad (4.71)\end{aligned}$$

$$Sekc_{2n+1}(\xi)\sim-(\pi/k)^{\frac{1}{2}}ce'_{2n+1}(-\tfrac{1}{2}\pi,\theta)\exp(-\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\times\left(1+\frac{4a_{2n+1}-1}{4k}e^{-\xi}+\ldots\right). \quad . \quad (4.72)$$

Finally, corresponding to the characteristic number b_{2n},

$$\begin{aligned}\tfrac{1}{8}k^2\mathrm{B}_2Seh_{2n}(\xi)&=-se'_{2n}(\tfrac{1}{2}\pi,\theta)Seis_{2n}(\xi)\\&=se'_{2n}(0,\theta)Seic_{2n}(\xi)\\&\sim-(\pi k)^{-\frac{1}{2}}se'_{2n}(0,\theta)se'_{2n}(\tfrac{1}{2}\pi,\theta)\exp(\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\\&\qquad\times\left(1-\frac{4b_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad . \quad . \quad (4.81)\end{aligned}$$

$$Sekc_{2n}(\xi)\sim(\pi/k)^{\frac{1}{2}}se'_{2n}(\tfrac{1}{2}\pi,\theta)\exp(-\tfrac{1}{2}ke^{\xi}-\tfrac{1}{2}\xi)\times\left(1+\frac{4b_{2n}-1}{4k}e^{-\xi}+\ldots\right). \quad \ldots \quad (4.82)$$

It appears from the above that we have obtained three forms of the solution which tends to infinity with ξ, but only one which tends to zero, for each characteristic number. In the Table these solutions are classified, and the number of the equation by which they are defined is given.

TABLE.

Showing the Various Solutions obtained.

Function.	cosh.	sinh.	I_m.	I_m.	K_m.	mI_m.	mI_m.	mK_m.	Behaviour at infinity.
Argument.	$n\xi$.	$n\xi$.	$k\cosh\xi$.	$k\sinh\xi$.	$k\cosh\xi$.	$k\cosh\xi$.	$k\sinh\xi$.	$k\cosh\xi$.	
Characteristic no. a_{2n}	Ceh_{2n} (1.91)		$Ceic_{2n}$ (2.11)	$Ceis_{2n}$ (2.51)	$Cekc_{2n}$ (2.12)				$\longrightarrow \infty$ $\longrightarrow 0$
b_{2n+1}	Ceh_{2n+1} (1.92)		$Ceic_{2n+1}$ (2.13)		$Cekc_{2n+1}$ (2.14)		$Ceis_{2n+1}$ (3.34)		$\longrightarrow \infty$ $\longrightarrow 0$
a_{2n+1}		Seh_{2n+1} (1.93)		$Seis_{2n+1}$ (2.52)		$Seic_{2n+1}$ (3.32)		$Sekc_{2n+1}$ (3.36)	$\longrightarrow \infty$ $\longrightarrow 0$
b_{2n+2}		Seh_{2n+2} (1.94)				$Seic_{2n+2}$ (3.31)	$Seis_{2n+2}$ (3.33)	$Sekc_{2n+2}$ (3.35)	$\longrightarrow \infty$ $\longrightarrow 0$

References.

(1) W. G. Bickley, "Some Solutions of the Problem of Forced Convection," Phil. Mag. xx. pp. 322–343 (1935).
(2) S. Goldstein, "Mathieu Functions," Trans. Camb. Phil. Soc. xxxiii. pp. 303–336 (1927).
(3) E. L. Ince, "Tables of Elliptic Cylinder Functions," Proc. Roy. Soc. Edin. lii. pp. 355–423 (1932).
(4) S. Goldstein, "The Free Oscillations of Water in a Canal of Elliptic Plan," Proc. Lond. Math. Soc. xxviii. pp. 91–101 (1928).
(5) H. Jeffreys, "On Certain Approximate Solutions of Mathieu's Equation," "On the Modified Mathieu Equation," and "The Free Oscillations of Water in an Elliptic Lake," Proc. Lond. Math. Soc. xxiii. pp. 437–476 (1924).
(6) R. V. Southwell and H. B. Squire. "A Modification of Oseen's Approximate Equation for the Motion in Two Dimensions of a Viscous Incompressible Fluid," Phil. Trans. Roy. Soc. A, ccxxxii. pp. 27–64 (1933).
(7) D. Meksyn, "Solution of Oseen's Equation for an Inclined Elliptic Cylinder in a Viscous Fluid," Proc. Roy. Soc. A, clxii. pp. 232–250 (1937).
(8) T. Lewis, "Solutions of Oseen's Extended Equation for Circular and Elliptic Cylinders and the Flat Plate," Q. J. M. ix. pp. 21–31 (1938).
(9) See, for instance, 'Modern Analysis,' 2nd ed. p. 400.
(10) E. C. G. Poole, "On certain Classes of Mathieu Functions," Proc. Lond. Math. Soc. xx. pp. 374–388 (1921).
(11) R. Maclaurin, "On the Solutions of the Equation $(\nabla^2+k^2)\psi=0$ in Elliptic Co-ordinates and their Physical Application," Trans. Camb. Phil. Soc. xvii. pp. 41–108 (1898).

XXIX. *The Thermal and Electrical Conductivities of Metals and Alloys.*

By W. F. Chubb, Ph.D., B.Sc.*

[Received September 1, 1939.]

The relationship between the thermal and electrical conductivities of metals and alloys has been the subject of numerous investigations, but it cannot yet be claimed that clarity has been reached. One reason for this is to be seen in the fact that many, if not all, of the difficulties encountered have arisen from the inaccuracies of the methods employed for the measurement of thermal conductivity, for while the corresponding electrical measurements may be made over a very wide range of temperature and to a very high degree of accuracy, very few thermal conductivity determinations higher in accuracy than about 1 per cent. have as yet been reported. It may also be observed that many of the discussions advanced as to the relationship between thermal and electrical conductivity have been based not upon actual measurements of thermal conductivity, but upon values calculated from thermo-electric measurements by a method due to Borelius [1]. The difficulties associated with this method may be illustrated by the fact that Sedstrom's calculated thermal conductivity values for gold-silver alloys [2] differ from the practical determinations of Schulze [3] by as much as 30 per cent., but the type of variation between thermal conductivity and composition is essentially similar in each instance. The existing position as regards these relationships may therefore be defined briefly.

Table I.

Metal.	K/C at 0° C.	Metal.	K/C at 0° C.
Aluminium	136×10^{-8}	Molybdenum ..	201×10^{-8}
Bismuth	214	Nickel	169
Copper	150	Rhodium	101
Iron	194	Tungsten	195
Magnesium	153	Zinc	160

It was noted by Wiedemann and Franz that at ordinary temperatures the ratio between thermal conductivity, K, and electrical conductivity, C,

* Communicated by the Author.

is approximately constant for all metals. At 0° C. this ratio equals 140×10^{-8}, and the Wiedemann-Franz law may thus be written Z=K/C. In Table I., giving data for the common metals, it will nevertheless be seen that the constancy of the Wiedemann-Franz ratio is far from exact, for the ratio K/C varies from 101×10^{-8} for rhodium to more than double that value for bismuth.

It has also been demonstrated by Bridgman [4] that thermal conductivity is influenced by crystal structure. In the case of single crystals of tin, zinc and cadmium, the thermal conductivity may be as much as 20 per cent. greater than for polycrystalline material, while in the case of bismuth the difference may be as high as 200 per cent. For copper, on the other hand, the difference between single crystal and polycrystalline metal is only about 4 per cent. As to electrical conductivity, both Scott and C. S. Smith have reported that measurements yield the same values for single crystal and polycrystalline materials.

The effects of deformation have also been studied, and it has been observed that both the thermal and the electrical conductivities are lowered by tension, except in the case of nickel. The variations occurring in each instance are not, however, in the same proportion, so that the Wiedemann-Franz ratio is again not constant. In some instances it is the thermal conductivity which is affected more than the electrical conductivity when the metal is subjected to stress, while in other instances the opposite tendency is displayed. Also in the case of gold, for example, the thermal conductivity has been observed to decrease with cold-working, whereas with aluminium it increases, but many of the discrepancies noted may be attributed to the effects of impurities. The results for single crystals subjected to deformation are equally inconclusive.

It has also been recorded that the Wiedemann-Franz ratio is not constant for metals under compression, for in the case of copper and silver the electrical conductivity increases while the thermal conductivity is reduced, the differences being greater than can be ascribed to the non-metallic part of the conductivity. Hence, in considering all the available evidence, it is clear that the Wiedemann-Franz ratio is by no means established for the pure metals.

As regards the relationship between thermal conductivity and temperature, it has often been stated that this property is almost independent of temperature in the case of a pure metal. The measurements reported, however, do not always justify this conclusion, but reveal, in fact, that conductivity generally decreases with increase of temperature, aluminium providing an exception. For example, in Schofield's measurements [5], reproduced herewith, it will be seen that for copper there is a decrease of 4·8 per cent. over a temperature range of 530° C. and for aluminium a change in conductivity amounting to 2·7 per cent. over a range of 240° C.

In particular, zinc shows a pronounced fall in thermal conductivity of 7·8 per cent. over a temperature interval of only 78° C. Since the method employed in these determinations cannot be claimed to provide an accuracy higher than about 1 per cent., the statement that thermal conductivity is practically independent of temperature is not necessarily correct in the case of those metals cited.

Lees [6] has also deduced, from measurements over the range 30° to −170° C., that the heat conductivity of pure metals has a tendency to increase with decrease of temperature. Onnes and Holst [7], on the other

TABLE II.

Schofield's Thermal Conductivity Results.

Metal.	Temperature, °C.	Conductivity, c.g.s. units.	Variation.
Aluminium, 99·7 per cent. purity.	130	0·528	+2·7 per cent. in 240° C.
	150	0·527	
	253	0·536	
	370	0·542	
Copper, 99·9 per cent. purity.	95	0·901	−4·8 per cent. over 530° C.
	178	0·893	
	266	0·888	
	473	0·858	
	625	0·842	
Zinc, 99·8 per cent. purity.	25	0·258	−7·8 per cent. over 78° C.
	54	0·246	
	80	0·241	
	103	0·238	

hand, have shown that down to −93° C. thermal conductivity is independent of temperature, and that below that temperature it begins to increase. This increase is stated to be very slight at first, but more rapid down to −269° C. Thus the thermal conductivity of a metal may become infinite at the absolute zero of temperature, but the available results do not display any discontinuity at the temperature at which electrical supraconductivity makes itself apparent. It will therefore be clear that the Wiedemann-Franz ratio does not apply at these low temperatures.

In the case of the electrical conductivity it is also established that this property decreases as the temperature is increased, so that since the thermal conductivity does not vary appreciably with temperature, the Wiedemann-Franz-Lorenz coefficient, $L=K/C\theta$, is an approximation

only. In fact, for nickel it increases from $6{\cdot}3\times10^{-9}$ at 30° C. to $11{\cdot}0\times10^{-9}$ at 500° C., according to the measurements of Saegar [8].

It is also clear that in the case of alloys the Wiedemann-Franz and Wiedemann-Franz-Lorenz relationships are very far from being exact, the results displaying considerable inconstancy. Thus, for copper-nickel alloys the conductivity-composition curves are of U-form, but the two conductivities are not lowered proportionately when one metal is added to the other, and for this reason the Wiedemann-Franz ratio is not constant. It rises to a maximum at about 55 per cent. nickel and then falls again, but these observations have been explained on the grounds that when the non-metallic part of the conductivity represents a large proportion of the total conductivity, as in the case of alloys of intermediate composition, the Wiedemann-Franz ratio increases considerably. Thus this theoretical prediction agrees well with actual observations.

In a recent paper on thermal conductivity, C. S. Smith and Earl Palmer [9] have shown that the Lorenz coefficient for the alpha-solid solutions of copper with silicon, aluminium, manganese, and nickel increases with increasing concentration of the solute atoms. On the other hand, Griffiths [10] states that the Lorenz coefficient for aluminium alloys and for the bronzes is constant within the limits of experimental error in the measurement of thermal conductivity, but this is not usually so. In the case of alloys examined by Smith and Palmer, the increases reported are at first slight, but become more rapid later, so that the slopes of the curves to the composition axis increase in the order zinc, tin, nickel, aluminium, manganese, silicon, and phosphorus. It is also shown that alloying produces a positive temperature coefficient of conductivity. Furthermore, the Wiedemann-Franz ratio is proved to vary considerably, and for the copper alloys it lies on a curve which is almost a straight line. C. S. Smith has also shown, from measurements over the range 20° to 200° C., that for copper alloys generally the thermal conductivity, K, and the electrical conductivity, λ, are related to temperature, T, by the relation

$$K=k+C.\lambda.T,$$

in which C is a constant which may be regarded as the true Lorenz coefficient for the metallic part of the conductivity and k represents the non-metallic part. The value of this relationship is, however, limited by the fact that it applies over a certain portion only of the above temperature range, and the deviations from this linear relationship become considerable for those alloys of high thermal conductivity. Smith's equation must, in fact, be considered as a modification of Eucken's original formula, in which thermal conductivity was written as the sum as two terms, one metallic, K_m, and the other non-metallic, K_i, such that

$$K=K_m+K_i.$$

The objection to this formula, as emphasized by Hume-Rothery, lies in the fact that thermal and electrical conductivities are not in strictly linear relationship except in those instances where the non-metallic part of the conductivity is relatively very small as compared with the metallic part. Despite this objection, however, there would appear to be a certain measure of justification for regarding thermal conductivity as the sum of two independent terms, for if heat is transferred down the lattice as elastic waves, as demanded by the Debye heat theorem, the energy of the wave must be continually reabsorbed as heat and is thus converted into atomic vibrations. Hence, if ρ is the density, C the specific heat, v the velocity of sound in the metal, and l is a dissipation factor denoting the distance which the wave travels before its energy has been reduced to $1/e$ of its value, it may be proved that the thermal conductivity, K_1, is given by the formula

$$K_1 = \rho.v.l.C/4.$$

Now the dissipation of energy will be greater at high than at low temperatures, so that since l will vary inversely with the temperature, $l = A/T$, where A is a constant, and the above equation then becomes

$$\frac{K}{a.C} = \frac{A.\rho.v}{4a.T}$$

by dividing by $a.C$, where a is the atomic weight. Bidwell [11] has shown from measurement of the thermal conductivities of lithium, zinc, lead, and sodium that if $K/a.C$ be plotted against $1/T$ a straight-line relationship is obtained, and that the $K/a.C$ axis is cut at a definite positive value in each instance, so that it is again evident that thermal conductivity is the sum of two terms, the theoretical argument outlined above being in agreement with practical determinations. It has also been suggested that this conductivity equation should include a term representing the propagation of heat by atomic collisions, this being regarded as independent of temperature. Hence, if this portion of the total conductivity be denoted by k_2, if L is the mean free path, and V is the root mean square velocity, it follows that

$$k_2 = \frac{\rho.V.L.C}{3}$$

or

$$\frac{k_2}{a.C} = \frac{\rho.V.L}{3a} = \text{constant.}$$

Hence

$$\frac{K}{a.C} = \frac{k_1}{a.C} + \frac{k_2}{a.C} = \frac{A.\rho.V}{4a.T} + \frac{\rho.V.L}{3a},$$

and therefore,

$$\frac{K}{a.C} = \frac{B}{T} + D,$$

where B and D are constants.

For the metals enumerated this law applies over a very wide temperature range—for example, down to $-200°$ C. for lithium, and to an even lower temperature for zinc, lead, and sodium.

Hume-Rothery [12] has attempted to identify the Bidwell equation with that of Eucken's by employing the relationship established by Gruneisen for electrical resistivity. This states that the electrical resistivity is proportional to the product of the specific heat and the temperature, but Hume-Rothery's treatment, which does not allow of any definite conclusion being reached, does not appear altogether justified, if only for the reason that the Gruneisen equation is essentially empirical, although its later modifications apply accurately over a very wide temperature range. A further consideration lies in the fact that the Gruneisen equation was deduced from electrical resistivity measurements made with metals of exceptionally high purity, free from strains and other physical abnormalities, but as regards the corresponding thermal conductivity measurements very few have as yet been made to a comparable accuracy with materials of like purity.

Although the methods used for the determination of thermal conductivity are insufficient in accuracy for fundamental purposes, the results obtained have nevertheless allowed certain deductions as to the form of curve between conductivity and composition in alloy series to be made. In the case of binary alloys forming solid solutions the curves obtained are of characteristic U-shape, but when the two metals are insoluble in one another or form a constituent which is insoluble in one of the components, the thermal conductivity stands in linear relationship to the amount of the second phase present over the limits of composition governing its existence. Electrical conductivity-composition curves are of similar form, although the variations with composition in the case of solid solutions are not exactly the same as the corresponding variations in the thermal conductivity. This again means that neither the Wiedemann-Franz nor the Wiedemann-Franz-Lorenz conception applies in the case of alloys. As an example it may be mentioned that for copper-nickel alloys the Wiedemann-Franz ratio is by no means constant, and increases from 68×10^{-7} for pure copper to over 100×10^{-7} for the alloy containing 50 atoms per cent. of nickel, the ratio falling again to 66×10^{-7} for pure nickel. Also, in the gold-silver series of solid solutions the Lorenz coefficient at 0° C. calculated from Sedstrom's figure for thermal and electrical conductivities increases from $5{\cdot}88 \times 10^{-9}$ for pure gold to $7{\cdot}49 \times 10^{-9}$ for the alloy of 60 atoms per cent.

silver, while at 100° C. it increases from $5{\cdot}88\times10^{-9}$ for gold to $7{\cdot}9\times10^{-9}$ with 70 atomic per cent. of silver, the curves not being amenable to rigid mathematical analysis. It is nevertheless clear that these deviations from the two laws are not ascribable to inaccuracies in the methods of determining thermal conductivity, and for this reason it would appear that the only conclusion possible is that the Wiedemann-Franz and Wiedemann-Franz-Lorenz laws are not strictly applicable to alloy series.

As regards intermetallic compounds, the significance and interpretation of these two relationships are even more difficult to define, this being again due to a dearth of accurate information concerning conductivity measurements, more particularly in respect of the thermal conductivities of intermetallic compounds. In those instances where it has been possible to deduce appropriate values, the Wiedemann-Franz ratios show no apparent relationship to the corresponding values for the constituent metals, the antimony-cadmium compound, SbCd, providing a typical example. In this system the Wiedemann-Franz ratio for the pure metals is $6{\cdot}2\times10^{-6}$ and $7{\cdot}9\times10^{-6}$ for antimony and cadmium respectively, whereas the corresponding ratio for the compound SbCd has been found to be 660×10^{-6}. Hence this law does not appear to apply to intermetallic compounds, but the unusual behaviour of intermetallic compounds may possibly be explained by the fact that the non-metallic part of the conductivity assumes a very much greater importance than in alloys forming solid solutions, and exceptionally high value for the Wiedemann-Franz ratio being obtained in consequence.

Hence, from this review of the relationships existing between the thermal and electrical conductivities of metals and alloys, it may be concluded that the chief reason for the many anomalies that have arisen in regard to their interpretation lies in the relative inaccuracy of thermal conductivity determinations. It is equally clear that in order to obtain a clearer explanation of the many conceptions advanced it would be necessary to develop an exceptionally accurate method of measuring thermal conductivity. Also, despite the many attempts to do so, it does not seem likely that an acceptable theory of metallic conduction can be formulated until such time as sufficient thermal conductivity data of the required accuracy are available, calculation from other physical properties being of little value for these purposes. A method and procedure based upon the Guard Ring principle and capable of providing data of fundamental accuracy is, however, available [13].

References.

(1) Borelius, *Ann. d. Physik*, lii. p. 398 (1917).
(2) Sedstrom, *Ann. d. Physik*, lix. p. 134 (1919) ; lxxv. p. 549 (1924).
(3) Schulze, *Phys. Zeitschr.* xii. p. 1028 (1911).
(4) Bridgman, Proc. Amer. Acad. Arts Sci. lvii. p. 80 (1922).

(5) Schofield, Proc. Roy. Soc. cvii. p. 206 (1925).
(6) Lees, Phil. Trans. A, ccviii. p. 381 (1908).
(7) Onnes & Holst, *Proc. Akad. Amsterdam*, p. 760 (1914).
(8) Saeger, Rensselaer Poly. Inst. Bull. No. 21, Eng. & Sci. Series, pp. 1–23 (1928).
(9) Smith & Palmer, ' Metals Technology ' pp. 1–19 (Sept. 1935).
(10) Griffiths, Proc. Roy. Soc. A, cxv. p. 236 (1927).
(11) Bidwell, Phys. Review, xxxii. p. 311 (1928).
(12) Hume-Rothery, ' The Metallic State,' Oxford University Press, pp. 79–80 (1931).
(13) Chubb, Metal Ind. (Lond.) (May 27th and June 3rd (1938).

XXX. *The Idea of Minimum Proper Time, and some Consequences of it.*

By M. Fahmy, The Egyptian University, Cairo *.

[Received September 1, 1939.]

Kaluza [1] introduced a principle to unify gravitation and electricity as an alternative to that of Weyl. According to this theory the track of a charged particle has been represented by a geodesic. He adopted an extra fifth dimension together with the four dimensions used in the theory of relativity, thus giving the particle an extra degree of freedom; in other words, he treated the motion of the charged particle as if it occurred in a five-dimensional continuum.

Later O. Klein [2] improved this theory and directed attention to the application of the notation to wave mechanics.

Fisher [3], Flint, and others have developed a theory of the electron on the principle that the electron's track is a null geodesic in a five-dimensional manifold.

In this theory the mass of the electron is regarded as a fundamental constant related to the constant α of Kaluza's theory, to which they assigned a definite value in order to make the analogy of the electron and photon very close.

Fermi has found the above analogy very useful in the treatment of β-ray activity.

We start by giving a brief summary of the theory :—

The line element $d\sigma$ in five dimensions is written in the form

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$=\left(\gamma_{mn}+\frac{\gamma_{m5}\gamma_{n5}}{\gamma_{55}}\right)dx^m dx^n+\frac{(\gamma_{\mu 5}dx^\mu)^2}{\gamma_{55}}, \quad . \quad . \quad . \quad . \quad (2)$$

where m and n are to be summed over the values 1 to 4 and μ, ν over the values 1 to 5.

If the coefficient $\left(\gamma_{mn}+\frac{\gamma_{m5}\gamma_{n5}}{\gamma_{55}}\right)$ is identified with g_{mn} of Einstein's theory, it follows that

$$d\sigma^2=ds^2+\frac{1}{\gamma_{55}}(\gamma_{\mu s}dx^\mu)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

---

* Communicated by Dr. H. T. Flint.

Where ds is the line element of Einstein's theory, $\gamma_{\mu 5}dx^{\mu}$ is replaced by $\sqrt{\gamma_{55}}d\theta$, so that $d\theta$ may be regarded as the projection of the line element $d\sigma$ perpendicular to the four-dimensional space in which ds lies. The coefficients γ_{m5} are related to the components of the electro-magnetic potential (ϕ_m) as follows :

$$\gamma_{m5}=\alpha\phi_m, \quad \ldots\ldots\ldots \quad (4)$$

where α is Kaluza's constant and according to Fisher, Flint and W. Wilson [4]

$$\alpha=\frac{e}{m_0c^2}, \quad \ldots\ldots\ldots \quad (5)$$

γ_{55} is regarded as constant and equal to 1.

Thus $$d\sigma^2=ds^2+d\theta^2. \quad \ldots\ldots\ldots \quad (6)$$

The equation of the geodesic in the 5-dimensional continuum is

$$\frac{d}{dt}\left\{\gamma_{\mu\nu}\frac{dx^{\nu}}{dt}\right\}=\frac{1}{2}\frac{\partial\gamma_{\alpha\beta}}{\partial x^{\mu}}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}. \quad \ldots\ldots\ldots \quad (7)$$

It can be reduced to the form

$$\frac{d}{dt}\left(g_{mn}\frac{dx^n}{dt}\right)-\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^m}\frac{dx^{\alpha}}{dt}dx^{\beta}=\sqrt{\gamma_{55}}\,d\theta\,dt\left[\frac{\partial}{\partial x^m}\left(\frac{\gamma_{n5}}{\gamma_{55}}\right)-\frac{\partial}{\partial x^n}\left(\frac{\gamma_{m5}}{\gamma_{55}}\right)\right]\frac{dx^n}{dt}.$$

This must be identified with the relativistic equation

$$\frac{d}{dt}\left(g_{mn}\frac{dx^n}{dt}\right)-\frac{1}{2}\frac{\partial g_{ab}}{\partial x^m}\frac{dx^a}{dt}\frac{dx^b}{dt}=\frac{e}{mc}\left(\frac{\partial\phi_n}{\partial x^m}-\frac{\partial\phi_m}{\partial x^n}\right)\frac{dx^n}{dt}, \quad \ldots \quad (8)$$

i. e., it is necessary to make the identification

$$\frac{\alpha}{\sqrt{\gamma_{55}}}\frac{d\theta}{dt}=\frac{e}{mc},$$

or, making use of equation (5),

$$\frac{d\theta}{dt}=\sqrt{\gamma_{55}}c\frac{m_0}{m}. \quad \ldots\ldots\ldots \quad (9)$$

But since $ds^2=-c^2d\tau_0{}^2$ where $d\tau_0$ denotes the proper time of the theory of relativity, it follows that

$$d\sigma^2=c^2d\tau_0{}^2+\gamma_{55}c^2\frac{m_0{}^2}{m^2}dt^2=c^2d\tau_0{}^2-\frac{m_0{}^2c^2}{m^2}dt^2. \quad \ldots \quad (10)$$

Thus if the track of the electron has a projection $d\theta$ perpendicular to ds, which reduces the left-hand side of equation (10) to zero, the track is that of a null geodesic

This is the case if $\frac{d\tau_0}{dt}=\frac{m_0}{m}$ as is required by the theory of relativity.

In this case $\frac{d\theta}{dt}=\frac{d\theta}{d\tau_0}\frac{m_0}{m}$, or by equation (9)

$$\frac{d\theta}{d\tau_0}=c\sqrt{\gamma_{55}}.$$

$$\therefore \quad \frac{d\theta}{ds}=\sqrt{-1}.$$

The geometrical picture is that of a track making a constant angle ϵ_0 with the normal to ds such that

$$\cot \epsilon_0=\sqrt{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It is interesting to note that the mass of the charged particle is not introduced as the mass of a test particle as in the usual presentation of the theory of relativity. In this case the constant m_0 is a constant of the continuum and the track of the null geodesic takes the form (8). This justifies the treatment of the electron as if it possessed mass m_0. Equations (9) and (10) then give the usual mass relation

$$\frac{m}{m_0}=\frac{dt}{d\tau},$$

and for the case of special relativity $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$. From the fact that

$$ds^2=-d\theta^2=-\frac{m_0{}^2c^2}{m^2}dt^2,$$

it follows that

$$m^2\frac{ds^2}{dt^2}=-m_0{}^2c^2,$$

or

$$-p_x{}^2-p_y{}^2-p_z{}^2+\frac{\mathrm{E}^2}{c^2}=m_0{}^2c^2,$$

where E is the energy mc^2 of the particle,

i. e.,

$$\frac{\mathrm{E}^2}{c^2}=p^2+m_0{}^2c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is the well-known equation for the energy of the particle and further justifies the application of this formula to the electron. These results are obtained without reference to electron structure; they follow from the fact that a constant α is introduced into Kaluza's relation which is identified with $\frac{e}{m_0c^2}$.

It is necessary to consider how other charged masses enter into the theory, *e. g.*, let us consider the case of a particle of mass M carrying a charge e.

The equation of the geodesic compared with equation (7) in which M must replace m, shows that it is necessary to make the identification

$$\frac{1}{\sqrt{\gamma_{55}}}\alpha\frac{d\theta}{dt}=\frac{e}{\mathrm{M}c},$$

i. e.,

$$\frac{d\theta}{dt}=\frac{m_0 c}{\mathrm{M}}\sqrt{\gamma_{55}}.$$

In this case

$$\frac{d\theta}{dt}=\frac{d\theta}{c\,d\tau_0}=\frac{m_0}{\mathrm{M}}\sqrt{\gamma_{55}}\frac{dt}{d\tau_0},$$

and putting

$$\frac{\mathrm{M}_0}{\mathrm{M}}=\frac{d\tau_0}{dt},$$

it follows that

$$\frac{d\theta}{ds}=\frac{m_0}{\mathrm{M}_0}\sqrt{-1}, \quad \ldots\ldots\ldots \quad (13)$$

which means that the particle occupies a path making an angle ϵ with the normal to ds, where

$$\cot\epsilon=\frac{m_0}{\mathrm{M}_0}\sqrt{-1}. \quad \ldots\ldots\ldots \quad (14)$$

In this case $d\sigma$ is not equal to zero; in fact,

$$d\sigma^2=ds^2+d\theta^2=-c^2 d\tau_0{}^2\left(1-\frac{m_0{}^2}{\mathrm{M}_0{}^2}\right). \quad \ldots\ldots \quad (15)$$

But the fifth component of the momentum

$$p_5=\mathrm{M}_0\gamma_{5\mu}\frac{dx^\mu}{d\tau_0}=\mathrm{M}_0\sqrt{\gamma_{55}}\frac{d\theta}{d\tau_0}=\sqrt{\gamma_{55}}m_0 c. \quad \ldots\ldots \quad (16)$$

This is a constant and is independent of the mass M. It depends only on the charge e through m_0 (equation 5). According to Flint the additional coordinate x^5 is cyclic in character, and from the quantum theory the suggestion is that dx^5 occurs in physics in multiples of $h/\mathrm{M}_0 c$, where M_0 is the mass of the particle whose motion is being considered.

It follows from

$$d\theta=\frac{\gamma_{5\mu}dx^\mu}{\sqrt{\gamma_{55}}}$$

that

$$d\theta=\sqrt{\gamma_{55}}(\alpha\phi_m dx^m+dx^5), \quad \ldots\ldots \quad (17)$$

i. e., $\left(\frac{d\theta}{\sqrt{\gamma_{55}}}-\alpha\phi_m dx^m\right)$ occurs in multiples of $\frac{h}{\mathrm{M}_0 c}$.

But from (13)

$$d\theta=\frac{m_0 c}{\mathrm{M}_0}d\tau_0,$$

so that

$$\frac{m_0 c}{\mathrm{M}_0}d\tau_0-\alpha\phi_m dx^m=n\frac{h}{\mathrm{M}_0 c}. \quad \ldots\ldots \quad (18)$$

In the absence of an electromagnetic field, it follows that $d\tau_0$ occurs in multiples of h/m_0c^2. This result has been applied to the case of the electron, but it has not been stated before for other particles. It has simply been concluded that the minimum proper time associated with any particle is h/m_0c^2. The above treatment of the question appears to indicate that h/m_0c^2 is the least proper time which can be associated with any particle, m_0 being the mass of the electron.

It follows from (18) that

$$\frac{m_0c}{\mathrm{M}_0}d\tau_0 - \alpha\phi_m dx^m$$

is a finite quantity.

Suppose that it is applied to the case where a charged particle of mass M and charge e moves in a potential field of magnitude $-\frac{e}{r}$. This is the case where a charged particle moves in the Coulomb field of another stationary particle :

$$\phi_4 dx^4 = -ce\phi\, dt = \frac{ec}{r}dt\,; \quad d\tau_0 = \frac{\mathrm{M}_0}{\mathrm{M}}dt.$$

Thus $\left(\frac{m_0c}{\mathrm{M}} - \frac{ce^2}{m_0c^2}\cdot\frac{1}{r}\right)dt$ is finite. In other words, r cannot attain the value $\frac{e^2}{m_0c^2}\cdot\frac{\mathrm{M}}{m_0}$ for, if it did so, the element of time dt would become very great, and the interpretation is that the particle could not be localized in time and so could not be treated by ordinary mechanical methods. The dynamical representation of the charge breaks down under these circumstances. It has been deduced that the smallest element of proper time associated with the particle in the absence of a field of force is h/m_0c^2, and thus the smallest element of ordinary time is $\frac{\mathrm{M}}{\mathrm{M}_0}\frac{h}{m_0c^2}$.

If the particle moves with velocity v the smallest element of distance is $\frac{\mathrm{M}}{\mathrm{M}_0}\frac{h}{m_0c^2}v$.

A new Definition of Proper Time and some Consequences of it.

In a recent paper [5] A. Proca and S. Goudsmit suggest an extension to the idea of proper time. If we apply their suggestion to the present case the proper time will be defined by means of $d\sigma$ and not by ds. Thus the proper time $d\tau$ will now be defined by $d\sigma^2 = -c^2d\tau^2$ instead of by $d\tau_0$ with $ds^2 = -c^2d\tau_0{}^2$.

Mass and rest mass are now defined by means of μ and μ_0 where $\frac{d\tau}{dt} = \frac{\mu_0}{\mu}$, dt denoting an element of ordinary time.

The identification of the geodesic with the track of a charged particle requires, as before, $\frac{d\theta}{dt}=\frac{m_0 c}{\mu}\sqrt{\gamma_{55}}$, where m_0 is a constant identified as the rest mass of the electron.

From $$d\sigma^2=ds^2+d\theta^2,$$

$$\mu^2\left(\frac{d\sigma}{dt}\right)^2=p_1^2+p_2^2+p_3^2+p_4^2+m_0{}^2c^2,$$

where $$p_4=\mu\frac{dx^4}{dt}=i\mu c=i\frac{E}{c},$$

and $E=\mu c^2$ the energy of the particle and μ is to be described as the mass of the particle:

$$\mu^2c^2\left(\frac{d\tau}{dt}\right)^2=\mu_0{}^2c^2=-p^2+\frac{E^2}{c^2}-m_0{}^2c^2,$$

$$\frac{E^2}{c^2}=\mu_0{}^2c^2+p^2+m_0{}^2c^2.$$

$$\mu^2=\mu_0{}^2+\frac{p^2}{c^2}+m_0{}^2.$$

But $p=\mu v$, so that

$$\mu^2\left(1-\frac{v^2}{c^2}\right)=\mu_0{}^2+m_0{}^2,$$

$$\mu=\frac{\sqrt{\mu_0{}^2+m_0{}^2}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad \ldots\ldots\ldots \quad (19)$$

The quantity μ is the ordinary mass, for it is related to the energy by means of the equation $E=\mu c^2$. The term $\sqrt{\mu_0{}^2+m_0{}^2}$ in equation (19) represents the rest mass.

It consists of two parts: m_0, which results from the existence of a charge e, and μ_0, which is described as the material mass.

Let M_0 denote $\sqrt{\mu_0{}^2+m_0{}^2}$. In the case of an electron $\mu_0=0$ and we are left with the rest mass m_0; in other words, no material mass exists.

The analogy between photons and electrons becomes very close, since in the case of the photon the rest mass vanishes and its ordinary mass equals $h\nu/c^2$, while in the case of an electron μ_0 vanishes, *i. e.*, no material mass exists, and its ordinary mass equals $h\nu/c^2$, where ν, the de Broglie frequency, equals mc^2/h.

Corresponding to this we have

$$i\frac{d\sigma}{dt}=c\frac{d\tau}{dt}=\frac{\mu_0 c}{\mu}$$

for the electron $\mu_0=0$; $d\sigma=0$; $d\tau=0$, but $d\tau/\mu_0$ is finite and neither μ nor dt vanishes. In fact, $\mu=h\nu/c^2$ in this case.

The coordinate x^5 is cyclic, whereas in the previous discussion

$$\frac{\partial}{\partial x_5}=\frac{2\pi i \mathrm{M}_0 c}{h}$$

this is now replaced by

$$\frac{\partial}{\partial x^5}=\frac{2\pi i(\sqrt{\mu_0{}^2+m_0{}^2})c}{h}.$$

Instead of μ, we shall now write M, since it represents the ordinary mass, but μ_0 will be retained to denote the material mass.

It is now necessary to regard dx^5 as occurring in multiples of

$$\frac{nh}{(\sqrt{\mu_0{}^2+m_0{}^2}).c}.$$

From (17) $$d\theta=\sqrt{\gamma_{55}}\left[\frac{e}{m_0c^2}\phi_m dx^m+dx^5,\right],$$

i. e., $$\frac{m_0c}{\mathrm{M}}dt-\frac{e}{m_0c^2}\phi_m dx^m=\frac{n.h}{(\sqrt{\mu_0{}^2+m_0{}^2})c}=\frac{nh}{\mathrm{M}_0c}.$$

For the case when there is no field

$$dt=n.\frac{\mathrm{M}}{\mathrm{M}_0}\frac{h}{m_0c^2},$$

as in the previous case.

When the field is a coulomb field, it follows as before that

$$\left(\frac{m_0c}{\mathrm{M}}-\frac{ec}{m_0c^2}\frac{e}{r}\right)dt \text{ is finite,}$$

i. e., $$r\not<\frac{\mathrm{M}}{m_0}\frac{e^2}{m_0c^2}.$$

Application to the Orbital Electrons in an Atom.

An opportunity of applying equation (18) is provided by the case of a charged particle in a field Ze/r, as in the case of the electrons of an atom. In this case the condition (18) becomes

$$\frac{m_0c}{\mathrm{M}}dt-\alpha c\frac{Ze}{r}dt=\frac{nh}{\mathrm{M}_0c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The factor of dt cannot become vanishingly small so that

$$\frac{m_0c}{\mathrm{M}}\not<\alpha c\frac{Ze}{r}.$$

According to Sommerfeld's theory [6]

$$\beta = \frac{v}{c} = \frac{2\pi e^2}{hc} Z,$$

$$\frac{1}{a} = \frac{4\pi^2 e^2 Z M_0}{h^2 \sqrt{1-\beta^2}},$$

where a is the radius of the innermost ring of the atomic orbit.

Thus according to equation (20)

$$\frac{m_0 c}{M} \not< \frac{Ze^2}{m_0 c} \cdot \frac{4\pi^2 e^2 Z M_0}{h^2 \sqrt{1-\beta^2}},$$

$$\left(\frac{m_0}{M_0}\right)^2 \not< \frac{\frac{4\pi^2 e^4 Z^2}{h^2 c^2}}{1-\beta^2},$$

i. e.,

$$\left(\frac{m_0}{M_0}\right)^2 \not< \frac{\beta^2}{1-\beta^2}.$$

If the value of β be examined for the innermost atomic orbits it is found that $\frac{\beta^2}{1-\beta^2}$ is always less than unity, the value $\beta = \frac{1}{\sqrt{2}}$ giving the value unity to this relation. The right-hand side is always a fraction up to this value for β. Thus $M_0 \not> m_0$. This result may be interpreted by stating that the dynamical description of the atomic orbits is impossible if they are regarded as occupied by particles more massive than the electron. If electrons are regarded as occupying these orbits, then $\beta \not> \frac{1}{\sqrt{2}}$ and thus

$$\frac{\pi e^2}{hc} Z \not> \frac{1}{\sqrt{2}} \quad \text{or} \quad Z \not> \frac{137}{\sqrt{2}}.$$

This is a well-known result [7] and indicates that the maximum number of chemical elements is 97. As will be seen from the above presentation of the argument, equation (20) may be regarded as presenting the limit to which dynamical description of the atom in terms of orbits occupied by electrons is possible.

Another suggested Application of the Limitation.

Suppose that a charge $-e$ is considered in motion about a charge $+e$.

The above considerations suggest that there may be a limit to the approach of the charges.

Suppose that a circle of radius r is occupied by the moving charge, the centre of the circle being at the fixed charge.

Thus $2\pi r$ cannot be less than $v\,dt$., *i. e.*,

$$2\pi r \leqslant \frac{hv}{m_0c^2\frac{M_0}{M} - \frac{e^2}{r}\frac{M_0}{m_0}},$$

$$r \leqslant \frac{hv + 2\pi e^2\frac{M_0}{m_0}}{2\pi m_0c^2\frac{M_0}{M}}.$$

This is smallest when $v=0$, and then the least value of r is

$$\frac{e^2}{m_0c^2}\frac{M}{m_0}$$

in accordance with the result already obtained. When the moving particle is an electron $M=m$ and the minimum distance is the radius

$$\frac{e^2}{m_0c^2}\cdot\frac{1}{\sqrt{1-\beta^2}}.$$

This is how the length usually described as the electron radius occurs in this theory.

When the two charges are of like sign the above result is changed to

$$r \leqslant \frac{hv - 2\pi e^2\frac{M_0}{m_0}}{2\pi m_0c^2\frac{M_0}{M}}.$$

In this case it is possible for r to assume the value zero in the limit. This is a noteworthy asymmetry in the behaviour of like and unlike charges. It is a remarkable fact that while it has been found impossible to include electrons in the structure of the nucleus, it is possible to consider it as the seat of charges all of one sign, viz., protons, together, of course, with neutrons.

References.

(1) Kaluza, *Sitzungsber. d. Berl. Akad.* p. 966 (1921).
(2) O. Klein, *Zts. f. Phys.* xxxvii. p. 895 (1926) ; xlvi. p. 188 (1927).
(3) Fisher and Flint, Proc. Roy. Soc. A, vol. cxxvi. (1930).
(4) Flint and Wilson, Proc. Phys. Soc. i. p. 340 (1938).
(5) A. Proca and S. Goudsmit, *Journal de Physique*, ser. vii. vol. x. p. 209.
(6) A. Sommerfeld, ' Atomic Structure and Spectral Lines.'
(7) O. W. Richardson and H. T. Flint, Roy. Soc. Proc. A, cxvii. p. 637.

21 Karnak Road,
Heliopolis,
Cairo.

XXXI. *The Mutual Inductance of Two Circles : a Problem in Approximation.*

By E. H. Neville *.

[Received June 13, 1940.]

The mutual inductance M of two coplanar concentric circles of radii a, b is

$$4\pi ab\int_0^{\pi}\frac{\cos\theta\,d\theta}{r},$$

where $r^2=a^2+b^2-2ab\cos\theta$. If $a=b$ the integral is, of course, infinite. The problem is to determine the dominant part of the function when $b-a$ is small.

It is convenient to replace θ by 2ϕ. Then if $ab=c^2$ and $b-a=q$, we have

$$\mathrm{M}=8\pi c^2\int_0^{\frac{1}{2}\pi}\frac{\cos 2\phi\,d\phi}{r},$$

where $r^2=q^2+4c^2\sin^2\phi$, and a method of solution is obvious: $\int\cos\phi\,d\phi/r$ is elementary and we can examine the evaluated definite integral as a function of q, while $(\cos\phi-\cos 2\phi)/r$ is bounded and we can put $q=0$ before integration. In detail, with the substitution $2c\sin\phi=q\sinh t$ we have $r=q\cosh t$, $r+2c\sin\phi=qe^t$, and therefore

$$2c\int_0^{\frac{1}{2}\pi}\frac{\cos\phi\,d\phi}{r}=\left[\log\frac{r+2c\sin\phi}{q}\right]=\log\frac{\sqrt{(4c^2+q^2)}+2c}{q},$$

and when $q=0$, $r=2c\sin\phi$ and

$$2c\int_0^{\frac{1}{2}\pi}\frac{(\cos\phi-\cos 2\phi)\,d\phi}{r}=\int_0^{\frac{1}{2}\pi}\left(2\sin\phi-\frac{\sin\phi}{1+\cos\phi}\right)d\phi=2-\log 2.$$

If we are to neglect terms of order q, we can replace $\log\{\sqrt{(4c^2+q^2)}+2c\}$ by $\log 4c$, and we have, finally,

$$\mathrm{M}=4\pi c\{\log(8c/q)-2\}+\mathrm{O}(q).$$

We can replace c inside the logarithm by a or b without affecting the order of the approximation, but in Maxwell's formula (Elect. and Mag. ii. p. 342), which has a outside the bracket, the error is $\mathrm{O}\{q\log(1/q)\}$.

The method used here is essentially that given in 1927 for the general problem of the quarter-period of the Jacobian elliptic functions (' Messenger of Mathematics,' lvii. p. 91), but the simplicity of the details when we apply the idea to the particular case is gratifying.

---

* Communicated by the Author.

XXXII. OBITUARY.

Sir OLIVER JOSEPH LODGE, D.Sc., Sc.D., LL.D., F.R.S.

Born, June 12, 1851. Died Aug. 22, 1940.

The memory of Oliver Lodge bridged the gulf of years that lies between the Indian Mutiny and the War which is now shaking the foundations of our civilisation ; between a world in which the electric telegraph and railway locomotion were still novelties, and one which is reaping the doubtful benefits of the discovery of aerial locomotion ; between a world in which the atomic theory was awaiting rehabilitation at the hands of Cannizzaro, and one where the classical concept of the atom, after suffering many a sea-change, has been engulfed by the rising tide of quantum-mechanical theory. In many of these changes, Oliver Lodge's work has played a prominent part.

Lodge was born at Penkhull, near Stoke-upon-Trent, where his father held a prosperous and growing agency, which he had built up himself, for the supply of materials used in the pottery industry. Unremitting work was the note of the Lodge household ; and Lodge's mother, a remarkable woman, found time, in a country home, to run the house, to manage stackyards and poultry-yards, paddocks and hayfields, to keep the books of the business, and, as a minor matter, to take up photography (by the wet process, be it borne in mind) as a hobby. Young Lodge spent the years from eight to fourteen as a boarder in schools at Newport (Salop) and at Combs (Suffolk), where he was put through the customary classical drill, and suffered many of the customary brutalities which were features of the schools of the period.

At fourteen his schooling was cut short ; his father expected him to inherit the business which he had developed, and brought him home to take part in it. From this age, until he had reached his twenty-second year, Lodge was immersed in office work.

But not wholly ; a kindly and farseeing aunt, who had already introduced him to some knowledge of astronomy, whisked him away on periodical visits to London, where he attended chemistry classes at the College of Chemistry in Great Marlborough Street, geology lectures at King's College, and, above all, Tyndall's lectures on heat at the Museum of Geology in Jermyn Street. This last-named course, as Lodge said, was an eye-opener, and probably was a deciding factor for his future. As Maxwell did, he began to experiment at home with primitive apparatus, and with the now almost forgotten but still useful *Penny Cyclopædia* as a work of reference. He attended Science and Art classes at Burslem, and he won an exhibition, under the Science and Art Department, which brought him, with his father's unwilling assent, to the Royal College of Science at South Kensington. Huxley, Frankland and Guthrie were the presiding deities ; but their lectures did not suffice to satiate Lodge's curiosity, and he added to their courses evening lectures in mathematics, mechanics and physics at King's College. Then followed a period of study at University College, where he read mathematics under Henrici and Clifford, and where Carey Foster gave him a post as demonstrator. The matriculation examination of London University was passed in 1871, the first and second examinations for the B.Sc.

degree followed rapidly, and in 1877 Lodge took his D.Sc. examination. He was wont to say that of all examinations, he found matriculation the hardest.

About this time he was appointed to teach physics at Bedford College under Barchesterian conditions. One spiral staircase in the vestibule took the (male). lecturer down to an austere waiting, or staff, room in the basement ; another spiral staircase took him direct to the lecture-room door. The bell rang ; the students entered, guarded by an elderly chaperon, complete with knitting. The lecture over, a reverse process, by way of the spiral staircases, decanted the lecturer into the street.

During the years 1875–80, Lodge published papers on the flow of electricity in plane sheets, on models illustrating the flow of electricity through various media, on thermo-electric phenomena, on thermal conductivity, on the foundations of mechanics and on Hughes' induction balance.

In 1881 he was appointed to the chair of physics at the newly-instituted University College in Liverpool, and was fairly embarked on his life's work. He built up the department *ab initio*, and there carried out those researches which resulted, in 1898, in the award of the Royal Society's Rumford medal. The official terms of the award tell the story of Lodge's chief contributions to the advancement of science. After remarking on the simultaneous work of Hertz and of Lodge the report proceeds :

" The researches of the English and German physicists were entirely independent, and, though the merit of the actual discovery of electro-magnetic radiation belongs certainly to Professor Hertz, there seems little reason to doubt that Professor Lodge's experiments would have led him eventually to the same result . . . When the discovery of electro-magnetic radiation was announced, Professor Lodge . . . by his experimental investigations . . . added considerably to the knowledge of the subject . . . Professor Lodge's introduction of the ' coherer ' . . . as a substitute for the Hertz ' resonator ' has increased in a marked degree the facility for reproducing and extending the experiments of Professor Hertz . . .

" With the object of obtaining some information as to the properties of the ether, and of ascertaining whether any mechanical connexion can be detected between matter and ether, Professor Lodge carried out an elaborate series of experiments . . . His method of observation is to divide a beam of light into two of equal intensity, and to cause the latter to traverse in opposite directions an annular space in a steel disk. The two beams are superposed so as to produce interference bands. If any appreciable motion can be communicated to the ether in the annulus by causing the disk to revolve rapidly round the common axis of the disk and annulus, it is practically certain that some change must be produced in the velocity of light in the two beams, and this change must show itself by an observable displacement of the interference bands."

As all the world knows, the displacement was *not* observed, and the interpretation then given to this negative result was that matter had no hold that could be detected on the ether in its vicinity.

Lodge made many contributions to the physical science of his day, but his work on electro-magnetic radiation, and his heroic experiment on the relative motion of ether and matter are outstanding.

With his appointment as Principal of the University of Birmingham in 1900, Lodge's major contributions to physical science ceased. As is well known,

he devoted much of his overflowing energy to psychical research, and held pronounced views on the possibility of human survival.

In the domain of physics, Lodge was himself his severest critic. Although he does not put it quite in this way, he had, like Henry Bradshaw, a tendency to be "always doing something else." The late Lord Rayleigh remarked to him, concerning his discovery of the coherer, "Now you can go ahead; there is your life-work." But, as Lodge said, "I didn't, . . . though I went on with it more or less at intervals; but I attended to many other things as well, and the result was that Maxwell's ether waves . . . were mainly worked out and developed practically by others."

The British Association influenced greatly Lodge's life and thought, and he was a faithful attender of its meetings for some sixty-three years—his first meeting was the Leeds meeting of 1873, his last, the meeting at Blackpool in 1936. He missed very few meetings, and was probably the last survivor of that body of members who listened in 1874 to Tyndall's Belfast Address, in an atmosphere which grew "more and more sulphurous as the materialistic utterances went on in that strongly Protestant atmosphere of Northern Ireland.' At the Cardiff meeting in 1891 he was President of Section A, and in 1913 he was President of the Association.

So with the Physical Society. As a young physicist in London in 1874, working with Guthrie and with Carey Foster, he was naturally in close touch with the movement which resulted in the formation of the Society. Though he attended its meetings from the beginning, his formal membership dates from 1875, and he was a regular attender at its meetings and contributor to the pages of its Proceedings, until he took up his work at Liverpool in 1881. He occupied the Presidential chair during the years 1899–1901, and to the last the Physical Society ever claimed, and held, a large share in his affections.

Lodge's connexion with the Philosophical Magazine was long and intimate. It began as far back as the middle 'seventies, when he published in the Magazine his early papers written in collaboration with Carey Foster, and continued to the day of his death. In 1912 he accepted an invitation to join the editorial board, and for twenty-eight years his wide knowledge of science and of men has been of inestimable service to the Magazine.

Lodge's impressive personality, his gifts of expression and his great power of exposition contributed to the commanding position which he held in the world of science. He will long be remembered as a great teacher; to those whose good fortune it was to know him well, he will live in memory as the kindly friend whose wise advice and counsel were ever freely placed at their disposal. Their world, and the world at large, is richer for his presence among them.

A. F.

An obituary notice of the late Sir J. J. Thomson, O.M, M.A., Sc.D., F.R.S., for many years one of the editors of this Magazine, will appear in the next issue.

XXXIII. *Notices respecting New Books.*

Mechanics Applied to Vibrations and Balancing. By D. Laugharne Thornton. (London : Chapman and Hall, Limited. Price 36*s*.)

The subject of vibrations is one of ever increasing importance to the engineer. The tendency in recent works on the subject has been to deal with only a few of the manifold aspects of the matter, and it is therefore very pleasing to see a work covering a wide field with a conciseness and originality in presentation such as has been achieved by Mr. Thornton. The introduction to Lagrange's equations and the notion of generalized coordinates is made quite early, and one of the chief merits of the book lies in the manner in which the attention of the reader is directed to this powerful means of investigating the problem of vibrations. The author has adopted the method of treating examples as an integral part of the work and has been highly successful in giving that continuity which is so necessary to maintain a lively interest.

The following brief outline may give some idea of the scope of the work. In the first two chapters the problem of the balancing of engines and locomotives is dealt with, in the early pages by the ordinary methods of dynamics and later, and in marked contrast, after a section introducing Lagrange's equations, by the use of generalized coordinates in conjunction with Lagrange's equations. The stability of governors and the effect of valve gears are also discussed. Chapter III. deals with the general theory of vibrations including damped and forced oscillations, interference, and resonance, followed by the application to structural frames. Later, Rayleigh's theorem for the fundamental mode is given and illustrated by examples previously solved by the more elaborate method. Chapter IV. is devoted to considerations of stress propagation in elastic materials and includes an instructive section on earthquakes.

Chapter V. begins with the problem of vibration of slender beams with the usual constraints, leading on to the question of struts and ties, continuous beams, and thence to framed structures with rigid joints. The small vibrations of a tightly stretched membrane are now considered as an example of motion involving two variables, solutions being given for the various modes of vibration of a rectangular and a circular membrane. Thin flat plates are treated by approximate methods, the exact solution being impracticable. A brief discussion of the causes and general character of vibration in aircraft follows, and the chapter concludes with the derivation of Hamilton's canonical equations of motion and a short account of the principle of least action.

Chapter VI. covers the problem of whirling of shafts, critical speeds in torsional oscillation and the effect of gearing, the vibration of turbine disks and blading, ending with considerations of the gyroscopic action of rotating bodies.

An excellent concluding chapter summarizes the main points in the work and indicates the general lines of approach to further problems not covered in the present work.

The student of the subject will be deeply impressed by the excellence of the treatment given by the author, but it is rather unfortunate that this should be marred, to some extent, by a lack of attention to detail. An example of this

is to be found in certain of the problems which form such a valuable part of the text. A clear statement of the problem with the relevant data is an essential to easy reading, but here it is sometimes necessary to work through the problem before all the data can be collected. Again, in the early sections, the author makes no distinction between vector and scalar quantities in his notation. A further regrettable detail is the tendency to confusion between mass and weight. The author states (p. 45) that "the 'absolute' system of units will be used in the following discussion," whereas, on p. 60 and onwards, without further reference, the quantity $1/g$ is found prefixed to every inertia force. Later (p. 205) the mass is given as the volume times the density divided by g

$$\left(\frac{A\rho\delta x}{g}\right).$$

Nevertheless, the book is a welcome and useful contribution to the literature on the subject of vibration.

Elementary Number Theory. By J. V. Uspensky and M. A. Heaslet. (McGraw-Hill Publishing Co., Ltd. Price 26*s.* net.)

The authors of this work disclose in their preface that they wished to fill the need of a class-room text-book in elementary number theory; certainly in the opening chapters, and in general throughout the book, this is noticeably the case. To a reviewer who received his mathematical training in England it comes as a surprise, if not as a shock, that a class-room text-book in Number Theory should be in demand; our tendency has always been to leave students of suitable ability and intelligence to stumble independently upon the subject, and a consequence of this has been to leave the subject largely untouched by the uninterested and by the incompetent. One has an uncomfortable feeling in reading this book that the authors in their University work have often to struggle with students of very ordinary ability without the proper flair for the subject. They have undoubtedly been at great pains in their writing to be as comfortable and attractive as possible, and the result is a very workmanlike account of the subject matter treated. The presentation will suit the average audience of unspecialized mathematical undergraduates, and is not of the kind reserved for first-class minds only. If its pace and methods are often pedestrian, they are deliberately so, and the tone of class hack work is avoided.

The book opens with four elementary chapters and proceeds to discuss congruence and residue theory. There are admirable chapters upon Bernoulli's Numbers, upon Diophantine Problems, and upon Liouville's Methods.

It is pleasant to note the steady flow of historical reference and the very balanced way in which it is used to enhance interest and to bind the subject together upon occasion. This book will be very welcome amongst students who are beginning a course upon the subject.

The printing and setting out are to be heartily commended.

Mathematical Recreations and Essays. By W. W. Rouse Ball; revised by H. S. M. Coxeter. [Pp. xvi+418.] (London: Macmillan & Co., 1939. Price 10*s.* 6*d.*)

The appearance of yet another edition, the eleventh, of this unique book bears witness to the width of its appeal. It is perhaps to the mathematical neophyte, not necessarily young, who has successfully negotiated the first hazards

on the course of technical mastery, that it will provide the most continuous excitement in opening up new and fascinating prospects ; but the sedate and established pure mathematician may often be provoked and stimulated by the perversity of particular problems which withstand attack from a well-prepared ground of general theorems. Nor must those who religiously attack and solve—or fail to solve—" to-day's problem," be forgotten ; they too will find much to enjoy, for the book contains no advanced mathematics (explicitly, at least), and will attack the next problem with renewed zest and critical appraisal, feeling that, after all, the great mathematicians are not a race apart.

It is unnecessary to recapitulate the contents of a book so well known—it must suffice to indicate some changes and additions. Coxeter has added a new chapter on polyhedra, written with the easy mastery which would be expected from an authority on the subject. There are extensive additions to the chapters on arithmetical and geometrical " recreations," and to that on magic squares. The final chapter, on cryptographs and ciphers, has been rewritten by Sinkov, of the United States War Department.

To some readers " recreations " may seem a quaint term for some of the problems proposed—they may well feel occasionally that " play is work." And it is work for which they could claim no merit. For did not the author, writing of the problems in an early preface, " hasten to add that the conclusions are of no practical use " ? It is to be hoped, however, that many things of " no practical use " will survive in the world, and that the enjoyment given in the past by this book will spread, through the new edition, to a still wider circle of readers. E. C. S.

Modern Elementary Theory of Numbers. By L. E. Dickson. (Publishers : University of Chicago Press (U.S.A.), Cambridge University Press (Great Britain and Eire)). Price 18*s.* net.)

This book confines its attention to those parts of number theory which do not call for limit theory, that is to say, it deals with the type of number theory generally called " classical," although much of the subject matter is modern. The author has been at considerable pains to make his book the basis of a systematic course of study, doubtless knowing full well how easily this subject can fall into a sequence of fragments. He provides examples, worked and unworked, for his student, who is, however, clearly expected to be possessed of interest and understanding suitable to the subject in hand. Although the book is self-contained and deliberately caters for a beginner, the tone of the volume never loses the highmindedness which characterizes the subject matter. Professor Dickson is careful to be simple, as a great worker in any subject is, and in assuming a proper respect and regard for the work assumes also that the foolish will leave it alone. He writes for those of appropriate but not necessarily of inordinate ability, and wishes to make the introduction to his pearl of mathematical subjects as comfortable and comprehensive as possible.

It is impossible to give in this short space a catalogue of the contents of the book. Chapters I.–IV. and IX. constitute an elementary course upon congruence theory, quadratic residues, binary quadratic forms and diophantine equations. The more advanced chapters of the work deal with advanced topics of representation, and include chapters on Waring's Problem and Positive Quadratic Forms in n Variables. Two appendices deal with the infinitude of primes in an arithmetical progression, the one analytical result used in the book.

The book will meet with rapid appreciation both from beginners and from mathematicians with years already behind them, for it is a work which reveals the flash of colour which lights up this mathematical subject.

Advanced Calculus. By Prof. I. S. Sokolnikoff. [Pp. x+446.] (London : McGraw-Hill Publishing Co., Ltd., 1939. Price 26*s.*)

This treatise on advanced calculus is one to be welcomed. True, " advanced " is a relative term, and may here be assessed by the author's statement that he has continually kept in view the needs and difficulties of the student who, after one year of preparation in the calculus, " is encountering for the first time a serious body of mathematical doctrine."

It is a thoroughly practical, and, at the same time, scholarly book. It insists on a study of the logic of the fundamentals sufficient to ensure that the student will have nothing to unlearn in the course of his future studies, and is not so prolix in its development of small points in theory as to confuse or discourage the reader who desires to come to grips with the practical details of his subject.

The book covers the usual topics of an advanced course, and the chapters on Fourier Series and on Implicit Functions are specially worthy of note.

The book has a large number of exercises of various degrees of difficulty. It may be strongly recommended.

Development of the Minkowski Geometry of Numbers. By Dr. Harris Hancock. [Pp. xxiii+839.] (London : Macmillan & Co., Ltd., 1939. Price £3 net.)

The genius of Minkowski manifested itself in many fields of mathematical thought, but the greatness of his contribution to this recondite and difficult subject is not well known outside the few whose interests lie in this direction ; for here, perhaps more than anywhere else, Minkowski's powers of exposition hardly matched his genius. None the less, when Hermite read the 'Geometrie der Zahlen,' it drew from him the enthusiastic remark, " je crois voir la terre promise."

Dr. Hancock has done much to clarify and to extend Minkowski's work. It would demand more space than could be spared were one to attempt even the most condensed analysis of his twenty chapters, which cover a range dealing with a criterion for algebraic numbers, the theory of continued fractions, periodic approximation of algebraic numbers, the approximation of a real quantity through rational numbers, the densest placement of congruent homologous bodies, and a new theory of quadratic forms. The work also includes a discussion of some short papers by Minkowski.

Dr. Hancock's massive study will attract the interest of all workers in this field of research.

The volume—a difficult one to print—is most admirably produced.

The Psychology of Physics. By Blamey Stevens. [Pp. xvi+282.] (Manchester, England : Sherratt & Hughes, 1939 ; New York City : G. E. Stechert & Co., Agent. Price 7*s.* 6*d.* net.)

In a primary sense every enquiry has its psychology, for enquirers think and thinking is psychological, but the " psychology " in the above title seems to have special implications.

"Undoubtedly there are things outside ourselves that really exist"—the author explicitly protects physics from solipsism. Space and time, however, are "states of consciousness" which have "nothing to do" with these "real objects." Sense impressions, those "messages of the objective," are "classified" by "the subjective theory of" that "objective." Such items, the emphasis on "psychological postulates," the commending of "pure deduction" from these postulates as the "ideal method" (though observation and experiment may be useful checks or aids), and the general tenour of the book suggest that *The Psychology of Physics* is symptomatic.

According to one trend of thought the physicist reveals his own ways of thinking at least as much as, and possibly much more than, he displays the ways of nature. An exposition in which the "laws of physics are a part of man's psychological nature" seems to be caught in this trend, and appears to be symptomatic of much modern partiality for a *subjective* physics.

JOSHUA C. GREGORY.

Reports on Progress in Physics.—Vol. VI. General Editor: J. H. AWBERY. [Pp. v+434.] (Physical Society, London, 1940. Price to non-Fellows, 22*s*. 6*d*. net.)

THE utility of the Physical Society's 'Progress Reports,' and the extent to which this utility is already recognized, are sufficiently demonstrated by the well-thumbed condition of library copies of the first five volumes. The new volume, number six in the series, has been edited by Mr. J. H. Awbery; it follows closely the pattern established by his predecessor in office, Professor Allan Ferguson, holding very nicely the balance between the most modern developments of atomic and nuclear physics and what Mr. Awbery calls the "traditional bread-and-butter subjects of Physics." The teaching of physics again receives attention—this time in an article by Dr. David Owen, who deals principally with advanced teaching of part-time students in Technical Institutions.

There are in all twenty-two sections, and the sections on "Spectroscopy" and "X-rays and crystals" are subdivided into three and four sub-sections respectively. It would require too much space (and necessitate the application of a too arbitrary selection rule) to attempt even merely to list the more important sections. Reference may, however, be permitted to the long article on induced radioactivity by the late Dr. H. J. Walke. The excellence of this article adds melancholy point to the editorial reference to the untimeliness of its young author's death last December.

The Physical Society's Council showed commendable vision in inaugurating this series some five years ago; it has displayed a similarly commendable courage in refusing to allow the present abnormal conditions to break the sequence. In subject matter and in production the new volume bears no mark of the difficulties of the times, and it well deserves its place alongside its five predecessors.

H. R. R.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

CONTENTS OF No. 202.

Page

XXXIV. Research on Combined Stress and Anisotropy. By James J. Guest 350

XXXV. The Matrix Theory of Four-Terminal Networks. By Louis A. Pipes, Ph.D., Graduate School of Engineering, Harvard University, Cambridge, Mass. 370

XXXVI. The Lateral Diffusion of a Stream of Ions in a Gas. By L. G. H. Huxley, M.A., D.Phil., F.Inst.P., F. W. Bennett Lecturer in Physics, University College, Leicester 396

XXXVII. A Simple Application of the Laplace Transformation. By H. S. Carslaw 414

XXXVIII. A Contribution to the Theory of Elasticity of Non-Isotropic Materials (with Applications to Problems of Bending and Torsion). By Eric Reissner, Ph.D., Massachusetts Institute of Technology 418

XXXIX. Note on the Barrier-layer Resistance of Selenium Photo-elements. By Arne Eld Sandström 428

XL. Obituary :—Sir J. J. Thomson, O.M., M.A., D.Sc., Sc.D., LL.D., F.R.S. 430

XLI. Notices respecting New Books :—

Mr. Bruce Chalmers's The Physical Examination of Metals.—Vol. I. Optical Methods 432

Messrs. I. H. B. and A. G. H. Spiers's The Physical Treatises of Pascal 433

Stephen Timoshenko : 60th Anniversary Volume 434

Otto Klemperer's Electron Optics (Cambridge Physical Tracts). 434

Drs. G. D. Parkes and J. W. Mellor's Mellor's Modern Inorganic Chemistry 435

Mr. N. Rashevsky's Advances and Applications of Mathematical Biology 435

Mr. J. W. L. Glaisher's British Association Mathematical Tables.—Vol. VIII. Number-Divisor Tables 435

Prof. Felix Klein's Elementary Mathematics from an Advanced Standpoint 436

The Raman Jubilee Volume 436

*** It is requested that all Communications for this Work may be addressed, post-paid, to the care of Messrs. Taylor and Francis, Ltd., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

XXXIV. *Research on Combined Stress and Anisotropy.*

By James J. Guest *.

[Received March 27, 1940.]

Abstract.

This paper discusses the effect of departure from isotropism in the specimen on the yield point under various stress combinations, and analyses the experimental results and deductions drawn therefrom given in papers by Prof. Mason, Dr. Becker, and Prof. G. Cook. For ascertaining the effects of the simple forms of anisotropy which manufacture tends to leave in the product, it is assumed that the primary factor in elastic failure is the maximum shearing stress.

The application of these deductions to the researches referred to show that, for close correlation, an effect due to volumetric stress must be included, and that the various unexplained difficulties encountered in these researches are then completely clarified and all the results are shown to be in close agreement with the Guest criterion of elastic failure.

The completeness of the explanations of difficulties so diverse is good evidence that it and the assumption are correct, and the effects are shown to be such as to merit careful examination in future research.

If a proposed law of Nature be actually true the results of all carefully conducted researches should give results in accordance with it, but in the investigation of elastic failure under combined stress there are many experimental results inconsistent with any one of the numerous criteria of failure which have been advanced. I here propose to discuss certain effects of anisotropy, considering three researches which have been carefully carried out and which have been widely discussed, but with the conclusions which have been drawn therefrom I do not wholly agree.

In "Mild Steel Tubes in Compression and under Combined Stress" [1], Dr. W. Mason, Professor in the University of Liverpool, describes a research schemed to be of a discriminative nature. The position then (1909) was that all engineering design in this country was carried out in accordance with the Rankine maximum stress law, while abroad it was governed by the St. Venant Maximum Extension theory, but that a research "On the Strength of Ductile Materials under Combined Stress" [2], which

* Communicated by the Author.

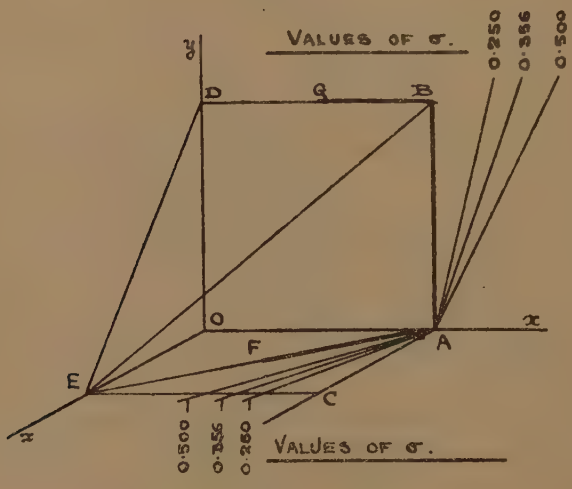
I had conducted, had proved that both these theories were widely in error. The results obtained had enabled me to formulate a generally applicable law of failure according to which shear stress was a primary factor in yield, but that volumetric stress of a certain type had also to be taken into account, the equation of the criterion being

$$p_1 - p_2 + \lambda(p_1 + p_2) = \text{Const.} = 2q_0, \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where p_1 and p_2 are the greatest and least principal stresses, q_0 the yield point in pure torsion, and λ a non-dimensional constant of the material, varying from zero to unity as the ductility decreases.

In order to present the results clearly I regarded the three principal stresses at yield in each experiment as the coordinates of a point with

Fig. 1.

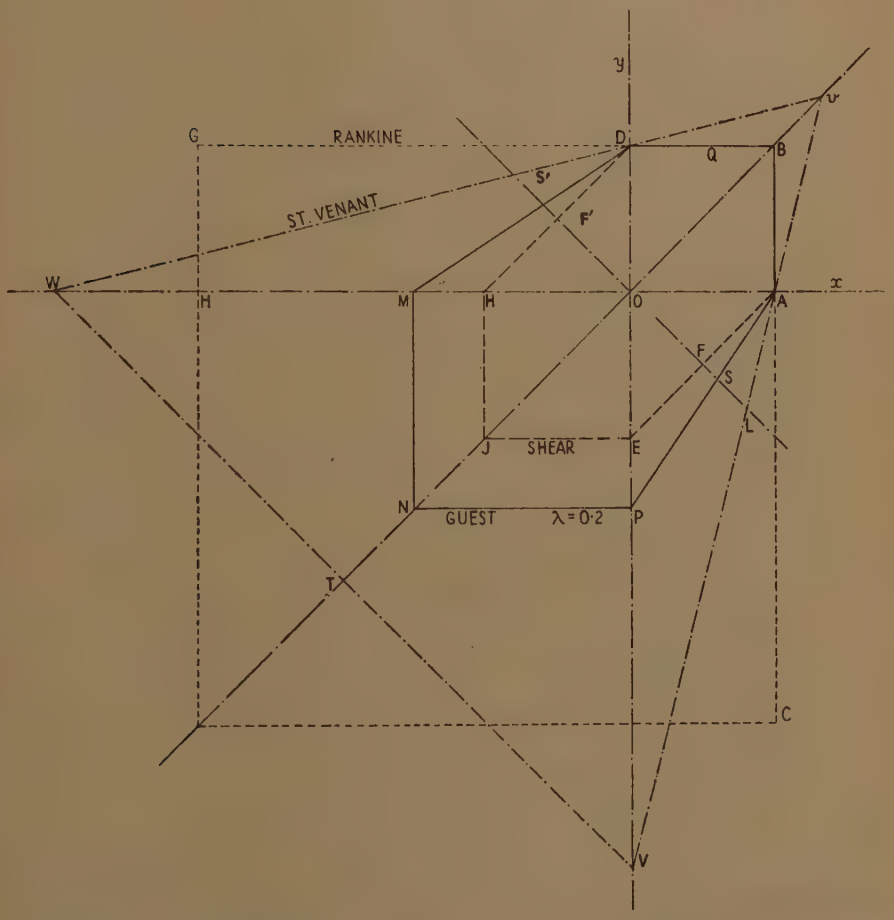


regard to three orthogonal axes, Ox, Oy, Oz, so that their complete locus would be a continuous surface —the yield surface. In fig. 1 (fig. 20 in [2], p. 93) the planes EFAB, EBQD are parts of two of the planes of the yield surface according to the shear stress criterion, and the lines marked with various values of σ—Poisson's ratio—are traces of the corresponding planes of the yield surface of St. Venant's theory.

When, as is most usual, one principal stress is zero, the relationship between the other two is similarly represented by a two-dimensional diagram, termed in that paper (fig. 19 in [2], p. 122) a synoptical diagram and being the section of the yield surface. The yield surface, and its sections, differ for the various criteria of failure which have been advanced. Sections according to several criteria are shown in fig. 2. If a material failed according to Rankine's law the observation points would be along

the sides of the square ABDGHCA, but if according to the St. Venant theory, accepting his theoretical value of Poisson's ratio ($\frac{1}{4}$), along the sides of the triangle UVTW. I have shown the Rankine square eccentric to O as the law is used in practice with different working stresses in tension and compression. All such diagrams must be symmetrical with

Fig. 2.

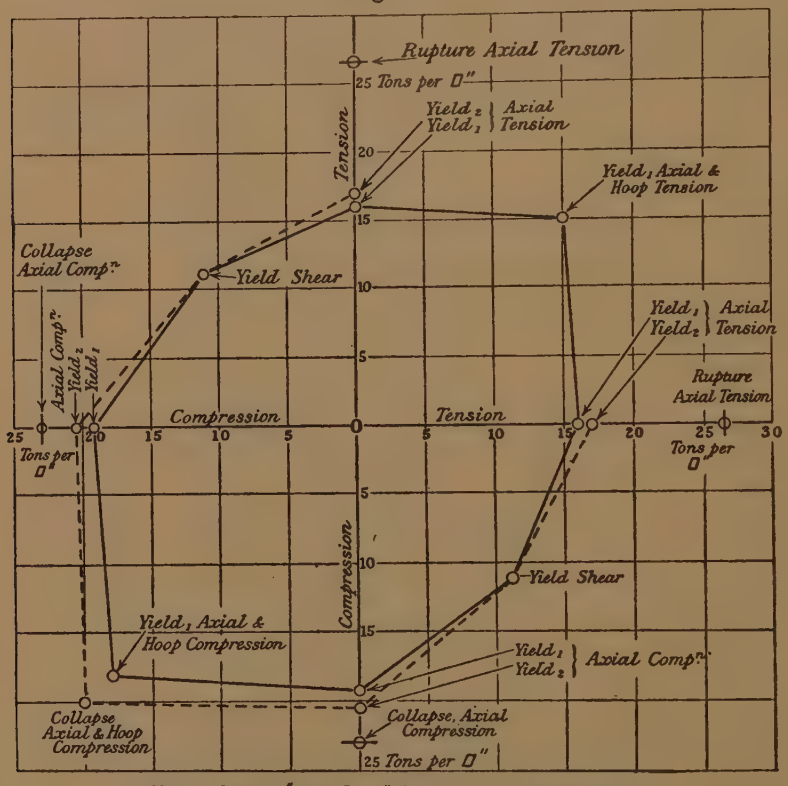


regard to the line UBOJNT. The simple shear stress law—which I suggested as accurate enough for use in design provided it were strictly confined to ductile metals and not used for cast iron as advocated by Coulomb—would give the diagram ABQDHJEFA. The alteration from it to give the criterion of equation A is obtained by swinging the line AFE round A through an angle dependent on λ, giving the figure

ABQDMNPSA. The portion QBAS of this was covered by the experimental determinations.

Professor Mason's research consisted in subjecting tubular specimens, 3″ bore × 0·08″ thick, to axial loads and fluid pressure (external and internal) and ascertaining the yield points under five combinations—simple axial tension, A; equal axial and circumferential tensions, B;

Fig. 3.



Yield 1 denotes "Yield Point" determined by Extensometer
Yield 2 " " " observed not using "

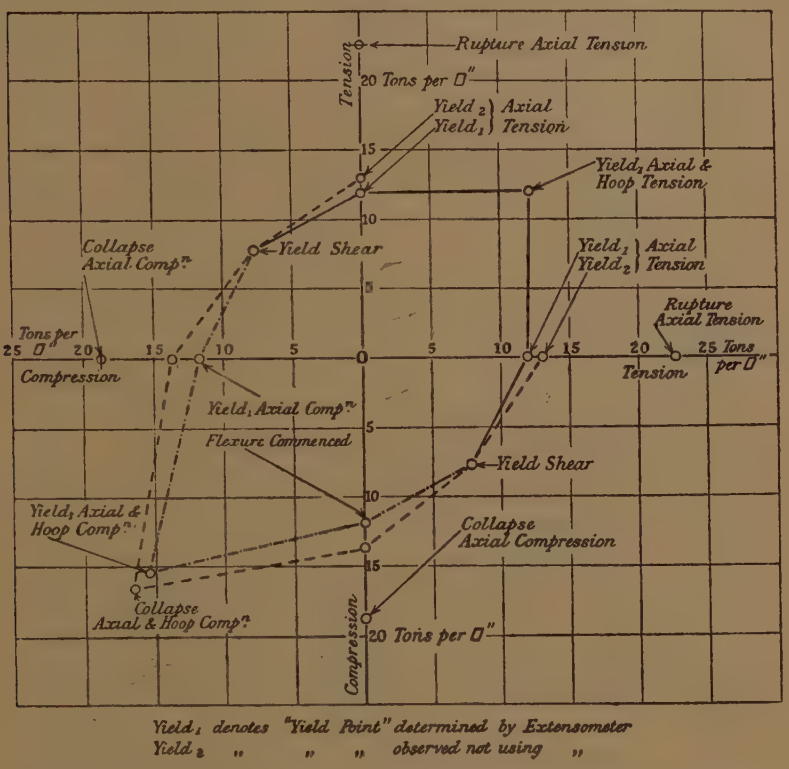
simple shear (produced by internal fluid pressure and axial compression), S; simple axial compression, M; and equal axial and circumferential compression, N. The experimental determination of the relation of the points M and N to the others would afford discriminating evidence.

In figs. 3 and 4 are reproduced two of Mason's diagrams (figs. 15 and 17 in [1]), the first for tubing as received from the makers and the second for tests taken after a special annealing. As the latter specimens buckled under simple compression before yielding, the yield point is doubtful,

but by comparison with Mason's other tests may be taken as about the same as the yield stress in double compression.

It is evident that these results agree with the Guest criterion to a first approximation, but the "yield-shear" point is high in both diagrams, and in the first the "yield, axial and hoop tension" point is low. That it becomes practically the same as the axial tension yield stress after

Fig. 4.



Yield 1 denotes "Yield Point" determined by Extensometer
Yield 2 " " " observed not using "

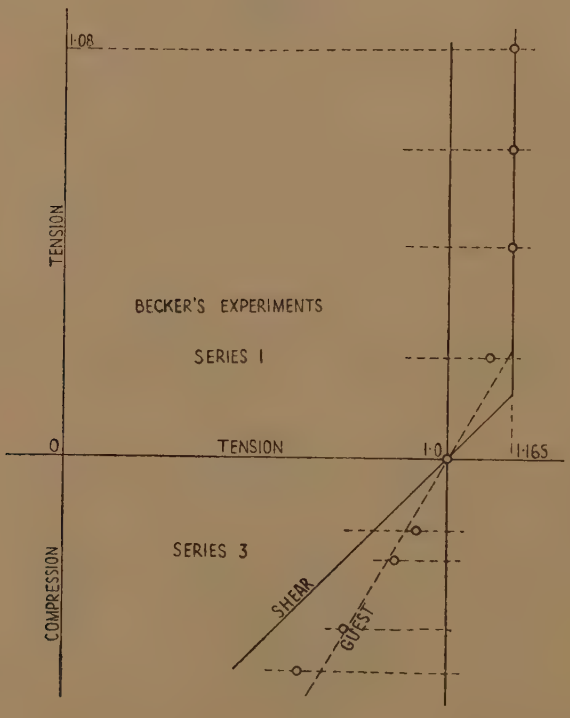
annealing, and so agreeing with the author's results, suggests that the divergence in fig. 3 may be due to anisotropy.

Prof. Mason expresses his conclusions thus :—"It appears, then, that mild steel in direct compression yields by shearing ; and, to a first approximation, that the value of this shear-stress is independent of any normal compressive stress on the planes of the slide"—that is, that the volumetric stress $\frac{1}{2}(p_1+p_2)$ of equation A is of no appreciable effect. With this latter often cited conclusion I disagree, considering the effect definitely in evidence, and to this point a return will be made later.

In "The Strength and Stiffness of Steel under Biaxial Loading" (3)

a research of a similar type on a large scale, using specimens $5\frac{1}{2}''$ diam. by 0·085″ thick, carefully machined all over from steel tubing, was carried out by Dr. A. J. Becker (Univ. of Illinois). Four varieties of steel were experimented upon, the yield points in tension being 21,500 ; 24,000 ; 42,500 ; and 50,000, lb./sq. in. respectively. The results generally were similar, but to obtain a complete diagram two sets of tests have to be combined ; those of series 1 and series 3 are combined to form fig. 5, the tension yield being taken as unity for each. Becker followed Mason in

Fig. 5.

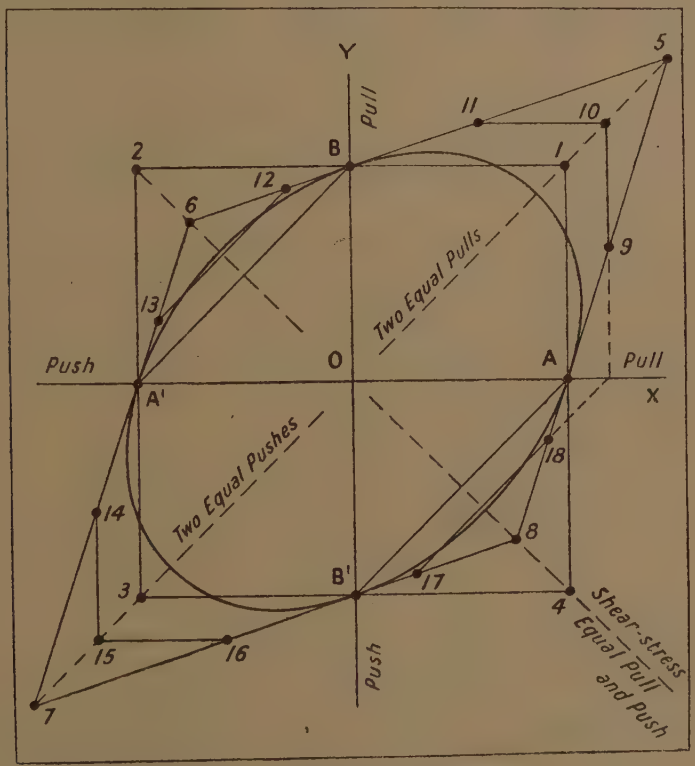


his mode of producing shear stress, all principal stresses in these experiments being parallel to and perpendicular to the axes of the tubes.

As a preliminary, Dr. Becker discusses various criteria of failure, expressing the unfavourable, but interesting, opinions that according to the St. Venant theory "two compressions will produce an increase of strength," and that in St. Venant's work ([3], p. 91), "The effect of shear in producing strain has been neglected." Comparing the results of his research with the predications of the various criteria he finds the Hancock relation the most satisfactory, but that his results cannot be accounted for by any of them and therefore all are untrue. He proceeds

to the formulation of a new law of elastic failure which he expresses as follows :—" No. 5. With increasing value of the ratio of biaxial stresses the yield point strength follows the maximum strain theory until the value of the shearing stress reaches the shearing yield point, then the shear stress controls according to the maximum shear theory. There are two independent laws each dominant within proper limits instead of some single law as has hitherto been assumed."

Fig. 6.



Strain-energy hypothesis is represented by ellipse

$$\left(\text{Equation } X^2+Y^2-2\sigma.X.Y=\overline{OA.}^2 \quad \text{Poisson's ratio } \sigma=\frac{1}{3}\right).$$

Rankine hypothesis for max. principal-stress, figure A1B2A′3B′4.
St. Venant hypothesis for max. principal-strain. figure A5B6A′7B′8.
Guest's hypothesis for max. shear-stress, figure A1BA′3B′.
Becker hypothesis for max. shear-stress and strain, figure A9, 10, 11,, B12, 13, A′.

In the ' Dictionary of Physics ' (4) Prof. R. G. Batson expresses his approval of this criterion, writing that it " appears to fit in very well

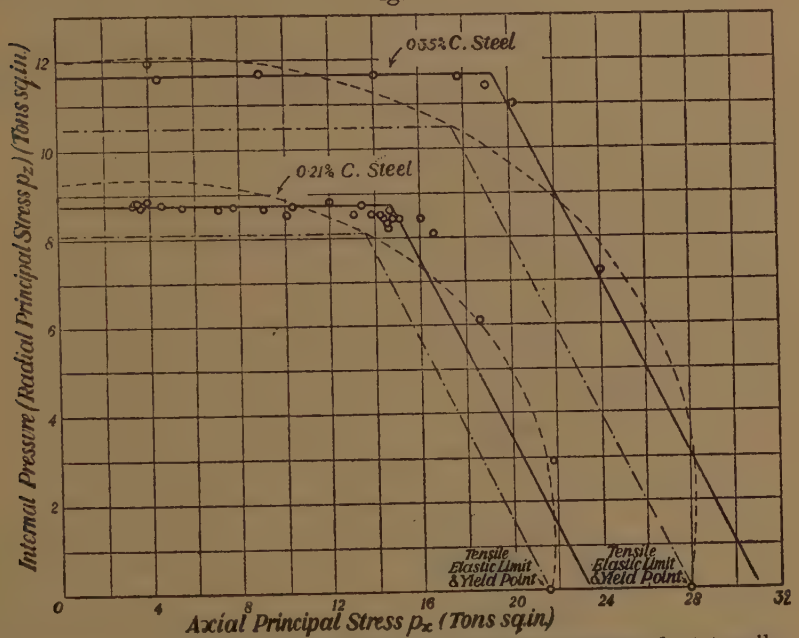
with existing experimental work "—which is the desirable test. In the United States it is used for practical design [5]. As this proposal regards stress and strain as fundamentally different concepts, while I regard them as alternative—the geometrical and the mechanical—aspects of the elastic state considered, I would, for a drawing of the three-dimensional yield surface of this theory, refer readers to that made [6] by Prof. Westergaard (Univ. of Ill.), who unreservedly accepts the Becker criterion as being the truth, and in fig. 6 is reproduced a synoptical diagram [7] by Dr. B. P. Haigh (Prof. Royal Naval Coll., Greenwich) which includes that for the Becker criterion. Concerning this Haigh observes that " where two nearly equal like stresses are combined, as in turbine discs, Dr. Becker's hypothesis appears to over-estimate the elastic limit, although not so greatly as does Saint Venant's." Prof. Haigh agrees with Dr. Becker that according to the St. Venant theory two compressions produce an increase in strength. I have drawn the triangle UVW in fig. 2 as expressing my view of this theory—the yield in compression being four times that in tension and a second compression reducing this amount. Prof. Haigh shows the Rankine square as centred at O agreeing with failure under a specific principal stress. In his diagram he includes the ellipse corresponding to the Beltrami-Haigh total energy theory of failure, which alone of the criteria of these diagrams gives a curved line. The ellipse of the Mises-Hencky hypothesis, which passes through the points AIBA′3B′, also gives curved lines.

In " The Elastic Limit of Metals exposed to Triaxial Stress " [8] Prof. G. Cook describes a research in which he submitted hollow specimens to combinations of internal pressure and axial loads, the specimens being so thick that the radial compressive stress was considerable. Prof. Cook maintains that it has been definitely established " that stress differences alone will be found in the expression of any theory which can be applicable," and hence the only criteria which can be true are " the Mises-Hencky and the Guest shear stress hypotheses "—the view being taken that ductile and brittle metals are essentially different in their mode of failure and that the Coulomb shear stress law, which was stated as applying to cast iron and hardstone, could not be considered. Cook experimented upon two steels, 0·21 and 0·35 per cent. carbon, and obtained results which he gives in tables and on a diagram, reproduced here (fig. 7), having as axes the axial principal stress and the radial principal stress. He notes the curious distribution of the points in the diagram, and observes that the shear stresses at the pure tension yields are low by about 10 per cent., so that it is evident that the point distribution is not due to mistakes. He concludes thus :—" The evidence therefore favours the view that, as regards the initial breakdown in mild steel, the maximum shear stress is the initial factor, that the Guest hypothesis

is true to the extent that, for the same kind of stress distribution the values of the principal stresses at the elastic limit are such as to give a constant value to the maximum shearing stress."

Despite the vague limitation " for the same kind of stress distribution," this conclusion has generally been taken as definitely that the shear stress law is true and accordingly has received much criticism. Thus in ' Theories of Strength ' [9], Prof. A. Nadai (Göttingen)—who states that the Mises-Hencky hypothesis is true within 3 per cent.—writes that " the whole comparison is unfortunately obscured by introducing the elastic limit . . the first yielding may have escaped observation . . . the upper and lower

Fig. 7.



Constant shear strain energy } equal to value observed at tensile
Constant shear stress } yield point.
Coustant shear stress, equal to mean of observed values.

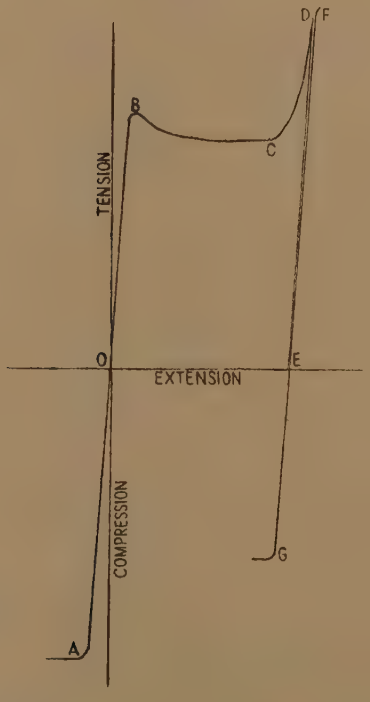
tension limits may have affected the matter, etc.," and finally argues that Cook's results, so far from supporting the shear stress criterion, actually are evidence in favour of the Mises-Hencky hypothesis.

These three researches have been carried out carefully, yet there are difficulties in the explanations given by the various authors and others of the results.

As metals are manufactured products it is not to be expected that they are perfectly homogeneous or isotropic, and the mode of production of a

steel bar tends to result in some anisotropy, which is produced by any overstrain or permanent deformation. In a tension test on a mild steel, as expressed in fig. 8, if the specimen be first loaded in compression to a yield indicated at A and there in tension, in which after a yield point at B a plastic extension BC occurs with little variation of load, a further increase of load becomes necessary at C to produce further extension. If after reaching a state D the load be removed, leaving the specimen with permanent extension OE, a re-test in tension will give the line EF, closely

Fig. 8.

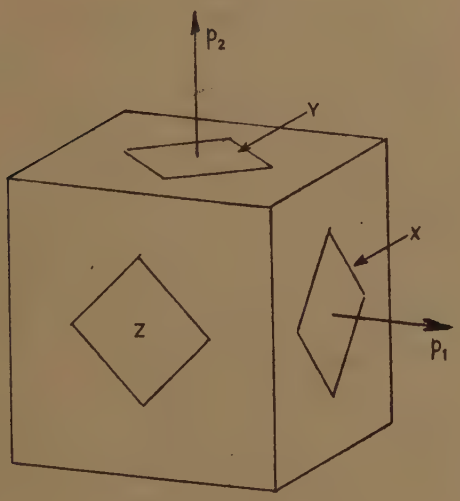


following DE and parallel to OB, with no appreciable time effect until F is reached, but if a re-test in compression be made yield will occur at G—a lower stress than at A. The overstrain along BCD has rendered the material anisotropic. Mechanical working in general thus tends to lower the compression yield relatively to the tension yield in tests taken along the direction of extension produced thereby. Annealing tends to rectify matters, but the extent to which it is possible depends upon the anisotropy produced, and compression test results generally are to be regarded as probably low in comparison with tension results.

The effects of anisotropy on test results has often been considered and

some tests made to determine its effect—such as those taken on steel plate along and across the direction of rolling, with results agreeing with the above conclusions. In my research (§§ 7 & 23 in [2]) the densities, Young's modulus, and the modulus of rigidity were determined for each specimen, as low values would suggest anisotropy, and methods of test to check isotropy were included in the experiments. Any such effect as an axial extension when a torque was added was looked for. The matter is also discussed in "The Plastic Distortion of Metals" [10] by Prof. G. I. Taylor (Cambridge Univ.) and H. Quinney, who, considering a research by E. Lode (Göttingen), decided "that either there was a considerable possible error in the measurements or else the tubes were not

Fig. 9.



isotropic," and first concentrated their attention on testing tubes for want of isotropy. Their test consisted in measuring the changes of internal volume of the tubes when subjected first to tension and afterwards to combined tension and torsion, when they found that there was an angular break in the curve, showing the existence of anisotropy.

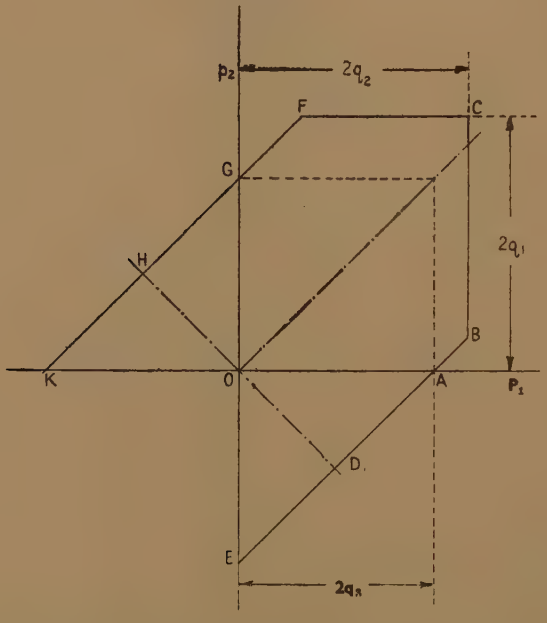
As, however, the tension first applied produced strains between 1/20 and 1/30 in amount, it would have produced the anisotropy described above in a tube previously isotropic. Thus it was not anisotropy of the tubes in their original state which Taylor and Quinney found to exist. The experiment, however, demonstrated that anisotropy existed after the post-yield extension.

Although the existence of anisotropy can be detected in various ways, in order to ascertain the effect of any anisotropy upon elastic failure

under a test, knowledge of the law of failure is necessary. In the following discussion I make several assumptions, and in particular that for the steels considered the elastic failure depends upon the shear stress to an accuracy sufficient for the purpose, and that, having regard to the methods of the production of steel bars and of the taking of test specimens, take the anisotropy to be symmetrical with regard to the principal stresses. This meets the three cases to be considered.

Thus in fig. 9, if p_1 and p_2 be two principal stresses, taking the third initially as zero, the elements with faces on which there are maximum shears are indicated by the squares X, Y, and Z drawn on the faces of the

Fig. 10.



primary element shown. Suppose that the material, owing to its anisotropy, be capable of resisting failure by shear to different extents on these elements, the yields being q_1, q_2, and q_3 on X, Y, and Z respectively.

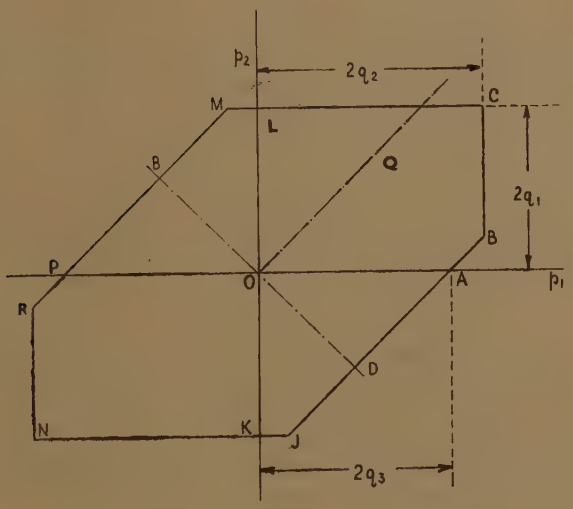
Firstly, suppose $q_1 > q_2 > q_3$. A single tension p, will cause yield on element Z, so that the yield stress will be $p_1 = 2q_3$, giving the point A on the synoptical diagram (fig. 10).

Now if a simultaneous tension p_2 be applied it will necessitate an increase in the value of p_1 in order to produce the shear stress q_3 on the element Z necessary for yield, and the new value of p_1 at yield will be $2q_3 + p_1$. As p_2 increases the action will continue giving the line AB at 45° to O p_1, until p_1 reaches the value $2q_2$ necessary to cause failure on

element Y, and for further increases of p_2 the material will continue to fail in this manner, giving the line BC on the diagram—until p_2 becomes so large that it produces failure by shear on X when $p_2=2q_1$ giving the point C. If, however, p_2 be negative, the yield will take place on the weakest element Z, so that $p_1-p_2=2q_3$, giving the line ADE on the diagram, D being the yield point in pure shear.

If p_2 be regarded as the main stress the action will be of a similar nature, the diagram being completed by the lines CFGHK, the point C being $2q_1$, $2q_2$. Any such diagram will clearly be symmetrical with regard to the point O, the shear stress being the controlling factor, but anisotropy renders it no longer symmetrical about OQ, the bisector of p_1Op_2.

Fig. 11.



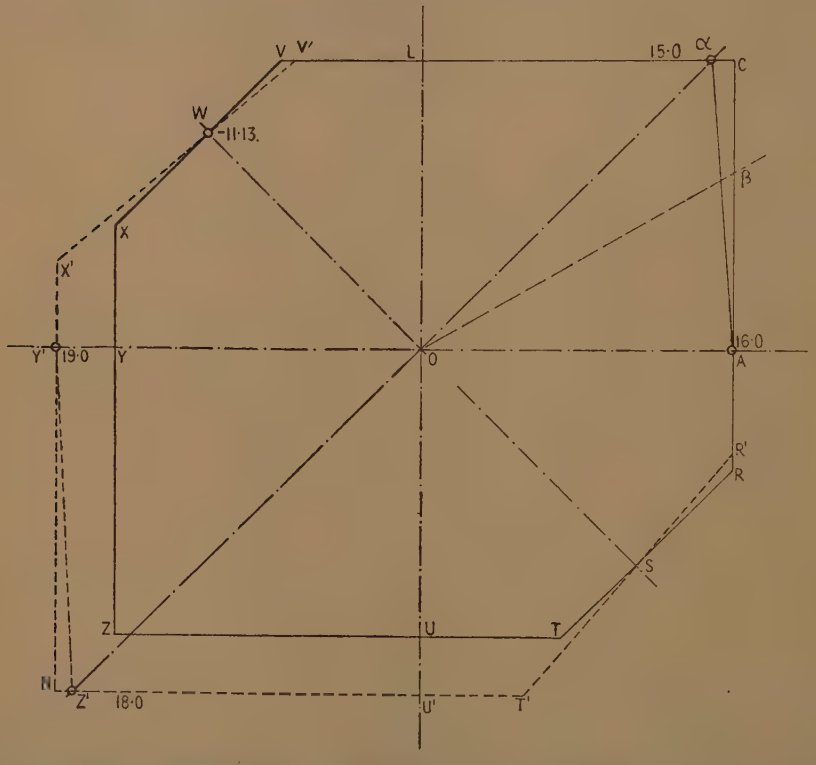
Thus a steel anistropic in this manner would give equal yield stresses under tension and compression—equal tension yields in two directions at right angles and a yield shear stress half the tension yield stress—to reveal the existence of anisotropy tests in which p_1 and p_2 were of the same sign, but of different amounts, would be necessary.

Now suppose that $q_2>q_3>q_1$. So far as A to B and B to C is concerned failure will be similar to the preceding, but when p_2 is negative the action, while the same until p_2 reaches the value of $2q_1$, will then change, as shown in fig. 11, and failure will afterwards take place on the element X, the corresponding diagram point being J—$2(q_2-q_3)$, $2q_1$—and the continuation along JK. If p_1 is zero, failure will occur on X, the point being L, corresponding to K, and for positive values of p_1 no change in the limiting

value of p_2 takes place, the line being LC, which continues negatively to M, symmetrical with J, and the diagram is completed along MPRN.

In this case the tension limits along p_1 and p_2 are different, and while the stress line ADJ is at 45°, this does not continue until the compression point is reached. The yield stress in pure shear is half the greater tension limit, but as it is greater than half the lesser tension limit OL, whether the material would appear to obey the shear stress law will depend upon the direction in which the tension test is taken.

Fig. 12.



If the shear stresses be in the order $q_3 > q_2 > q_1$, where p_1 is the main tension, the yield value is $2q_2$, giving the point A, and the superposition of a positive value of p_2 does not affect this, the line being AC as shown in fig. 12. Nor does a negative value until it is sufficient to cause failure on the element Z at the shear stress q_3, the point being R. Further negative increase of p_2 results in the line RST, failure along which results from the combined action ou element Z, until at T the stress p_2 alone produces failure on the element X. This continues along the line TUZ, and the diagram is completed in a similar manner along CLVWXYZUT.

Generally the tension yields in the directions p_1 and p_2 are unequal, but if $q_1=q_2$ they become equal, In either case the yield point in pure shear is high compared with the tension yield stresses.

It is to be observed that these three diagrams are interchangeable as regards p_1 and p_2.

If a test in combined stress is made, the loading will be applied in some particular sequence, which can be indicated by a line on the diagram. Thus, if the ratio p_1/p_2 be maintained constant the line is straight, as is shown by Oα (fig. 12) if the ratio is unity, or by Oβ if the ratio be 2, as would be the case for a thin tube under internal fluid pressure : if one type of load be applied first the line has a change point in it. Elastic failure occurs where the loading line cuts the synoptical curve, and there the results, in the case of anisotropy only, depend to a certain extent on the loading sequence.

If the material actually fails according to the Guest criterion, the effects of anisotropy will be of the same type except that the inclined lines, instead of being at 45°, will slope as indicated in fig. 2, and the yield stresses in compression will be greater than those in tension.

Returning to Mason's research, it is first to be observed that his paper and diagrams represent the limits in simple tension and in simple compression as determined in both the axial and in the circumferential directions. Actually Mason did not determine both, but assumed them to be equal. When anisotropy is considered this is not permissible. Taking p_1 as the axial stress the points actually determined by Mason correspond to the points ACWY and Z in fig. 12.

In determining the yield under axial and hoop tension, the stresses p_1 and p_2 were maintained equal while they were being gradually increased, giving the load line Oα, resulting in yield at the point α—at a stress of 15·0 tons per sq. in., while at A it was 16·0 tons per sq. in. for the tests shown in fig. 3. This diagram is drawn to scale, OA being set off as 16·0, OL as 15·0, and the shear points W and S as 11·13, 11·13. The diagram ACLVWXYZUTSRA is thus that for the simple shear stress criterion as affected by this type of anisotropy.

Now let OY′ and OU′ be set out as 19·0 and 18·0, Mason's results in single and double compression, and the diagram completed by drawing through the shear points W and S the broken lines V′WX′, R′ST′, parallel to AU′, to meet the broken lines drawn through Y′ and U′ parallel to the axes. This will be the diagram for the Guest criterion, taking account of a volumetric stress and distorted by this type of anisotropy. Mason's double compression point will occur where the 45° loading line cuts the diagram, *i. e.*, at Z′ and not at N. The results of Mason's experiments thus entirely agree with this explanation, and the points at AαWY′ and Z′ constitute a definite check agreeing within 1 per cent.

As Mason's results in single and in double tension were 16 and 15 tons per sq. in., and in single and in double compression were 19 and 18 tons per sq. in., they indicate most definitely the effect of the volumetric tension. As it has been pointed out above that the process of manufacture tends to lower the compression yield relatively to the tension yield in tests such as these, there is no evading this conclusion. The ratio of compression to tension yield is thus about 1·2 and the corresponding value of λ in equation A is 0·1, giving the measure of the effect of the volumetric stress.

The results for the tubes after the further annealing show that all yield stresses were considerably reduced thereby, and that the lesser shears became practically equal; the relative anisotropy with regard to the greatest shear was unaltered, and the effect of the volumetric stress slightly increased.

Mason's results, being completely accounted for by the existence of this type of anisotropy in his specimens and by the effect of the volumetric stress, provide strong evidence as to the truth of the Guest criterion, particularly as they show that the failure under two equal compressions was exactly as predicated by the law.

Anisotropy of this type may lead to the tension yields in two directions being unequal—as they were slightly in Mason's work—but always leads to a high value of the shear stress yield as compared with that in tension, "and produces the curious phenomenon that a small perpendicular stress of the opposite sense is without effect on the yield under a main principal stress." The diagram shows how large the effects may be.

Turning to the research of Dr. Becker, it is evident that the unusual nature of his results is due to anisotropy of the type indicated by fig 10, or by fig. 11 supposing q_1 to be nearly or quite equal to q_3—which is a simple form of anisotropy. No new law of elastic failure is required. Apart from the anisotropy, the inclination of the line in the lower quadrant of fig. 5 departs definitely from the 45° angle of the shear stress law, and the broken line corresponding to the Guest criterion is in good agreement with the observation points.

The evidence afforded by this research that the results plotted on principal stress axes lie along straight lines is very strong. There is no suggestion that any line is curved, and the research is therefore strong evidence against both the Beltrami-Haigh and the Mises-Hencky hypotheses, which give curved lines on the diagrams (see fig. 3), and the existence of anisotropy would simply distort these into other curved lines and not rectify them.

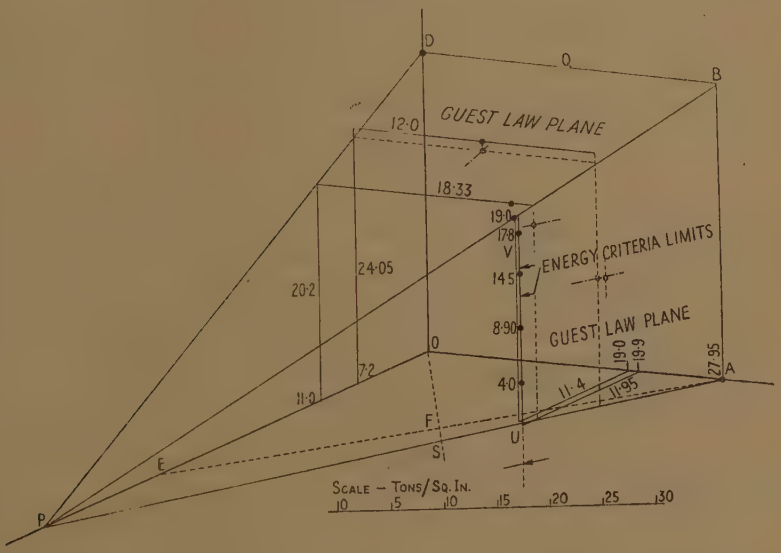
In considering Cook's results, as the three principal stresses are all of importance, a three-dimensional plotting on the lines given in fig. 1 is here employed as revealing the relations of the yield conditions more

clearly than that employed by Cook. Such a diagram for the results obtained for the 0·35 per cent. carbon steel is given in fig. 13, the drawing being to scale, but not standard isometric, as that would obscure certain features.

The planes corresponding to the Guest law for this steel are ASPB and PBQD, and it will at once be seen that every observation point, indicated by circular dots, lies on one or other of these planes, and the accuracy with which they so lie is, by calculation, within $2\frac{1}{2}$ per cent., which is within the experimental error.

The lines AB, BQD, correspond to those in fig. 1, and the plane BASP of the Guest criterion is obtained by swinging the plane BAFE round AB

Fig. 13.



through an angle dependent on the λ for the steel. The actual shear yield stress is represented by the point S, and the line OFS bisects the angle AOP. The deviations of the points from the planes are indicated by the double line UV and the double point at 12·0, 24·05, —7·2. Thereby the correlation between Cook's results and the Guest law can be visualized.

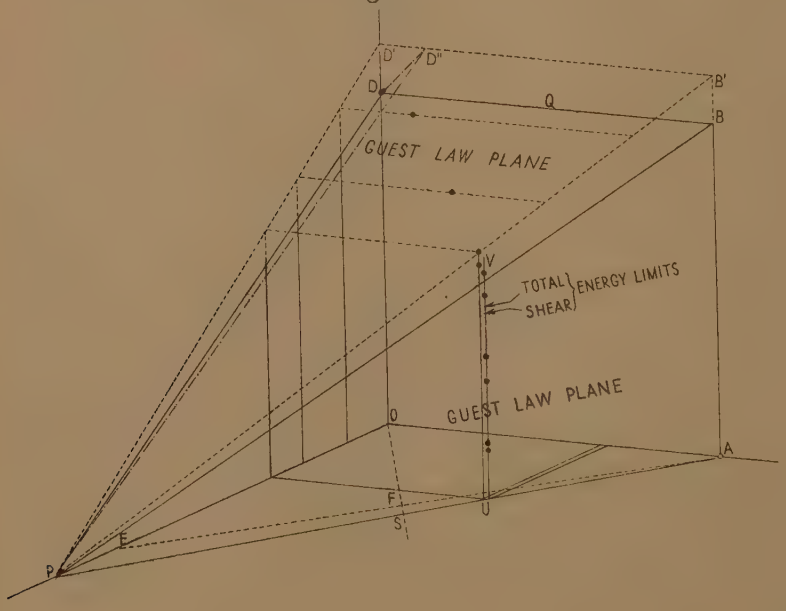
The symmetry permits points on the plane PBQD to be transferred to the plane BASP, in which they are indicated by circles. The points on this plane indicate its existence as part of the yield surface to be a physical fact. The perpendicularity of the line of observations UV to the plane AOP, shows that the plane BASP, with the line AB are per-

pendicular to AOP, which is in complete accord with the Guest criterion. and with the absence of the intermediate stress from equation A.

There is here no 10 per cent. disagreement as Prof. Cook found to exist in the correlation of his results with the shear stress theory: every observation agrees with the Guest criterion.

The results for the 0·21 per cent. carbon steel are shown in fig. 14. where it will be noticed that the points again lie nearly on one or other of two planes. The upper plane PB′D′, however, is slightly above its theoretical (symmetrical) position. This, being the effect which anisotropy would produce, indicates that the material is not quite isotropic.

Fig. 14.



All the points on the upper plane PB′D′ lie closely on it, except the point on OD, which lies at D and not at D′. If p_1, p_2, and p_3 (negative) be along OA, OD, and OP respectively, and $q_1 > q_2 > q_3$ in fig. 9, then, as in fig. 10, the tension yields OD and OA would be equal, but the existence of a tension along OA would necessitate an increase of the tension OD in order to cause elastic failure, thus raising the necessary yield stress along a line at 45°, until it reached the line B′D′ and the upper plane PB′D′. Thus while the point at D is below the plane, the other points, with higher values of the stress along OA, all lie upon it. Allowing for this effect of the anisotropy, all the observations agree with the Guest criterion within $1\frac{1}{2}$ per cent. Again the 10 per cent. variation

found to exist in their agreement with the shear stress theory does not exist.

As proof that no such variation from the shear stress hypotheses as is contained in the Guest criterion can be possible, Prof. Cook relies on the proposition that "Shearing Stress Alone is Effective in Causing Elastic Failure." This has been shown [11] by the writer to be of more than doubtful validity. That Cook's results cannot be explained in accordance with this proposition, while they can most completely by the Guest law, which includes a volumetric stress effect, is strong evidence that the proposition is unsound.

It now becomes clear that Prof. Nadai's criticisms of Cook's research are unwarranted, particularly his maintenance that the results actually support the Mises-Hencky hypothesis. For the two principal stresses at U, there can be only one positive and one negative third principal stress just to cause yield on that hypothesis, and, similarly, on the Bethrami-Haigh theory—the positive values for these are indicated in the diagrams. This research, however, finds that the whole line UV, parallel to OD, lies on the yield surface. That the experimental points from this research lie upon the two planes in figs. 12 and 13, is, however, the most complete evidence against the truth of either energy theory.

From the above it becomes clear that anisotropy of the simple types considered may influence experimental results so as to suggest conclusions which are unjustifiable. In "The Yielding of Mild Steel, with Particular Reference to the Effect of Size of Specimen" [12], Dr. J. L. M. Morrison, obtaining equal yield stresses in compression tests made for this special purpose upon specimens taken along and perpendicular to the axis of the bar, states that this proves the material to be isotropic. His tests in torsion determined the value of the shear/tension yield ratio to be 0·525. Having complete reliance upon "the proposition that Shearing Stress alone is effective in producing Elastic Failure," [11], Dr. Morrison concludes that the shear stress theory must be the truth and that the actual ratio is 0·5, and decides that the appreciable difference between these values is only to be accounted for by the "effect of random orientation of the crystals of the material." Such a suggestion cannot be valid: if there is no special arrangement of the structure the value of the ratio for this material must be 0·525, while if there is some special arrangement it constitutes anistropy. The above analysis indicates that the impasse arises from the conclusion that the special tests prove the material to be isotropic, and that the material actually had anisotropism of the type shown in fig. 12, the two principal stress yields being equal, but the shear stress yield unduly high.

Now let it be supposed that the principal stresses are not so simply related to the anisotropy as above, but that a plane having direction

cosines, l, m, n, is so weak as regards shear that failure occurs on it. Under the two principal stresses p_1 and p_2, the total stress on the plane is $\sqrt{l^2p_1^2+m^2p_2^2}$ and the normal stress $l^2p_1+m^2p_2$, so that the shear stress q is given by

$$q^2=l^2(1-l^2)p_1^2+m^2(1-m^2)p_2^2-2l^2m^2p_1p_2.$$

If it be taken that failure occurs when the shear stress reaches the value q_0 independently of its direction on this plane—which, in treating of anisotropy, is an assumption—the synoptical diagram would be the corresponding ellipse. Thus the diagram, instead of consisting of portions of straight lines, would consist of curved lines—meeting at points corresponding to changes of plane of failure. If one plane were always that of failure the diagram would be an ellipse, displaced if failure depended on direction of stress along the plane.

If the plane of weakness be equally inclined to Op_1 and Op_2, the ellipse will be symmetrical about the bisector of the angle between the axes, and the synoptical diagram will simulate those of the energy hypotheses. Thus if $n=\frac{3}{4}$, so that $l^2=m^2=7/32$, the equation becomes

$$p_1^2+p_2^2-2.\frac{7}{25}.p_1p_2=\frac{32.32}{7.25}.q_0^2,$$

which is the ellipse corresponding to the total energy theory for the value of $\sigma=7/25=0{\cdot}28$, a commonly accepted value.

As the value of n changes, the ellipse obtained corresponds to the Beltrami-Haigh hypothesis for varying values of σ, and when the plane of weakness is equally inclined to the axes, it becomes that of the von Mises hypothesis.

Thus although the material in its isotropic state may follow the shear stress criterion and the synoptical diagram consist of straight lines only, anisotropy may lead to experimental results which simulate the ellipses of the two energy hypotheses. The converse—that, if either of these hypotheses were the truth, anisotropy would lead to synoptical diagrams of experimental results consisting of straight lines is not true. Thus the diagrams in the latter papers discussed, as they consist of straight lines and planes, are evidence of a higher order than a diagram showing curved lines would afford. If an elliptic diagram is obtained from experimental results, before any conclusion that the results support an energy criterion can be considered, it is incumbent that experiments be carried out proving that no such interrelation of anisotropy and principal stresses as considered above, exists. Hitherto no such determination appears to have been made.

The evidence of each of the three researches above discussed is strongly in favour of the Guest criterion; all agreeing generally, while that of

Prof. Mason furnishes definite evidence as to the agreement of failure under double comparison and that the volumetric stress is of appreciable effect, that of Dr. Becker that the synoptical diagram consists of straight lines implying a linear relationship, and that of Prof. Cook furnishing explicit evidence that the yield surface consists of a series of planes corresponding to equation A.

The assumption was made that the shear stress was the primary factor in the failure of the results of these researches, and that a simple form of anisotropy existed: the very complete explanation afforded thereby of various hitherto unexplained features of the results of these researches is good evidence that the assumption and procedure are correct and that the various effects predicted exist and are to be guarded against in future experimental work and in the drawing of deductions therefrom.

References.

(1) Proc. Inst. Mech. Eng. pp. 1205–1236 (Dec. 1909).
(2) Phil. Mag. pp. 69–132 (July 1900).
(3) Ill. Univ. Eng. Dept. Bulletin, no. 85, pp. 84–93 (1916).
(4) Vol. I. Elastic Constants, p. 186.
(5) See 'Engineering,' p. 350 (Apr. 5, 1940).
(6) 'Journal Franklin Institute,' clxxxix. pp. 627–640 (1920).
(7) Brit. Assoc. Reprints, new series, no. 4, pp. 324–329.
(8) Proc. Roy. Soc. A, cxxxvii. p. 559 (1932).
(9) Trans. Amer. Soc. Mech. Eng. lv. p. 111 (1933).
(10) Phil. Trans. Roy. Soc. ccxxx. pp. 323–362.
(11) Phil. Mag. pp. 289–297 (Mar. 1940).
(12) Proc. Inst. Mech. Eng. cxlii. no. 3, p. 193 (Jan. 1940).

XXXV. *The Matrix Theory of Four-Terminal Networks.*

By Louis A. Pipes, Ph.D., Graduate School of Engineering, Harvard University, Cambridge, Mass. *

[Received March 31, 1939.]

Abstract.

Starting with the canonical equations of a general, linear, bilateral network in the steady state and by the use of matrix algebra, the general theory of four terminal networks is deduced. By the use of simple matrix multiplication the parameters of complicated structures are determined, and the exposition is concluded with a discussion of the matrix theory of the general artificial line, the smooth line, and the general ladder structure.

Introduction.

There has been considerable work done recently on the application of tensors, dyadics, and matrices to electric circuit theory by many investigators. Quite considerable work has been done in the application of matrix algebra to electrical filter theory (1–9), and it is the purpose of this discussion to extend the application and to illustrate the intimate connexion between certain well-known functions of filter theory and the fundamental quantities of matrix algebra. The extreme simplicity and homogeneity of the equations when formulated in matrix form, it is hoped, will be of service. To understand the discussion, a knowledge of the fundamentals of matrix algebra covered in section II. of (10) will be assumed.

I. The General Equations.

Let us consider a general four-terminal network as shown in fig. 1.

We will assume the reference directions for voltage and current as shown in fig. 1. The four-terminal network will be supposed to consist of an n mesh network having only two accessible elements as shown. The voltages E_1 and E_2 are complex voltages produced by sources external to the network, and the currents I_1 and I_2 are the mesh currents in meshes

* Communicated by the Author.

1 and 2 produced by the application of the external electromotive forces in the sense shown in the figure.

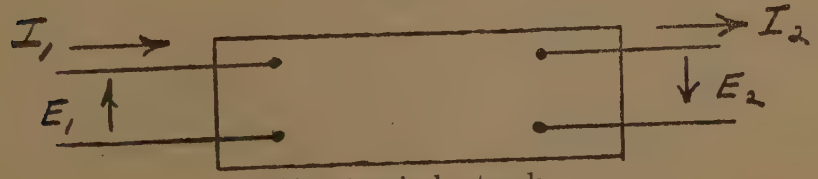
The internal structure of the four-terminal network will be assumed to be quite general, and the n meshes will be supposed to have complex mesh impedances Z_{mm} and complex mutual impedances between meshes Z_{mn}. The restrictions on the component elements of the network are that they be linear, constant, and bilateral. It will be further assumed that the network is passive, that is, the only sources of electromotive force are E_1 and E_2 produced by external agencies.

The canonical equations for the complex voltages and currents of such a network may be written in the following matrix form :

$$(E)=[Z](I), \quad \ldots\ldots\ldots \quad (1)$$

where (E) is a columnar matrix whose n elements are the n mesh voltages, [Z] is a square symmetric matrix of order n whose n elements are the self and mutual mesh impedances of the n mesh network, and

Fig. 1.



Four-terminal network.

(I) is a columnar matrix whose n elements are the n mesh currents of the network. The matrix equation (1) is readily obtained by an application of Kirchhoff's second law to the complex mesh voltages and mesh currents of the system. In view of the symmetry conditions of the mutual impedance elements,

$$Z_{mn}=Z_{nm}, \quad \ldots\ldots\ldots \quad (2)$$

we have

$$[Z]'=[Z], \quad \ldots\ldots\ldots \quad (3)$$

where [Z]′ is the transpose of the matrix [Z] and is obtained from [Z] by interchanging its rows and columns.

In the case under consideration the system has accessible terminals in only two meshes, the input mesh 1 and the output mesh 2. Since the system is supposed devoid of internal electromotive forces, the voltage matrix has all its elements equal to zero with the exception of its first two. If we pre-multiply (1) by $[Z]^{-1}$, the inverse of [Z], we have an explicit relation for the current matrix.

$$(I)=[Z]^{-1}(E). \quad \ldots\ldots\ldots \quad (4)$$

Let us introduce a square matrix of the nth order [Y] defined by

$$[Y]=[Z]^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since [Y] is the inverse of [Z], and [Z] is a symmetric matrix, we have

$$Y_{rs}=\frac{\text{cofactor of } Z_{rs}}{|\ Z\ |}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where Y_{rs} is the general element of the matrix [Y] and $|\ Z\ |$ is the determinant of the matrix [Z]. Hence we write concisely :

$$(I)=[Y](E). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

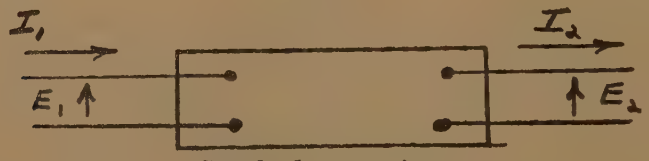
From (7) we may write the following two equations for the currents I_1 and I_2 of the accessible terminals.

$$\left.\begin{aligned} I_1&=Y_{11}E_1+Y_{12}E_2,\\ I_2&=Y_{21}E_1+Y_{22}E_2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $Y_{12}=Y_{21}$ as a consequence of the symmetry of [Z].

In the above discussion it has been assumed that the positive currents in each mesh have clockwise directions and that positive electromotive

Fig. 2.



Standard convention.

forces tend to circulate positive currents in the meshes where they are applied. It is easy to see that, with this convention of signs, all the mutual impedances Z_{mn} are negative. Now in the theory of our terminal networks it is convenient to use a somewhat different reference direction for the voltage E_2. The standard convention is shown in fig. 2.

The only alteration necessary in the above equations to make them conform to the standard reference directions as shown in fig. 2 is the reversal of the sign of E_2. Hence if we write

$$\left.\begin{aligned} I_1&=Y_{11}E_1-Y_{12}E_2,\\ I_2&=Y_{21}E_1-Y_{22}E_2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we conform to the standard reference directions of fig. 2.

Solving for E_1 and I_1 in the equations (9) we arrive at the standard form of the equations

$$\left.\begin{aligned} E_1&=AE_2+BI_2,\\ I_1&=CE_2+DI_2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where
$$\left.\begin{aligned} A&=Y_{22}/Y_{21}, & B&=1/Y_{12},\\ C&=(Y_{11}Y_{22}-Y_{12}^2)/Y_{12}, & D&=Y_{11}/Y_{12}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

and by direct multiplication it is seen that we have the identity

$$AD-BC=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It may be noted that the parameters A and D have the dimensions of pure numbers, B has the dimension of an impedance, and C has the dimension of an admittance. Since no restriction has been placed on the number of internal meshes of the structure it follows that the relation (12) is true in general. This relation holds in the limiting case where the four-terminal network is a smooth transmission line which may be regarded as a circuit of an infinite number of internal meshes. A discussion of a transmission line from this point of view will be given in section VIII.

The general theory thus shows that any linear, passive, bilateral four-terminal network is characterized by a square matrix of the second order, $[u]$, given by the following equation :

$$[u]=\left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The elements of $[u]$ are related by the equation (12), and hence the determinant of the matrix $[u]$ is equal to unity. In the remainder of the discussion the matrix $[u]$ will be called the associated matrix of the four-terminal network. The associated matrices of several common structures will be given in section IV.

II. Alternative Forms of the Equations.

The standard equations (10) which refer to fig. 2 may be put into the following alternative matrix forms :

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ I_1 \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} D/B & -1/B \\ 1/B & -A/B \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad . \quad . \quad (16)$$

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} A/C & -1/C \\ 1/C & -D/C \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad . \quad . \quad . \quad (17)$$

$$\begin{pmatrix} I_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} C/A & 1/A \\ 1/A & B/A \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ I_2 \end{pmatrix}, \quad . \quad . \quad . \quad (18)$$

$$\begin{pmatrix} E_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} B/D & 1/D \\ 1/D & -C/D \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ E_2 \end{pmatrix} \quad . \quad . \quad . \quad . \quad (19)$$

The equations (14–19) will be found very useful in many applications.

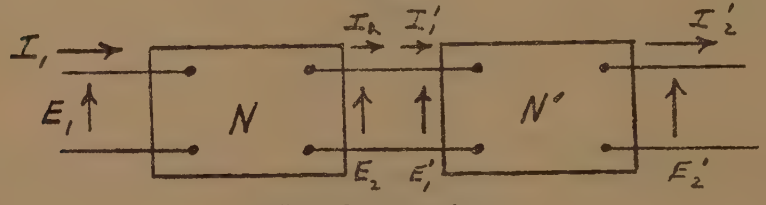
III. Interconnexion of Four-Terminal Networks.

The various matrix equations of the preceding article may be used to advantage in finding the resultant four-terminal constants when several four-terminal networks whose associated matrices are known are connected in various manners. In this section we shall consider the usual connexions of two four-terminal networks whose associated matrices are supposed known and determine the associated matrix of the resultant interconnected system.

A. *The Cascade Connexion.*

Let us first consider two four-terminal networks N and N′ whose associated matrices $[u]$ and $[u']$ are supposed known. Let these two networks be connected in cascade as shown in fig. 3.

Fig. 3.



Cascade connexion.

In this case we have the equations

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\begin{bmatrix} E_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} E_2' \\ I_2' \end{bmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where (A, B, C, D) are the known parameters of the network N and (A′, B′, C′, D′) are the parameters of the network N′, and the various currents and voltages have the sense shown in fig. 3. Since for this type of connexion we have

$$\begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1' \\ I_1' \end{bmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

it follows that, making use of this relation in equations (20) and (21), we obtain

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} E_2' \\ I_2' \end{bmatrix}, \quad . \quad . \quad (23)$$

or, carrying out the matrix multiplication, we obtain

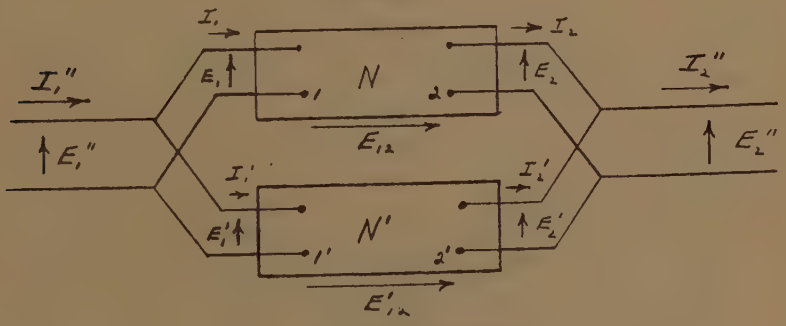
$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} (AA'+BC') & (AB'+BD') \\ (CA'+DC') & (CB'+DD') \end{bmatrix} \begin{bmatrix} E_2' \\ I_2' \end{bmatrix} \quad . \quad . \quad (24)$$

The four parameters of the resulting network are given by the elements of the square matrix of equation (24).

B. *The Parallel Connexion.*

Let us now consider the two networks N and N′ interconnected in parallel as shown in fig. 4. It is obvious that when two networks operate in cascade, then the current emerging from the two input terminals is equal to the current entering the two input terminals, and the same is true for the current emerging from the output terminals. Now in the more complicated cases to be treated, such as the parallel connexion, etc., the currents entering the respective input terminals are not, in general, equal to the currents emerging from the corresponding input terminals, and the same is true with regard to the output terminals. In such a case of unbalanced currents the matrices $[u]$ and $[u']$ of the respective

Fig. 4.



The parallel connexion.

networks no longer give the relations (14–19), and it is not possible to deduce the elements of the associated matrix of the resultant interconnected system from a knowledge of the elements of the matrices $[u]$ and $[u']$ of the component systems.

If there is no current unbalance within the system, the entering and emerging currents of the input and output terminals are equal. For such a balanced condition, it is necessary that $E_{12}=E_{12}'$, as shown in fig. 4. In this case we may write equation (16) for the upper and lower networks in the form

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} D/B \\ 1/B \end{bmatrix} \begin{bmatrix} -1/B \\ -A/B \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad . \quad . \quad . \quad (25)$$

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} D'/B' \\ 1/B' \end{bmatrix} \begin{bmatrix} -1/B' \\ -A'/B' \end{bmatrix} \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} \quad . \quad . \quad . \quad (26)$$

For this connexion we have the relations

$$\left.\begin{array}{l} I_1''=I_1+I_1' \\ I_2''=I_2+I_2' \end{array}\right\} \quad \begin{bmatrix} E_1'' \\ E_2'' \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} \quad . \quad . \quad (27)$$

If we add equation (25) to (26), then, in view of (27), we obtain

$$\begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} (D/B+D'/B') & -(1/B+1/B') \\ (1/B+1/B') & -(A/B+A'/B') \end{bmatrix} = \begin{bmatrix} E_1'' \\ E_2'' \end{bmatrix} . \quad (28)$$

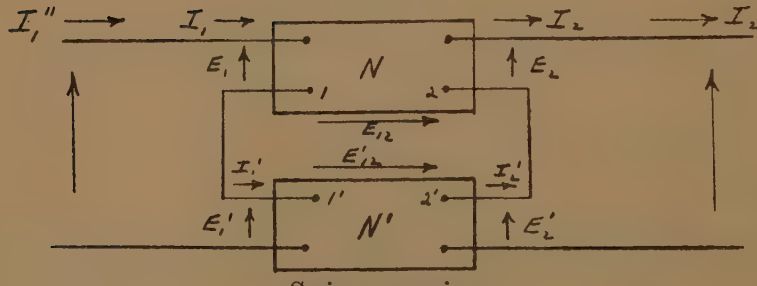
If we call (A'', B'', C'', D'') the parameters of the composite system, we have

$$\begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} D''/B'' & -1/B'' \\ 1/B'' & -A''/B'' \end{bmatrix} \cdot \begin{bmatrix} E_1'' \\ E_2'' \end{bmatrix} , \quad . \quad . \quad (29)$$

and if we compare the coefficients of the square matrices (29) and (28) we have

$$\left.\begin{array}{l} A''=(AB'+A'B)/(B+B'), \\ B''=BB'/(B+B'), \\ C''=(A-A')(D'-D)/(B+B'), \\ D''=(B'D+BD')/(B+B'). \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Fig. 5.



Series connexion.

It must be noted, however, that these results are valid only if $E_{12}=E_{12}'$. This condition is automatically satisfied in a large class of networks that have one terminal in common. Typical examples of this are the usual T and Pi networks.

C. *The Series Connexion.*

Fig. 5 shows the two four-terminal networks N and N' connected in series.

The criterion for current balance in this case is that $E_{12}=E_{12}'$ as shown in the figure. In this case there is no circulating current in the centre

loop, and the derived equations for the individual networks are valid. If this criterion is satisfied we may write the equations (17) in the form

$$\left.\begin{aligned} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} &= \begin{bmatrix} A/C & -1/C \\ 1/C & -D/C \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} &= \begin{bmatrix} A'/C' & -1/C' \\ 1/C' & -D'/C' \end{bmatrix} \cdot \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

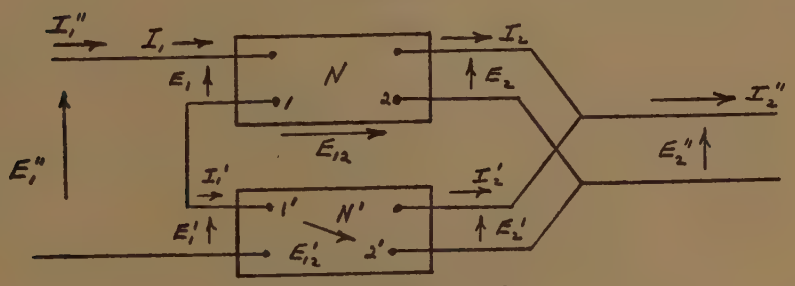
For this connexion we have the relations

$$\left.\begin{aligned} E_1''=E_1'+E_1 \qquad I_1''=I_1=I_1' \\ E_2''=E_2'+E_2 \qquad I_2''=I_2=I_2' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If we add the equations (31), then, in view of the relations (32), we obtain the equation

$$\begin{bmatrix} E_1'' \\ E_2'' \end{bmatrix} = \begin{bmatrix} (A/C+A'/C') & -(1/C+1/C') \\ (1/C+1/C') & -(D/C+D'/C') \end{bmatrix} \begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} . \quad (33)$$

Fig. 6.



Series-parallel connexion.

Calling (A'', B'', C'', D'') the parameters of the composite system, we have

$$\begin{bmatrix} E_1'' \\ E_2'' \end{bmatrix} = \begin{bmatrix} A''/C'' & -1/C'' \\ 1/C'' & -D''/C'' \end{bmatrix} \cdot \begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} . \quad . \quad . \quad (34)$$

Comparing the coefficients of the square matrices of equations (33) and (34) we obtain :

$$\left.\begin{aligned} A''&=(AC'+A'C)/(C+C'), \\ B''&=(A''D''-1)/C'', \\ C''&=CC'/(C+C'), \\ D''&=(DC'+CD')/(C+C'). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

D. *Series-Parallel Connexion.*

Fig. 6 shows the two four-terminal networks N and N' connected in series-parallel.

The criterion for current balance in this case is that $E_{12}=E_{12}'$ as marked in fig. 6. To treat this case, we write the equations (19) in the form

$$\begin{bmatrix} E_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} B/D & 1/D \\ 1/D & -C/D \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ E_2 \end{bmatrix}$$
$$\begin{bmatrix} E_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} B'/D' & 1/D' \\ 1/D' & -C'/D' \end{bmatrix} \cdot \begin{bmatrix} I_1' \\ E_2' \end{bmatrix} \quad \ldots\ldots (36)$$

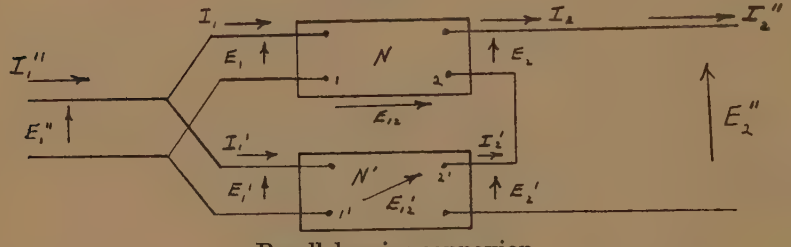
The relations between the voltages and currents for this connexion are :

$$\left.\begin{aligned} E_1''&=E_1'+E_1 \\ I_2''&=I_2+I_2' \\ I_1''&=I_1=I_1' \\ E_2''&=E_2=E_2' \end{aligned}\right\} \quad \ldots\ldots\ldots (37)$$

If we add the equation (36), then, in view of (37), we obtain

$$\begin{bmatrix} E_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} (B/D+B'/D') & (1/D+1/D') \\ (1/D+1/D') & -(C/D+C'/D') \end{bmatrix} \cdot \begin{bmatrix} I_1'' \\ E_2'' \end{bmatrix} . \quad (38)$$

Fig. 7.



Parallel-series connexion.

If we call (A″, B″, C″, D″) the parameters of the composite system, we have :

$$\begin{bmatrix} E_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} B''/D'' & 1/D'' \\ 1/D'' & -C''/D'' \end{bmatrix} \cdot \begin{bmatrix} I_1'' \\ E_2'' \end{bmatrix} . \quad . \quad . \quad (39)$$

Comparing the coefficients of the square matrices of (39) and (38) we obtain

$$\left.\begin{aligned} A''&=(B''C''+1)/D'', \\ B''&=(BD'+B'D)/(D+D'), \\ C''&=(CD'+C'D)/(D+D'), \\ D''&=DD'/(D+D'). \end{aligned}\right\} \quad \ldots\ldots (40)$$

E. *The Parallel-Series Connexion.*

Fig. 7 represents the two four-terminal networks N and N′ connected in parallel-series,

The criterion for current balance in this case is $E_{12}=E_{12}'$ as shown in the figure. If this is met, then we may write the equations (18) in the form

$$\left.\begin{aligned}\begin{bmatrix} I_1 \\ E_2 \end{bmatrix} &= \begin{bmatrix} C/A & 1/A \\ 1/A & B/A \end{bmatrix} \begin{bmatrix} E_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} I_1' \\ E_2' \end{bmatrix} &= \begin{bmatrix} C'/A' & 1/A' \\ 1/A' & B'/A' \end{bmatrix} \begin{bmatrix} E_1' \\ I_2' \end{bmatrix}\end{aligned}\right\} \quad . \quad . \quad . \quad (41)$$

The relations between the voltages and currents in this connexion are :

$$\left.\begin{aligned} E_1''&=E_1=E_1', \\ I_1''&=I_1+I_1', \\ E_2''&=E_2'+E_2, \\ I_2''&=I_2=I_2'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If we add the equations (41), then, in view of the relations (42), we obtain

$$\begin{bmatrix} I_1'' \\ E_2'' \end{bmatrix} = \begin{bmatrix} (C/A+C'/A') & (1/A+1/A') \\ (1/A+1/A') & (B/A+B'A') \end{bmatrix} \cdot \begin{bmatrix} E_1'' \\ I_2'' \end{bmatrix} . \quad (43)$$

If we now let the parameters of the composite system be (A'', B'', C'', D'') we have

$$\begin{bmatrix} I_1'' \\ E_2'' \end{bmatrix} = \begin{bmatrix} C''/A'' & 1/A'' \\ 1/A'' & B''/A'' \end{bmatrix} \cdot \begin{bmatrix} E_1'' \\ I_2'' \end{bmatrix} . \quad . \quad (44)$$

Comparing the coefficients of the square matrices of (43) and (44) we obtain

$$\left.\begin{aligned} A''&=AA'/(A+A'), \\ B''&=(BA'+B'A)/(A+A'), \\ C''&=(CA'+C'A)/(A+A'), \\ D''&=(B''C''+1)/A''. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

In applying these formulas to the calculation of the parameters of composite systems caution must be exercised in being certain that the various criteria for current balance are met in each case.

IV. The Associated Matrices of Common Four-Terminal Structures.

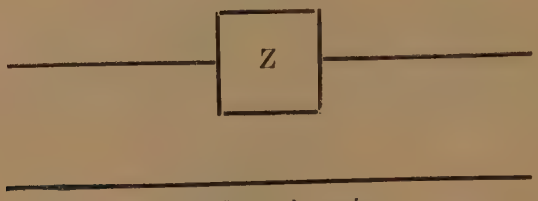
It will now be demonstrated how by matrix multiplication of the associated matrices of several simple structures, it is possible to determine the associated matrix of more complicated structures.

A. *Impedance in Series.*

By a comparison of the coefficients of the currents and voltages of equation (14) with those obtained by a direct application of Kirchhoff's

laws to the circuit of fig. 8, we obtain the associated matrix of this simple circuit $[u]$.

Fig. 8.



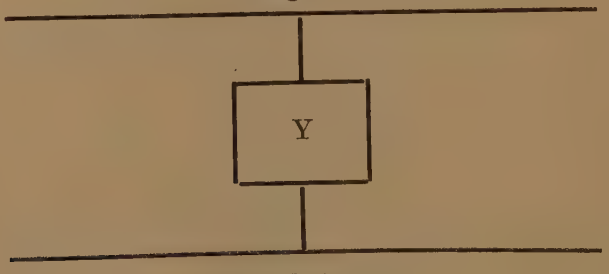
Impedance in series.

$$[u]=\begin{bmatrix}1 & Z\\ 0 & 1\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46\,a)$$

B. *Shunt Admittance.*

Following the same procedure, we determine the associated matrix of the shunt admittance of fig. 9.

Fig. 9.



Shunt admittance.

$$[u]=\begin{bmatrix}1 & 0\\ Y & 1\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46\,b)$$

C. *T. Network.*

The associated matrix of the T network of fig. 10 may be easily obtained by a matrix multiplication, since the T network is formed by a cascade connexion of three four-terminal networks of the types of figs. 8 and 9.

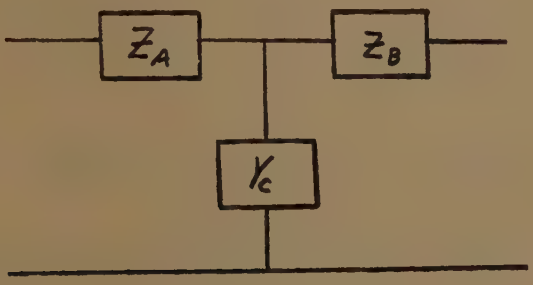
$$[u]=\begin{bmatrix}1 & Z_a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ Y_c & 1\end{bmatrix}\begin{bmatrix}1 & Z_b\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}(1+Z_aY_c) & (Z_b+Z_aZ_bY_c+Z_a)\\ (Y_c) & (Z_bY_c+1)\end{bmatrix} \quad . \quad . \quad . \quad . \quad (47)$$

D. *Pi Network.*

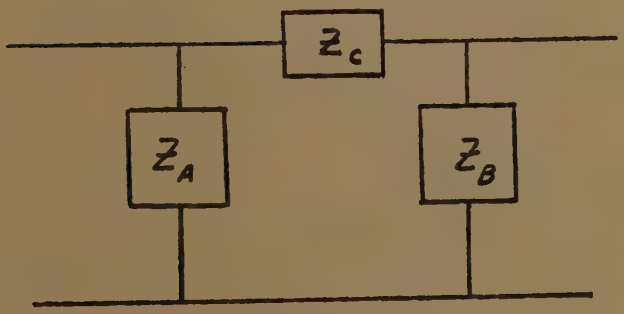
The associated matrix of the Pi Network of fig. 11 is obtained by matrix multiplication, since it is composed of three four-terminal networks in cascade.

Fig. 10.



T network.

Fig. 11.



Pi network.

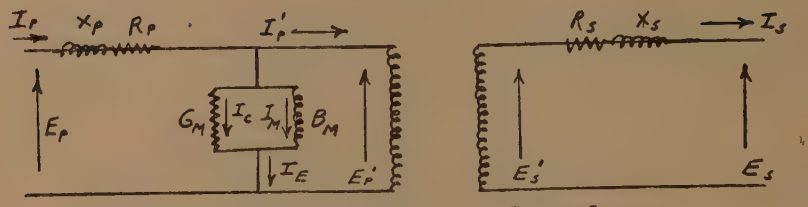
$$[u]=\begin{pmatrix}1 & 0\\ 1/Z_a & 1\end{pmatrix}\begin{pmatrix}1 & Z_c\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1/Z_b & 1\end{pmatrix}$$

$$[u]=\begin{pmatrix}(Z_b+Z_c)/Z_b & Z_c\\ \dfrac{(Z_a+Z_b+Z_c)}{Z_aZ_b} & \dfrac{(Z_a+Z_c)}{Z_a}\end{pmatrix} \quad \ldots \quad (48)$$

E. *The Real Transformer.*

The equivalent circuit for a real transformer is shown in fig. 12. The over-all four-terminal parameters will be obtained by a matrix multiplication.

Fig. 12.



Equivalent circuit diagram of real transformer.

We use the following notation :—

N_p=the number of turns in the primary winding.

N_s=the number of turns in the secondary winding.

$a=N_s'/N_p$=the ratio of transformation.

G_m=the equivalent conductance of the magnetizing circuit.

B_m=equivalent susceptance of the magnetizing circuit.

I_p=the primary current.

E_p=the primary voltage.

I_s=the secondary current.

E_s'=the secondary voltage.

I_c=the current representing the core loss.

I_m=the magnetizing current.

I_e=the exciting current.

X_p=the primary leakage reactance.

X_s=the secondary leakage reactance.

R_p=the primary winding resistance.

R_s'=the secondary winding resistance.

Now if we let

$$Z_p=R_p+jX_p, \quad \ldots\ldots\ldots \quad (49)$$

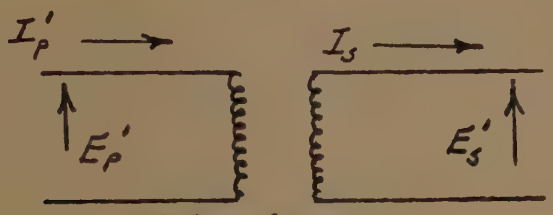
$$Z_s=R_s+jX_s, \quad \ldots\ldots\ldots \quad (50)$$

$$Y_m=G_m+jB_m. \quad \ldots\ldots\ldots \quad (51)$$

then the four-terminal network representing the transformer may be regarded as the cascade connexion of four individual circuits. The first circuit consists of a series of impedance Z_p, the second of a shunt admittance Y_m, the third of an ideal transformer, and the fourth of a series impedance Z_s.

Fig. 12 is the circuit diagram of the ideal transformer.

Fig. 13.



Ideal transformer.

Since the ratio of transformation is a, we have the equations

$$E_p' = -E_s'/a \quad I_p' = -aI_s'. \quad \ldots\ldots \quad (52)$$

Accordingly, the associated matrix $[u]$ for the four-terminal network representing the ideal transformer is

$$[u] = \begin{pmatrix} -1/a & 0 \\ 0 & -a \end{pmatrix} \quad \ldots\ldots\ldots \quad (53)$$

If we let $[u]_t$ represent the associated matrix of the real transformer, we have

$$[u]_t = \begin{pmatrix} 1 & Z_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y_m & 1 \end{pmatrix} \begin{pmatrix} -1/a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} 1 & Z_s \\ 0 & 1 \end{pmatrix}$$

$$= \frac{-1}{a} \begin{pmatrix} (Z_p Y_m + 1) & (Z_s + Z_p Z_s Y_m + a^2 Z_p) \\ Y_m & (Y_m Z_s + a^2) \end{pmatrix}, \quad \ldots \quad (54)$$

and we have the following relation between the primary quantities and the secondary quantities :

$$\begin{pmatrix} E_p \\ I_p \end{pmatrix} = [u]_t \begin{pmatrix} E_s \\ I_s \end{pmatrix} \quad \ldots\ldots \quad (55)$$

and

$$\begin{pmatrix} E_s \\ I_s \end{pmatrix} = [u]_t^{-1} \begin{pmatrix} E_p \\ I_p \end{pmatrix} \quad \ldots\ldots \quad (56)$$

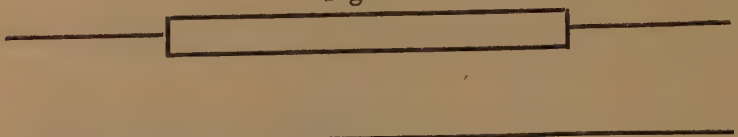
It thus appears that when the individual parameters of a complex four-terminal system are known, then the parameters of the complex system formed by the interconnexion of the simple constituent elements may be determined by simple matrix multiplication.

F. *The Smooth Transmission Line.*

In section VIII., the associated matrix of the smooth transmission line of fig. 14 will be deduced by regarding the transmission line as con-

structed of an infinite number of four-terminal networks connected in cascade. The associated matrix is given here for convenience.

Fig. 14.



The smooth line.

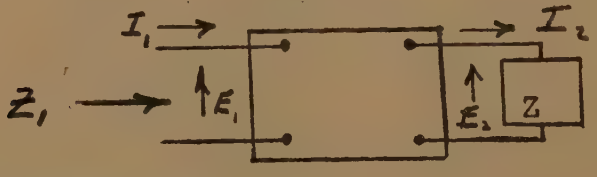
$$[u]=\begin{bmatrix}\cosh(\alpha L) & Z_0\sinh(\alpha L),\\ \dfrac{\sinh(\alpha L)}{Z_0} & \cosh(\alpha L)_z\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where α is the propagation constant of the line defined in section VIII., L is the length of the line, and Z_0 is the characteristic impedance of the line.

V. The Homographic Transformation.

Let us consider a four-terminal network terminated by an impedance Z as shown in fig. 15.

Fig. 15.



We have our general equation :

$$\begin{bmatrix}E_1\\ I_1\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}E_2\\ I_2\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The termination introduces the relation

$$E_2=ZI_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Hence, substituting this value of E_2 into (14), we obtain

$$\begin{bmatrix}E_1\\ I_1\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}Z\\ 1\end{bmatrix}I_2. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If we call Z_1 the impedance looking into the network from the left when the network is terminated by the impedance Z as shown, we obtain

$$Z_1=E_1/I_1=\frac{AZ+B}{CZ+D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

It is thus apparent that the insertion of the four-terminal network transforms the impedance Z into the impedance Z_1 by a homographic transformation. The properties of the homographic transformation are well known, and an account of them may be found in (11), (12). The chief properties may be briefly summarized by the following statements :—

a. The homographic transformation (60), where A, B, C, and D are complex constants transforms circles in the Z plane into circles in the Z_1 plane if straight lines are regarded as degenerate circles. It is this property that accounts for the many circle diagrams that occur in circuit theory.

b. The homographic transformation is a one to one transformation of the complete Z plane into the complete Z_1 plane which leaves the angle between any two intersecting curves unaltered.

c. The homographic transformation may be generated by an even number of inversions with respect to circles or straight lines.

If we place $Z_1=Z$ in the transformation (60), we shall obtain two different values of Z which will be unaltered by the transformation. These are given by the equation

$$CZ^2-(A-D)Z-B=0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The two roots of this equation may be written in the form

$$s_1=\frac{(A-D)+\sqrt{(A+D)^2-4}}{2C} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and

$$s_2=\frac{(A-D)-\sqrt{(A+D)^2-4}}{2C} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

If s_1 and s_2 are distinct, we may write

$$(Z_1-s_1)/(Z_1-s_2)=K(Z-s_1)/(Z-s_2), \quad . \quad . \quad . \quad . \quad (64)$$

where

$$K=(A-Cs_1)/(A-Cs_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

K is called the multiplier of the transformation.

If we now have a chain of n identical circuits all having the associated matrix $[u]$ as shown in fig. 16, then the impedance Z_n looking into the cascade chain from the left is given by the transformation

$$(Z_n-s_1)/(Z_n-s_2)=K^n(Z-s_1)/(Z-s_2). \quad . \quad . \quad . \quad . \quad (66)$$

The result (66) is obtained by a repeated application of (64). Solving for Z_n in (66) we obtain

$$Z_n=\frac{Z(s_1-s_2K^n)+(K^n-1)s_1s_2}{Z(1-K^n)+(s_1K^n-s_2)}; \quad . \quad . \quad . \quad . \quad . \quad (67)$$

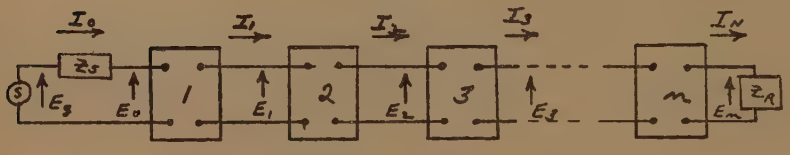
We shall see in a later section that the quantities s_1, s_2, and K are intimately related to the matrix $[u]$. The problem of determining the sending end impedance of the cascade arrangement of fig. 16 is thus completely solved by equation (67).

VI. Cascade Connexion of Dissymmetrical Networks.

The problem of determining the voltages and currents when n identical dissymmetrical networks are arranged as shown in fig. 17 will now be considered.

Fig. 16.

Fig. 17.



If we apply the method of Part III. A, and consider the equations for the cascade connexion of the element Z_s and the n identical four-terminal networks whose associated matrix is $[u]$, we obtain the following equation :

$$\begin{pmatrix} E_g \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Z_s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \cdot \begin{pmatrix} E_n \\ I_n \end{pmatrix} \cdot \quad . \quad . \quad . \quad (68)$$

We could, of course, raise the matrix $[u]$ to any integral power n by direct multiplication, but such a procedure is tedious, and we may therefore make use of the basic theorems of matrix algebra.

As a first step in the procedure of raising the matrix $[u]$ to a high power, let us construct the characteristic matrix of the matrix $[u]$ defined by the equation

$$[f(q)] = \begin{pmatrix} (q-A) & -B \\ -C & (q-D) \end{pmatrix} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The characteristic equation of $[u]$ is in this case given by

$$|f(q)| = q^2 - q(A+D) + 1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and the latent roots of $[u]$ are given by the equations

$$\left.\begin{aligned} q_1 &= \frac{(A+D)}{2} + \sqrt{\frac{(A+D)^2}{4} - 1}, \\ q_2 &= \frac{(A+D)}{2} - \sqrt{\frac{(A+D)^2}{2} - 1}. \end{aligned}\right\} \quad \ldots \quad (71)$$

We note that the product of the latent roots satisfies the equation

$$q_1 . q_2 = 1. \quad \ldots \quad (72)$$

The sum of the latent roots satisfies the equation

$$q_1 + q_2 = (A+D). \quad \ldots \quad (73)$$

Accordingly, if we let

$$q_1 = e^{a}, \quad \ldots \quad (74)$$

where a is a complex number, we have, in view of (72), the relation

$$q_2 = e^{-a}. \quad \ldots \quad (75)$$

As a consequence of the relation (73) we have

$$\cosh(a) = \frac{(A+D)}{2}. \quad \ldots \quad (76)$$

The adjoint matrix of the characteristic matrix of $[u]$ is

$$[F(q)] = \begin{bmatrix} (q-D) & B \\ C & (q-A) \end{bmatrix}. \quad \ldots \quad (77)$$

Let us now form the vector $(k)_1$ by substituting $q = q_1$ in the first column of (77), that is, let

$$(k)_1 = \begin{bmatrix} (e^{-a}-D) \\ C \end{bmatrix}. \quad \ldots \quad (78)$$

Now let the vector $(k)_2$ be formed by substituting $q = q_2$ into the first column of (77), that is,

$$(k)_2 = \begin{bmatrix} (e^{-a}-D) \\ C \end{bmatrix}. \quad \ldots \quad (79)$$

From the two columnar matrices $(k)_1$ and $(k)_2$, let the square matrix $[k]$ be formed in the following manner :

$$[k] = \begin{bmatrix} (e^{a}-D) & (e^{-a}-D) \\ C & C \end{bmatrix}. \quad \ldots \quad (80)$$

Now if we substitute (76) into (62) and (63) we obtain

$$s_1 = (e^{a}-D)/C. \quad \ldots \quad (81)$$

$$s_2 = (e^{-a}-D)/C. \quad \ldots \quad (82)$$

Accordingly, we may write $[k]$ in the form

$$[k]=C\begin{bmatrix} s_1 & s_2 \\ 1 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and we find $[k]^{-1}$ to be

$$[k]^{-1}=\frac{1}{C(s_1-s_2)}\begin{bmatrix} 1 & -s_2 \\ -1 & s_1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Now there is a basic theorem in matrix algebra that states that any matrix $[u]$ may be transformed by pre-multiplying by $[k]$ and post-multiplying by $[k]^{-1}$ into the following canonical form :

$$[u]=[k]\begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}[k]^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

provided that the two latent roots q_1 and q_2 are distinct. In this case the roots are distinct if

$$(A+D)\neq\pm 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Substituting in the values of $[k]$ and $[k]^{-1}$ we write the matrix $[u]$ by the collineatory transformation (85) in the form

$$[u]=\frac{1}{(s_1-s_2)}\begin{bmatrix} s_1 & s_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{a} & 0 \\ 0 & e^{-a} \end{bmatrix}\begin{bmatrix} 1 & -s_2 \\ -1 & s_1 \end{bmatrix} \quad . \quad . \quad (85)$$

The form (85) has the advantage that by repeated multiplication we obtain

$$[u]^n=\frac{1}{(s_1-s_2)}\begin{bmatrix} s_1 & s_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{an} & 0 \\ 0 & e^{-an} \end{bmatrix}\begin{bmatrix} 1 & -s_2 \\ -1 & s_1 \end{bmatrix}, \quad . \quad (87)$$

where n may be a positive or negative integer. To revert to the standard notation of the literature we note that

$$s_1=Z_{01}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

$$s_2=-Z_{02}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where Z_{01} is the iterative impedance in the $+$sense and Z_{02} is the iterative impedance in the negative sense of the four-terminal network. Carrying out the matrix multiplication of (86) we obtain

$$[u]^n=\frac{1}{(Z_{01}+Z_{02})}\begin{bmatrix} (Z_{01}e^{an}+Z_{02}e^{-an}) & (2Z_{01}Z_{02}\sinh(an)) \\ 2\sinh(an) & (Z_{02}e^{an}+Z_{01}e^{-an}) \end{bmatrix}, \quad (90)$$

where n is either a positive or negative integer. If the individual quadripoles are symmetric, then $(A=D)$ and hence $Z_{01}=Z_{02}$. In this special case we have

$$[u]^n=\begin{bmatrix} \cosh(an) & Z_{01}\sinh(an) \\ \dfrac{\sinh(an)}{Z_{01}} & \cosh(an) \end{bmatrix}, \quad . \quad . \quad . \quad . \quad (91)$$

The case of equal roots arises when

$$(A+D)=\pm 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

As a consequence of (92) we have

$$Z_{02}=-Z_{01}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

$$\cosh(a)=\pm 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

$$\sinh(a)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

This case may be treated as a limiting case of (90). To continue the investigation of the various currents and voltages of fig. 17, let us introduce the notation

$$[u]^n=\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

we then have

$$\begin{bmatrix} E_g \\ I_0 \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} E_n \\ I_n \end{bmatrix} . \quad . \quad . \quad . \quad (97)$$

Now since we have the relation

$$E_n=I_nZ_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

we have the following equation for the impedance looking at the system from the generator end :

$$Z_g=E_g/I_0=\frac{(A_n+Z_sC_n)Z_r+(B_n+Z_sD_n)}{(C_nZ_r+D_n)} . \quad . \quad . \quad . \quad . \quad (99)$$

To find the general voltage E_k and the current I_k along the network, we write

$$\begin{bmatrix} E_g \\ I_0 \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^K \begin{bmatrix} E_k \\ I_k \end{bmatrix}, \quad . \quad . \quad (100)$$

or, on finding the inverse of the resultant square matrix on the right side, we have

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-K} \begin{bmatrix} 1 & -Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_g \\ I_0 \end{bmatrix} . \quad . \quad . \quad (101)$$

In view of (96) and (99), this may be written in the form

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = \begin{bmatrix} A_{-k} & B_{-k} \\ C_{-k} & D_{-k} \end{bmatrix} \begin{bmatrix} 1 & -Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/Z_g \end{bmatrix} E_g. \quad . \quad (102)$$

We thus have obtained all the relations in terms of E_g and the constants of the systems. Substituting the relation for Z_g from (99) and carrying out the matrix multiplication of (102), we obtain, after some algebraic

reductions, the two relations for the voltages and currents along the line in the form

$$E_k = \frac{E_g[Z_{01}e^{(n-k)a} + Z_{02}R_r e^{-(n-k)a}]}{(Z_s+Z_{01})(e^{na} - R_s R_r e^{-na})}, \quad . \quad . \quad . \quad . \quad . \quad (103)$$

$$I_k = \frac{E_g[e^{(n-k)a} - R_r e^{-(n-k)a}]}{(Z_s+Z_{01})(e^{na} - R_s R_r e^{-na})}, \quad . \quad . \quad . \quad . \quad . \quad (104)$$

where

$$R_s = (Z_s - Z_{02})/(Z_s + Z_{01}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$R_r = (Z_r - Z_{01})/(Z_s + Z_{02}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

This is the conventional form of the solution where R_s and R_r are the well-known reflexion coefficients. The simplicity by which this solution is obtained by matrix multiplication may be noted. The boundary conditions enter in a most natural manner.

VII. Attenuation and Pass Bands.

The prominent rôle that the latent roots of the associated matrix of the four-terminal network play in determining the attenuation and pass bands of the system will now be demonstrated. Let us consider an isolated quadripole in the chain of fig. 17. Now let it be assumed that we have the following relations between the output and input currents and voltages :

$$\left.\begin{aligned} E_{k+1} &= mE_k \\ I_{k+1} &= mI_k \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

where m is a complex number to be determined. In view of the relation

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = [u] \begin{bmatrix} E_{k+1} \\ I_{k+1} \end{bmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

we have

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = [u] \begin{bmatrix} E_k \\ I_k \end{bmatrix} m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

or

$$(U - m[u]) \begin{bmatrix} E_k \\ I_k \end{bmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

where U is the unit matrix of the second order defined by

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

If we write $m = 1/q$, we obtain the following equation as a necessary condition for the consistency of (110) :

$$\begin{vmatrix} (q-A) & -B \\ -C & (q-D) \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

This is the characteristic equation of the associated matrix $[u]$ discussed in section VI. Using the notation of (74) and (75) and the relation $m=1/q$, we obtain

$$m_1=e^{-a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

$$m_2=e^{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

m_1 is a constant associated with the propagation of voltages and currents to the right in the cascade connexion of fig. 17, and m_2 is associated with propagation to the left. Since in general A and D are complex numbers and

$$\cosh(a)=(A+D)/2=b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

it follows that in general a is complex. Let us assume

$$a=\alpha+j\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

and

$$b=b_0e^{j\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

where α, β, b_0, and ϕ are real numbers. Substituting this into (76), we obtain

$$\cosh(\alpha)\cos(\beta)+j\sinh(\alpha)\sin(\beta)=b_0[\cos(\phi)+j\sin(\phi)]. \quad . \quad (116)$$

Equating real coefficients, we obtain

$$\cos(\beta)=b_0\cos(\phi)/\cosh(\alpha). \quad . \quad . \quad . \quad . \quad . \quad (117)$$

Equating imaginary coefficients, we have

$$\sin(\beta)=b_0\sin(\phi)/\sinh(\alpha). \quad . \quad . \quad . \quad . \quad . \quad (118)$$

If $\alpha=0$, then m_1 and m_2 introduce changes in phase, but no attenuation, as the voltage and current are propagated along the chain of circuits. Now since $|\sin(\beta)| \leqq 1$, and $\sinh(0)=0$, it follows from (118) that $\sin(\phi)=0$ and therefore

$$\phi=k\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

where $k=0, \pm1, \pm2$, etc., as a necessary condition that there be no attenuation. In view of (115) this restricts b to real values. Now we have

$$|\cos(\beta)| = |b_0\cos(\phi)_0/\cosh(0)| \leqq 1; \quad . \quad . \quad . \quad (119\,a)$$

then, since ϕ is restricted to take on the values (119), it follows that we must have

$$|b| \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

Therefore, in terms of A and B we have the following two conditions for a pass band

$$(A+D)=\text{Real number} \quad . \quad . \quad . \quad . \quad . \quad (121)$$

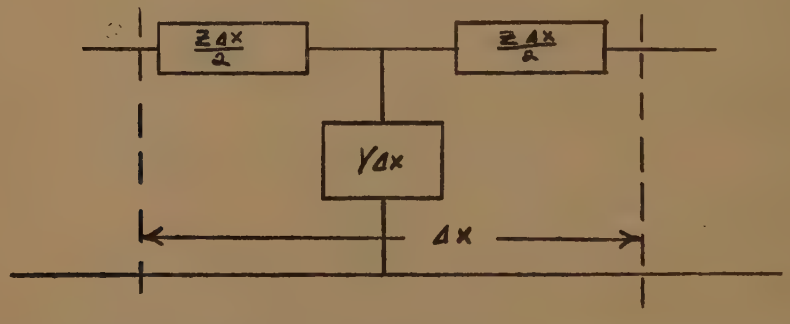
$$|(A+D)| \leqq 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

If the component parts of the four-terminal network are non-dissipative, then it may be shown that (121) is always satisfied. The condition (122) is true only for certain frequencies; currents and voltages having frequencies for which (122) is satisfied are passed by the cascade arrangement and currents and voltages of other frequencies are attenuated.

VIII. The Smooth Transmission Line.

The smooth transmission line will now be regarded as the limiting case of an infinite number of symmetrical, identical four-terminal networks. We shall consider that the transmission line has a series impedance Z per unit length and a shunt admittance of Y per unit length. If we consider a section of the line of length Δx long, then this infinitesimal section of the line may be regarded as the four-terminal network represented in fig. 18.

Fig. 18.



The transmission line may be regarded as the cascade connexion of an infinite number of four-terminal networks of the type of fig. 18. The associated matrix of the four-terminal network of fig. 18 is

$$[u]=\begin{pmatrix}(1+ZY\overline{\Delta x^2}/2) & (Zx+Z^2Y\overline{\Delta x^2}/4)\\ Y\Delta x & (1+ZY\Delta x^2/2)\end{pmatrix} \quad . \quad . \quad . \quad (123)$$

In terms of the associated matrix $[u]$ of the infinitesimal four-terminal network, we have the following relation between the input and output currents and voltage of the entire transmission line:

$$\begin{pmatrix}E_1\\ I_1\end{pmatrix}=\underset{\substack{n\to\infty\\ \Delta x\to\infty}}{\text{Limit}}\,[u]^n\begin{pmatrix}E_2\\ I_2\end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

Since the infinitesimal network is symmetric, that is, A=D, we may make use of formula (91), where Z_{01} is given by

$$Z_{01}=\sqrt{[(A+D)^2-4]}/2C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Substituting the values of A, D, and C from (123) into this equation, we obtain, on neglecting high order terms, the equation

$$Z_{01}=\sqrt{Z/Y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

To calculate the latent roots of the matrix (123) we use

$$q=(A+D)/2+\sqrt{(A+D)^2/4-1}=e^{a}. \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Here we have

$$(A+D)/2=1+ZYx^2/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

Hence, on substituting this into (71) and neglecting high order terms, we obtain

$$e^{a}=1+\Delta x\sqrt{ZY}+\frac{\overline{\Delta x^2\sqrt{ZY}}}{2}+\ldots. \quad . \quad . \quad . \quad (127)$$

However, we have the relation

$$e^{a}=1+a+a^2/2+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127\,a)$$

Hence, comparing this expansion with that of (127), we obtain

$$a=\Delta x\sqrt{ZY}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Substituting this into (91), we have

$$[u]^{n}=\begin{bmatrix}\cosh(n\Delta x\sqrt{ZY}) & \sqrt{(Z/Y)^{\frac{1}{2}}}\sinh(n\Delta x\sqrt{ZY})\\ (Y/Z)^{\frac{1}{2}}\sinh(n\Delta x\sqrt{ZY}) & \cosh(n\Delta x\sqrt{ZY})\end{bmatrix}. \quad (129)$$

On passing to the limit we make use of the relation

$$\underset{\substack{n\to\infty\\ \Delta x\to\infty}}{\text{Limit}}\,(n\Delta x)=L, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

where L is the length of the transmission line. Using the notation

$$L\sqrt{ZY}=\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

and

$$(Z/Y)^{\frac{1}{2}}=Z_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

and substituting this into (124), we have the usual relation for the smooth transmission line

$$\begin{bmatrix}E_1\\ I_1\end{bmatrix}=\begin{bmatrix}\cosh(\theta) & Z_0\sinh(\theta)\\ \dfrac{\sinh(\theta)}{Z_0} & \cosh(\theta)\end{bmatrix}\cdot\begin{bmatrix}E_2\\ I_2\end{bmatrix} \quad . \quad . \quad . \quad (133)$$

This result is thus obtained as a special case of the equation (91) without the use of differential equations.

IX. The General Ladder Network.

The matrix method of analysis appears to have its greatest power in the theory of the general ladder network. A special form of this network has been solved by A. C. Bartlett [12] by using continued fractions

and continuants. In the interest of unity this discussion will be concluded with a matrix analysis of such a network. Let us consider the general network of fig. 19.

This is a generalization of the network of fig. 17 in that we will now assume the n networks to have different associated matrices. Let $[u]_r$ be the matrix associated with the rth network. We then may write the equation

$$\begin{bmatrix} E_g \\ I_0 \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \prod_{r=1}^{n} [u]_n \begin{bmatrix} E_n \\ I_n \end{bmatrix} \quad . \quad . \quad . \quad . \quad (134)$$

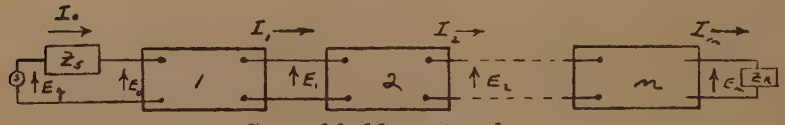
If we let,

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \prod_{r=1}^{n} [u]_n, \quad . \quad . \quad . \quad . \quad (135)$$

then, in view of (60), we have the following equation for Z_g, the impedance of the system looking at it from the generator end

$$Z_g = E_g/I_g = (A_n Z_r + B_n)/(C_n Z_r + D_n). \quad . \quad . \quad . \quad . \quad (136)$$

Fig. 19.



General ladder network.

To find the voltage and current distribution at any point of the net work we use the equation

$$\begin{bmatrix} E_g \\ I_0 \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} E_k \\ I_k \end{bmatrix}, \quad . \quad . \quad . \quad . \quad (137)$$

or

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = \begin{bmatrix} D_k & -B_k \\ -C_k & A_k \end{bmatrix} \begin{bmatrix} E_g \\ I_0 \end{bmatrix}; \quad . \quad . \quad . \quad (138)$$

but in view of (136), we may write

$$\begin{bmatrix} E_k \\ I_k \end{bmatrix} = \begin{bmatrix} D_k & -B_k \\ -C_k & A_k \end{bmatrix} \begin{bmatrix} 1 \\ 1/Z_g \end{bmatrix} E_g, \quad . \quad . \quad (139)$$

and we thus have the currents and voltages in terms of the generator voltage and the constants of the system. It may be remarked that this analysis parallels that of the series of identical elements given in section VI. In that case the matrix product (135) is reduced to the matrix power (68). Since, however, in order to solve the problem of the general ladder structure it is only necessary to carry out the chain of matrix multiplication (134), and matrix multiplication of second order

matrices is particularly simple, it appears that the method given here possesses many advantages over the usual one of continued fractions or continuants.

X. Conclusion.

The natural manner that matrix algebra enters into the analysis of four-terminal networks is evident from this discussion. The conciseness of the results and the unity of approach should prove helpful to workers in this field. The simple operation of matrix multiplication replaces the more complex methods of differential equations, difference equations, continued fractions, and continuants used in the conventional treatment of the subject.

XI. Bibliography.

(1) Ph. Le Corbeiller et Ch. Lange, "Etude sur les lignes en T. dissymetriques et application aux filtres de bandes," *L'Onde Electrique*, ii. pp. 560–570 (Oct. 1923).
(2) F. Strecker and R. Feldtkeller, *Electrische Nachrichten Technik*, vi. pp. 93–112 (March 1929).
(3) R. Feldtkeller and F. Strecker, *Archiv für Electrotechnik*, xxiv. pp. 425–468 (Nov. 7, 1930).
(4) R. Feldtkeller, *Zeitschrift für Hochfrequenztechnik*, xxxv. pp. 45–55 (Feb. 1930).
(5) R. Feldtkeller and W. Kautter, *Electrische Nachrichten Technik*, viii. pp. 93–103 (March 1931).
(6) E. Selach, *Electrische Nachrichten Technik*, viii. pp. 304–306 (July 1931).
(7) H. G. Gaerwald, *Electrische Nachrichten Technik*, ix. pp. 31–38 (Jan. 1932).
(8) H. Konig, *Helvetica Physica Acta*, iii. p. 459 (1930).
(9) Leon Brillouin, *Revue Générale de L'Electricite*, pp. 3–16 (Jan. 4, 1936).
(10) L. A. Pipes, 'Electrical Engineering,' pp. 1177–1190 (Sept. 1937).
(11) G. H. Hardy, 'Pure Mathematics,' Cambridge, 1921, pp. 94–96.
(12) A. C. Bartlett, 'The Theory of Electrical Artificial Lines and Filters.' John Wiley, 1930, Chapt. viii.

XXXVI. *The Lateral Diffusion of a Stream of Ions in a Gas.*

By L. G. H. Huxley, M.A., D.Phil., F.Inst.P., F. W. Bennett
Lecturer in Physics, University College, Leicester *.

[Received April 22, 1940.]

Summary.

The problem of determining theoretically the spatial distribution of ions or of electrons moving in a stream in a steady state of motion through a gas under the action of a uniform electric field is of considerable practical importance. It is generally solved by employing the standard methods of the theory of the conduction of heat and of diffusion, but in general the solutions appear in series form from which the labour of computing the distribution of the stream over a receiving electrode may be very considerable, although in special circumstances simplifications can be introduced.

In this paper an alternative analysis is presented which affords solutions in a form notably more convenient for the purposes of practical computation, and leads to some generalizations of theory.

The more important cases encountered in practice are discussed, and the paper also includes a section on the effects of magnetic fields.

1. *Introduction.*

The investigation of the lateral spreading by diffusion of a stream of ions or electrons moving through a gas under the action of a constant and uniform electric field is an essential feature of the well-known methods employed by Townsend [1], [2], [4], Bailey [3], [15] and their associates to study the motions of electrons in gases and the attachment of electrons to molecules.

The spreading of a stream is determined in these experiments through the ratio R of the current to an insulated central portion of a large plane electrode on which the stream impinges to the total current to the whole electrode. The effects of an applied magnetic field are also observed.

The ratio R is a function of the quantity Z/k (Z is the electric force, and k Townsend's energy factor [1], [2], [4]), which can be found from that

* Communicated by the Author.

solution of the basic differential equation for the distribution of the electrons in the stream which is appropriate to the boundary conditions and dimensions of the particular apparatus employed.

The relation between R and Z/k is usually not simple, and is generally represented by a curve.

When the stream of ions is slightly divergent R may be found conveniently and accurately [1] from approximate solutions, but in general the solutions are obtained in the form of series [1], [5] from which the evaluation of R as a function of Z/k can be a laborious and lengthy process which must be repeated whenever a change of dimensions or of form of the apparatus is made.

It was found, in connexion with an experimental investigation in progress in this laboratory, that an alternative mathematical analysis, the advantages of which have been discussed by G. Green [16] in another connexion, led to solutions of greater convenience and generality, and some of the simpler and more useful examples are discussed below.

II. *Diffusion in Three Dimensions.*

The basic differential equation for the distribution of ions or electrons in a stream moving in a uniform electric field Z along Oz is [2], [5], [6]

$$\nabla^2 n = 2\lambda \frac{\partial n}{\partial z}, \quad \ldots \ldots \ldots \quad (1)$$

where n is the number of ions per cubic centimetre and

$$2\lambda = \frac{W}{K} = \frac{Ne}{\Pi}\left(\frac{Z}{k}\right);$$

W is the velocity of drift in the direction of Z and K the diffusion coefficient of the ions.

When Z is in volts per centimetre it is known [1], [2] that at a temperature of 15° C.

$$2\lambda = \frac{W}{K} = 40{\cdot}3\frac{Z}{k}. \quad \ldots \ldots \ldots \quad (2)$$

In equation (1) write

$$n = e^{\lambda z}V. \quad \ldots \ldots \ldots \quad (3)$$

The equation satisfied by V is, therefore,

$$\nabla^2 V = \lambda^2 V. \quad \ldots \ldots \ldots \quad (4)$$

This equation is similar to an equation in the theory of wave-propagation [8] and may be treated by the usual methods.

If U and V are two functions which are solutions of (4) which, together

with their first and second derivatives, are finite and single-valued throughout any finite region, then it follows from Green's theorem that

$$\iint\left(U\frac{\partial V}{\partial \nu}-V\frac{\partial U}{\partial \nu}\right)dS=\iiint(U\nabla^2V-V\nabla^2U)\,\partial\tau=0, \quad . \quad . \quad . \quad (5)$$

in which the positive direction of the normal to a surface element dS is taken outwards from the enclosed volume.

Put $U=\frac{e^{\pm\lambda r}}{r}$, so that U becomes a solution of (4) depending only upon r, the distance from a fixed point. When the fixed point P falls within the bounding surface S, it may be isolated by drawing a small sphere about it, and we deduce [8] from (5)

$$4\pi V_P=\iint\left[\frac{e^{\pm\lambda r}}{r}\frac{\partial V}{\partial \nu}-V\frac{\partial}{\partial \nu}\left(\frac{e^{\pm\lambda r}}{r}\right)\right]dS, \quad . \quad . \quad . \quad . \quad (6)$$

where V_P is the value of V at the point P and r the distance of P from the surface element dS.

V_P may consequently be regarded as arising from the combined contributions of elementary sources over S, and it may be concluded that elementary solutions of (4) exist, of the form

$$V=\frac{Ae^{\lambda r}+Be^{-\lambda r}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$V=C\frac{d}{dr}\left(\frac{e^{\pm\lambda r}}{r}\right)\cos(\nu, r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

On multiplying these expressions by $e^{\lambda z}$ they are converted into solutions of (1).

These solutions and others may be obtained [9] directly from (4) by solving (4) in polar coordinates :

$$r^2\frac{\partial^2V}{\partial r^2}+2r\frac{\partial V}{\partial r}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2V}{\partial\phi^2}-\lambda^2r^2V=0.$$

We seek solutions of the form

$$V=R_kS_k,$$

where R_k depends only upon r and S_k is a surface harmonic of order k. The equation satisfied by S_k is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_k}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2S_k}{\partial\phi^2}+k(k+1)S_k=0,$$

and since k is an integer in physical applications, S_k is an associated Legendre function.

The equation for R_k is

$$r^2\frac{d^2R_k}{dr^2}+2r\frac{dR_k}{dr}-[k(k+1)+\lambda^2r^2]R_k=0. \quad . \quad . \quad . \quad . \quad (9)$$

In (9) put $t=\lambda r$ to obtain

$$t^2\frac{d^2R_k}{dt^2}+2t\frac{dR_k}{dt}-[k(k+1)+t^2]R_k=0. \quad . \quad . \quad . \quad . \quad (10)$$

A general solution of (10) may be expressed conveniently in the form [10]

$$R_k=(-1)^k\left(\frac{\pi}{2}\right)^{\frac{1}{2}}t^k\left(\frac{1}{t}\frac{d}{dt}\right)^k\left[\frac{Ae^t+Be^{-t}}{t}\right], \quad . \quad . \quad . \quad . \quad (11)$$

where A and B are arbitrary constants.

A special case of (11) which makes R_k finite as t becomes infinite is

$$R_k=(-1)^k\left(\frac{\pi}{2}\right)^{\frac{1}{2}}t^k\left(\frac{1}{t}\frac{d}{dt}\right)^k\left(\frac{e^{-t}}{t}\right)$$

$$=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-t}}{t}\left[1+\frac{k(k+1)}{2t}+\frac{(k-1)k(k+1)(k+2)}{(2\,!)(2t)^2}+\ldots\right]=t^{-\frac{1}{2}}K_{k+\frac{1}{2}}(t),$$

where $K_{k+\frac{1}{2}}(t)$ is a modified Bessel function of the second kind and of half-integral order [11]. Thus

$$R_0=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-t}}{t}=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-\lambda r}}{\lambda r},$$

$$R_1=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-\lambda r}}{\lambda r}\left[1+\frac{1}{\lambda r}\right],$$

$$R_2=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-\lambda r}}{\lambda r}\left[1+\frac{3}{\lambda r}+\frac{3}{(\lambda r)^2}\right].$$

Thus a general solution of equation (1) which remains finite at infinity may be obtained in the form of a series whose $(k+1)$th term is

$$A_k e^{\lambda z}(\lambda r)^{-\frac{1}{2}}K_{k+\frac{1}{2}}(\lambda r)P_k^m(\cos\theta)\cos m\phi.$$

The simple solutions (7) and (8) are the terms for which $m=0$ and $k=0$ and 1 respectively.

Applications.

(*a*) The solution of (1) which is obtained from (7) is

$$n=A\frac{e^{-\lambda(r-z)}}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and represents the distribution due to an isolated point source of ions placed in a gas in a uniform electric field Z. This solution has been employed by H. A. Wilson and H. Mache [7] to account for the distribution of the yellow coloration of a Bunsen flame by sodium vapour originating in a bead of glass placed just above the inner cone of the flame.

(*b*) The solution

$$n=Ae^{\lambda z}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r}\right)\cos\theta=Ae^{\lambda z}\frac{z}{r}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r}\right), \quad . \quad . \quad . \quad (13)$$

derived from (8), represents the distribution due to a doublet source placed at the origin when a field Z acts along Oz. It follows that the concentration of ions in the plane $z=0$ is everywhere zero except at the origin.

This solution is of interest in representing the lateral diffusion of a stream of ions moving through a gas under the action of an electric field, after having passed through a small hole in a metal plate, if the usual plausible assumption is made that the concentration is zero at the surface of a metal electrode.

In the Townsend diffusion apparatus the stream of ions is received by a parallel electrode over which we assume n to be zero. To find the appropriate solution, we suppose the first plane electrode to occupy the plane $z=0$, with the hole in it at the origin, and the receiving electrode the plane $z=h$.

We require n to be zero when $z=0$ for all values of r except $r=0$, while at the plane $z=h$ n is zero everywhere.

These conditions may be satisfied by employing image sources suitably placed along Oz.

The first, placed at the point $(0, 0, 2h)$, is of strength $Ae^{\lambda.2h}$. It produces at a point $P(x, y, z)$ at a distance r_1 from it a concentration

$$Ae^{2\lambda h}\frac{(z-2h)}{r_1}e^{\lambda(z-2h)}\frac{d}{dr_1}\left(\frac{e^{-\lambda r_1}}{r_1}\right)=Ae^{\lambda z}\frac{(z-2h)}{r_1}\frac{d}{dr_1}\left(\frac{e^{-\lambda r_1}}{r_1}\right).$$

The original source and the image together give rise to a concentration at P

$$n=Ae^{\lambda z}\left[\frac{z}{r}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r_1}\right)+\frac{(z-2h)}{r_1}\frac{d}{dr_1}\left(\frac{e^{-\lambda r_1}}{r_1}\right)\right],$$

which makes $n=0$ over the plane $z=h$, since here $r=r_1$.

The image source is itself compensated by a second in order to make n zero at $z=0$, and the second image by a third, and so on.

The final expression for the concentration at a point P between the plates is the following series of diminishing terms :

$$n=Ae^{\lambda z}\left[\frac{z}{r}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r}\right)+\frac{(z-2h)}{r_1}\frac{d}{dr_1}\left(\frac{e^{-\lambda r_1}}{r_1}\right)\right.$$
$$\left.+\frac{(z+2h)}{r_2}\frac{d}{dr_2}\left(\frac{e^{-\lambda r_2}}{r_2}\right)+\frac{(z-4h)}{r_3}\frac{d}{dr_3}\left(\frac{e^{-\lambda r_3}}{r_3}\right)+\ldots+\cdot-:\right],$$

where

$$r^2=\rho^2+z^2\ ;\ r_1^2=\rho^2+(z-2h)^2\ ;\ r_2^2=\rho^2+(z+2h)^2\ ;\ \text{etc.},$$

and

$$\rho^2=x^2+y^2.$$

The current to an element dS of the lower electrode situated at a distance r from the origin of coordinates is proportional to

$$-K\left(\frac{\partial n}{\partial z}\right)_{z=h}dS.$$

We have

$$\left(\frac{\partial n}{\partial z}\right)_{z=h} = 2Ae^{\lambda h}\left[\frac{1}{r}\frac{d}{dr}\left(\frac{e^{-\lambda r}}{r}\right)+\frac{1}{r_2}\frac{d}{dr_2}\left(\frac{e^{-\lambda r_2}}{r_2}\right)+\cdots\right].$$

In practice the effect of the second and higher terms in the brackets would be inappreciable, since over the central portion of the plane $z=h$, $r_2 \doteqdot 3r$.

Thus the current to dS may be taken to be proportional to $n\,d$S, where n is obtained by putting $z=h$ in the simple solution (13). This procedure is known in practice to give correct results.

The ratio of the current to a circular portion of the lower electrode of radius b with its centre on Oz to the total current to the plane $z=h$ is

$$\mathrm{R}=\frac{\int_0^b n\rho\,d\rho}{\int_0^\infty n\rho\,d\rho}=\frac{\int_h^d nr\,dr}{\int_h^\infty nr\,dr}=1-\frac{h}{d}e^{-\lambda(d-h)},$$

where

$$d^2=h^2+b^2.$$

The calculation of R in this case is consequently a simple procedure.

By employing a spatial lattice of suitable image sources the distribution of ions within the diffusion chamber could be determined when the latter has the form of a rectangular parallelopiped of finite dimensions over whose boundaries n is zero. The receiving electrodes would here be rectangular and not circular.

III. *Diffusion in Two Dimensions.*

In the more usual form of Townsend diffusion apparatus the ions enter the diffusion chamber through a short slit (parallel to Oy) at the centre of the upper electrode (occupying the plane $z=0$) and move towards a second plane electrode (the plane $z=h$) under the action of a uniform field Z. The second electrode consists of a central strip parallel to the slit of width $2b$ flanked by plane conductors of sufficient extent to contain the whole stream.

The ratio R of the current received by the central strip parallel to Oy to the total current received by the whole plane $z=h$ is measured.

In order to obtain a solution of equation (1) which satisfies the boundary conditions of this apparatus, the equation is first transformed to one in two dimensions [6].

The current to the receiving electrode between the parallel lines x and $(x+dx)$ is proportional to $q=\int_{-\infty}^{+\infty} n\,dy$.

It is assumed that at the boundaries of this electrode both n and dn/dy

are zero, so that on integrating equation (1) with respect to y between the limits $\pm\infty$ we find

$$-\left[\frac{\partial n}{\partial y}\right]_{-\infty}^{\infty}=\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial z^2}-2\gamma\frac{\partial q}{\partial z}=0. \quad . \quad . \quad . \quad . \quad (14)$$

The appropriate solution is usually obtained in the form of a Fourier series of sinusoidal functions [5], [9], the slow convergence of which makes the calculation of the ratios R a laborious process. The solution given below has proved more convenient and flexible.

We follow the method of section II. in writing

$$q=e^{\lambda z}\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that from (14) the equation satisfied by V is

$$\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial z^2}=\lambda^2\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which becomes in polar coordinates ($r^2=x^2+y^2$; $\theta=\tan^{-1}x/z$)

$$r^2\frac{\partial^2 \mathrm{V}}{\partial r^2}+r\frac{\partial \mathrm{V}}{\partial r}+\frac{\partial^2 \mathrm{V}}{\partial \theta^2}-\lambda^2 r^2\mathrm{V}=0$$

or

$$t^2\frac{\partial^2 \mathrm{V}}{\partial t^2}+t\frac{\partial \mathrm{V}}{\partial t}+\frac{\partial^2 \mathrm{V}}{\partial \theta^2}-t^2\mathrm{V}=0, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $t=\lambda r$.

In (17) put $\mathrm{V}=\mathrm{R}_k e^{ik\theta}$, where k is an integer. The equation satisfied by R_k is

$$t^2\frac{d^2\mathrm{R}_k}{dt^2}+t\frac{d\mathrm{R}_k}{dt}-(t^2+k^2)\mathrm{R}_k=0. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Consequently

$$\mathrm{R}_k=\mathrm{A}_k\mathrm{I}_k(t)+\mathrm{B}_k\mathrm{K}_k(t),$$

where A_k and B_k are arbitrary constants and $\mathrm{I}_k(t)$ and $\mathrm{K}_k(t)$ are modified Bessel functions [11] of order k and of the first and second kinds respectively.

A solution of (16) is, therefore,

$$q=e^{\lambda z}\sum_{k=0}^{\infty}[\mathrm{A}_k\mathrm{I}_k(\lambda r)+\mathrm{B}_k\mathrm{K}_k(\lambda r)][\mathrm{C}_k\cos k\theta+\mathrm{D}_k\sin k\theta]. \quad . \quad . \quad (19)$$

The function $\mathrm{I}_k(t)=(-i)^k\mathrm{J}_k(it)$ and is finite when t is zero, but becomes infinite with t.

The functions $\mathrm{K}_k(t)$ are important among Bessel functions in remaining finite as t becomes infinite, and are appropriate for the problem in question.

Some of the more useful properties of these functions for the present applications are the following :—

$$K_k(t)=\int_0 e^{-t\cosh\phi}\cosh k\phi\, d\phi. \quad . \quad . \quad . \quad . \quad . \quad (20)\ *$$

Other integral representations are to be found in treatises on Bessel functions.

$$\left.\begin{aligned} &K_0(t) \text{ tends to infinity like } \log t, \\ \text{and} \quad &K_k(t)_{k>0} \text{ like } (k\,!)\frac{2^{k-1}}{t^k}, \text{ as } t \text{ approaches zero.} \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

For the larger values of t the following asymptotic expansion is important :

$$K_k(t)=\left(\frac{\pi}{2t}\right)^{\frac{1}{2}}e^{-t}\left[1+\frac{(4k^2-1)}{8t}+\frac{(4k^2-1)(4k^2-9)}{(2\,!)(8t)^2}+\ldots+R_s\right]=K_{-k}(t),$$

$$. \quad . \quad . \quad (22)$$

in which the remainder $R_s \leqslant \frac{\Gamma(k+s+\frac{1}{2})}{(s\,!)\Gamma(k-s+\frac{1}{2})}\cdot\frac{1}{(2t)^s}=(s+1)$th term in the expansion.

When k is half-integral the series in the brackets terminates and the expansion gives the Bessel functions already encountered in section II.

The following relations are also useful :

$$\left.\begin{aligned} 2kK_k(t)&=t[K_{k+1}(t)-K_{k-1}(t)], \\ 2K'_k(t)&=-K_{k-1}(t)-K_{k+1}(t), \\ K'_0(t)&=-K_1(t), \\ K_k(t)I'_k(t)-K'_k(t)I_k(t)&=1/t. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The functions $K_k(t)$ are related to the Hankel cylindrical functions [12] as follows :

$$\frac{2}{\pi}K_k(t)=\frac{2}{\pi}\int_0^\infty e^{-t\cosh\phi}\cosh k\phi\, d\phi=i^{k+1}H_k^{(1)}(it)=(-i)^{-(k+1)}H_k^{(2)}(it). \quad (24)$$

The functions $\frac{2}{\pi}K_0(t)$ and $\frac{2}{\pi}K_1(t)$ are tabulated in Janke-Emde's Tables of Functions [13] as $iH_0^{(1)}(it)$ and $-H_1^{(1)}(it)$ respectively.

Reverting to the problem of the Townsend slit apparatus, it is evident that the appropriate solution for the case of a narrow slit is the cosine term of (19) for which $k=1$ and $A_k=0$.

We write, accordingly,

$$q=Ae^{\lambda z}K_1(\lambda r)\cos\theta=Ae^{\lambda z}\frac{z}{r}K_1(\lambda r). \quad . \quad . \quad . \quad . \quad (25)$$

This solution makes q zero over the axis $z=0$ except at the origin

---

* Some writers multiply the integral by a factor $\frac{2}{\pi}$, *e. g.*, reference (14).

In order to reduce q to zero over the receiving electrode at $z=h$, the device of image sources is again used, and the final expression for the quantity q at any point within the diffusion chamber is found to be

$$q=Ae^{\lambda z}\left[\frac{z}{r}K_1(\lambda r)+\frac{(z-2h)}{r_2}K_1(\lambda r_2)+\frac{(z+2h)}{r_3}K_1(\lambda r_3)+\ldots\right],$$

where

$$r^2=x^2+z^2,$$

$$r_2^2=x^2+(z-2h)^2, \text{ etc.}$$

Proceeding as in section II., it appears that the ratio R of the current to the central strip of width $2b$ to the total current to the receiving electrode is

$$R_{z=h}=\frac{\int_0^b \frac{K_1(\lambda r)}{r}dx}{\int_0^\infty \frac{K_1(\lambda r)}{r}dx} \quad \ldots\ldots\ldots\ldots \quad (26)$$

The integrals in (26) are transformed by writing

$$\frac{r}{h}=\sec\theta=\cosh\phi,$$

$$\frac{x}{h}=\tan\theta=\sinh\phi\ ;\ \chi=\lambda h.$$

Thus

$$\theta=\tan^{-1}\frac{x}{h}=gd.\phi.$$

At

$$x=b,$$

$$\theta_b=\tan^{-1}\frac{b}{h}=gd.\phi_b,$$

where $2\theta_b$ is the angle subtended by the central strip of the receiving electrode at the slit.

Expression (26) becomes

$$R=\frac{\int_0^{\theta_b} K_1(\chi\sec\theta).\sec\theta\,d\theta}{\int_0^{\pi/2} K_1(\chi\sec\theta).\sec\theta\,d\theta}=\frac{\int_0^{\phi_b} K_1(\chi\cosh\phi)\,d\phi}{\int_0^{\infty} K_1(\chi\cosh\phi)\,d\phi} \quad \ldots\ldots \quad (27)$$

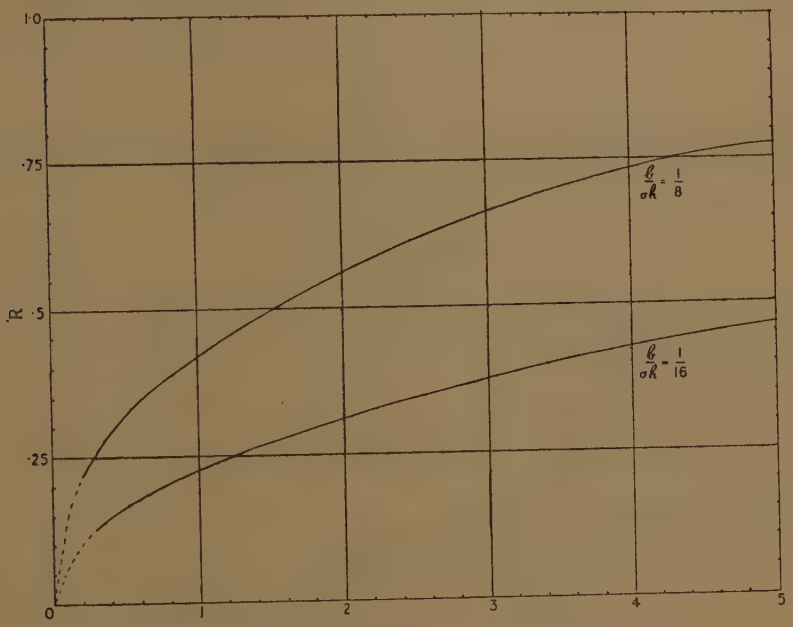
Since $\frac{2}{\pi}K_1(t)$ is a tabulated function [13], as already mentioned, the integrals may be computed when $\frac{b}{h}=\tan\theta_b=\cosh\phi_b$ is preassigned for a range of values of χ.

Since $\theta_b=\tan^{-1}\frac{b}{h}$, $\phi_b=\cosh^{-1}\frac{b}{h}$, and $\chi=\lambda h=20{\cdot}15\frac{Zh}{k}$, it follows that

R may be obtained as a function $R\left(\frac{b}{h}, \frac{Zh}{k}\right)$ of the quantities $\frac{b}{h}$ and $\frac{Zh}{k}$ *.

The two curves shown in the figure represent R as a function of $\left(\frac{Zh}{k}\right)$, $\frac{b}{h}$ having the constant value $\frac{1}{16}$ for the lower curve and $\frac{1}{8}$ for the upper.

This is a convenient mode of representation since the curves do not relate merely to a single apparatus.



$\frac{Zh}{k}$ (Z in volts per centimetre; h in centimetres).

The points on the curve were found by computing the integrals for each specified value of $\frac{b}{h}$ and χ by means of Simpson's formula, using a sufficient number of intervals to give good accuracy.

As the function $\frac{2}{\pi} K_1(t)$ is tabulated as far as $t = 16$, the tables were used for values of χ less than 15.

---

* Zh is evidently the total potential difference across the diffusion chamber.

For values of t greater than 16, $K_1(t)$ is virtually identical (for the present purposes) with $\left(\frac{\pi}{2t}\right)^{\frac{1}{2}} e^{-t}$, and R is given by

$$R = \frac{e^{\lambda h}\int_0^b \frac{e^{-\lambda r}}{r^{3/2}}dx}{e^{\lambda h}\int_0^\infty \frac{e^{-\lambda r}}{r^{3/2}}dx} = \frac{\int_0^{\theta b} e^{-\chi(\sec\theta-1)}(\sec\theta)^{\frac{1}{2}}d\theta}{\int_0^{\pi/2} e^{-\chi(\sec\theta-1)}(\sec\theta)^{\frac{1}{2}}d\theta}$$

$$= \frac{\int_0^{\phi b} e^{-\chi(\cosh\phi-1)}(\cosh\phi)^{-\frac{1}{2}}d\phi}{\int_0^\infty e^{-\chi(\cosh\phi-1)}(\cosh\phi)^{-\frac{1}{2}}d\phi}. \quad (28)$$

For values of χ greater than 150 the stream of ions is not widely divergent at the receiving electrode, and a further simplification is possible. We have

$$q = \text{const.}\frac{e^{-\chi(\cosh\phi-1)}}{(\cosh\phi)^{3/2}}, \quad (29)$$

and when χ is large we may consider ϕ to remain small over the region in which q is appreciable.

Thus

$$q \doteqdot \text{const.}\, e^{-\frac{\chi\phi^2}{2}} \doteqdot \text{const.}\, e^{-\frac{\chi x^2}{2h^2}} = \text{const.}\, e^{-\frac{\lambda x^2}{2h}},$$

an approximate form of solution which has frequently been employed [3], [6]. The expression for R becomes

$$R = \frac{2}{\pi}\int_{\bar{h}}^{\bar{b}}\left(\frac{\chi}{2}\right)^{\frac{1}{2}} e^{-y^2}dy = \text{Erf}\left[\frac{b}{h}\left(\frac{\chi}{2}\right)^{\frac{1}{2}}\right]^{(14a)}, \quad (30)$$

and the value of R is obtained directly from tables [12].

A comparison of the lower curve in the figure with the usually published curve [2], [5] obtained by the standard method, relating to an apparatus for which $h=4$ cm. and $\frac{b}{h}=\frac{1}{16}$, shows that the two curves are virtually identical.

The advantage of the present method lies in the relative rapidity with which the points may be found, the ease with which the portions of the curve for the smaller values of $\frac{Zh}{k}$ may be obtained, and in the generalization in the representation of the results to which it leads.

Effect of Finite Width of the Slit.

In practice the slit is not a line source, and it is of interest to inquire with what accuracy the solution given above represents the distribution due to a narrow slit of finite width. The following treatment is suggested.

We consider streams of electrons, for which the values of λ are small compared with those for ions, and assume as before that the concentration n is zero or very small at the surface of an electrode, so that near the electrode when λ is sufficiently small n is determined principally by the process of diffusion, and only to a minor extent by the action of the applied field Z.

Equation (14) then assumes the simplified form

$$\nabla^2 q = 0. \quad \ldots\ldots\ldots \quad (31)$$

It is supposed that q satisfies (31) for values of r, measured from the centre of the slit, at least as great as the width of the slit.

The problem of determining q near the slit is related to a similar problem in electrostatics.

Consider an infinite plane conducting sheet in which is cut a long parallel-sided aperture of width $2a$ small compared with its length taken parallel to Oy. Let the sheet occupy the plane $z=0$ and take the centre of the aperture as origin of coordinates. The sheet is supposed negatively charged and at zero potential.

The determination of the potential at any point in the central plane $y=0$ is a two-dimensional problem which is solved by the method of conjugate functions.

Put

$$U+iV = E_0[(x+iy)^2-a^2]^{\frac{1}{2}}, \quad \ldots\ldots \quad (32)$$

where U is the flux function and V the potential.

In the plane $z=0$ it follows that

$$U=0\ ;\ V=E_0(a^2-x^2)^{\frac{1}{2}} \text{ when } -a<x<+a,$$

$$U=E_0(x^2-a^2)^{\frac{1}{2}}\ ;\ V=0 \text{ when } x<-a \text{ and } x>+a.$$

The field-strength at the surface of the plate on either face is given by

$$|E| = \left|\frac{\partial U}{\partial x}\right|_{z=0} = \left|\frac{E_0 x}{(x^2-a^2)^{\frac{1}{2}}}\right|.$$

Thus $|E|$ is infinite at the edges of the aperture and approaches E_0 at great distances from it.

At great distances from the conducting sheet

$$U \to E_0 x\ ;\ V \to E_0 z,$$

and the strength of the field at infinity directed away from the plate on either side is E_0.

By superimposing everywhere a uniform field E_0 the field at infinity may be doubled on one side of the plane and reduced to zero on the other.

Considering the latter region, it is evident that the field is to be derived from

$$U+iV=E_0[(x+iz)^2-a^2]^{\frac{1}{2}}-E_0(x+iz), \quad . \quad . \quad . \quad . \quad (33)$$

an expression which represents the penetration of the field $2E_0$ through the aperture.

In (33) put $(x+iz)=re^{i\alpha}$ and expand the radical in a binomial series. On separating the real and imaginary parts and writing

$$\theta=\tan^{-1}\frac{x}{z}=\left(\frac{\pi}{2}-\alpha\right),$$

the following expression is obtained for V at distances $r>a$:

$$V=E_0\left[\frac{a^2}{2r}\cos\theta+\frac{a^4}{8r^3}\cos 3\theta+\frac{a^6}{16r^5}\cos 5\theta+\ldots\right]. \quad . \quad . \quad (34)$$

This is also the form of expression which gives the distribution of electrons near the aperture when λ is small provided $r>a$.

It has been shown that a series $e^{\lambda z}\Sigma_k B_k K_k(\lambda r)\cos k\theta$ is a solution of (14), so that from (34) and (21) we may infer the following solution of the problem in question, since it satisfies (31) when λ or r are small, together with the boundary conditions

$$q=\text{const.}\frac{e^{\lambda z}}{\lambda}\left[\frac{(\lambda a)^2}{2}K_1(\lambda r)\cos\theta+\frac{(\lambda a)^4}{8}\cdot\frac{K_3(\lambda r)}{2^2(2\,!)}\cos 3\theta\right.$$
$$\left.+\frac{(\lambda a)^6}{16}\cdot\frac{K_5(\lambda r)}{2^4(4\,!)}\cos 5\theta+\ldots\right]. \quad (35)$$

Thus the term involving $(\lambda a)^{n+1}$ becomes associated with the factor

$$\frac{K_n(\lambda r)}{2^{n-1}[(n-1)\,!]}.$$

It may be shown from (20) that

$$K_n(t)\underset{n\to\infty}{\to}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{2}{t}\right)^n e^{-n}n^{n-\frac{1}{2}}\underset{n\to\infty}{\to}2^{n-1}[(n-1)\,!],$$

the equality on the right deriving from Stirling's formula. Thus as n increases the factor in question tends to unity and the terms in the bracket in (35) when divided by λ tend to the value of the corresponding terms in (34).

The expansion (35) is therefore convergent when (34) is. If we suppose, for example, that λ assumes a value as large as 10, and that the slit-width $2a$ is 2 millimetres, then $\lambda a=1$. In this case an error of a few per cent. would be committed in identifying q with the first term only in (35).

With a narrower slit q is accurately represented by the single term and the treatment of the previous section affords a satisfactory solution. Where streams of ions rather than of electrons are concerned the values of λ are notably greater, and it would be necessary to use narrower slits than with electrons if the first term only of (35) is to represent the distribution accurately.

It may be noted that the strength of the source represented by the first term is proportional to the square of the width of the slit.

IV. *Diffusion in a Magnetic Field.*

There are two principal cases to consider :—

(*a*) The magnetic field H is applied along the direction of the electric field Z.

(*b*) The magnetic field is at right-angles to the direction of Z. This case is of significance only for the slit form of diffusion apparatus.

These cases will be considered in turn.

(*a*) When H acts along Z the coefficient of diffusion in a direction normal to Z is reduced to a value K_H, depending on H, whereas that along Z remains unchanged.

Considering first the type of apparatus in which the ions enter the diffusion chamber through a small hole and are received on a circular central electrode, it can be seen that the differential equation to be solved is

$$K_H\left[\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right]+K\frac{\partial^2 n}{\partial z^2}=W\frac{\partial n}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In (36) we put

$$\sigma^2=\frac{K_H}{K}; \quad \frac{W}{K}=2\lambda ; \quad n=e^{\lambda z}V$$

to obtain

$$\sigma^2\left[\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right]+\frac{\partial^2 V}{\partial z^2}=\lambda^2 V. \quad . \quad . \quad . \quad . \quad (37)$$

We introduce new variables :

$$x'=\frac{x}{\sigma}; \quad y'=\frac{y}{\sigma}; \quad \rho'=(x'^2+y'^2)^{\frac{1}{2}}=\frac{(x^2+y^2)^{\frac{1}{2}}}{\sigma}=\frac{\rho}{\sigma};$$

$$r'=(\rho'^2+z^2)^{\frac{1}{2}}=\left(\frac{\rho^2}{\sigma^2}+z^2\right)^{\frac{1}{2}}.$$

Equation (37) may now be written :

$$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z^2}=\lambda^2 V. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

It is clear, on comparison with (13), that the appropriate solution of (36) is

$$n=Ae^{\lambda z}\frac{z}{r'}\frac{d}{dr'}\left(\frac{e^{-\lambda r'}}{r'}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

When z is constant, $r'dr'=\frac{\rho\, d\rho}{\sigma^2}=\rho'\, d\rho'$, and the following expression is obtained for the proportion R of the whole current received by a central disc of radius b of the receiving electrode:

$$R=\frac{\int_0^b \frac{1}{r'}\frac{d}{dr'}\left(\frac{e^{-\lambda r'}}{r'}\right)\rho\, d\rho}{\int_0^\infty \frac{1}{r'}\frac{d}{dr'}\left(\frac{e^{-\lambda r'}}{r'}\right)\rho\, d\rho}=\frac{\int_h^{d'} d\left(\frac{e^{-\lambda r}}{r}\right)}{\int_h^\infty d\left(\frac{e^{-\lambda r}}{r}\right)},$$

where $$d'=\left(\frac{b^2}{\sigma^2}+h^2\right)^{\frac{1}{2}}.$$

Thus $$R=1-\frac{h}{d'}e^{-\lambda(d'-h)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It is clear from (40) that on applying the field H the proportion of the current taken by the disk of radius b is increased to a value equal to that previously taken by a disc of radius $b\left(\frac{K}{K_H}\right)^{\frac{1}{2}}$ when H was zero.

The behaviour of the stream of electrons or ions in an apparatus in which the ions enter the diffusion chamber through a slit may be similarly investigated. We start with the basic differential equation

$$K_H\frac{\partial^2 q}{\partial x^2}+K\frac{\partial^2 q}{\partial z^2}=W\frac{\partial q}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in place of (14), which, when treated according to the procedure carried out on (36), gives the solution

$$q=Ae^{\lambda z}\frac{z}{r'}K_1(\lambda r'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

in which $$r'=(x'^2+z^2)^{\frac{1}{2}}$$

and $$x'=\frac{x}{u}.$$

The ratio R is given by the following expression, analogous to (27):

$$R=\frac{\int_0^{\tan^{-1}\frac{b}{\sigma h}} K_1(\chi\sec\theta).\sec\theta\, d\theta}{\int_0^{\pi/2} K_1(\chi\sec\theta).\sec\theta\, d\theta}=\frac{\int_0^{\phi_{b'}} K_1(\chi\cosh\phi)\, d\phi}{\int_0^\infty K_1(\chi\cosh\phi)\, d\phi^2} \quad . \quad (43)$$

where $$\cosh\phi_{b'}=\tan\theta_{b'}=\frac{b}{\sigma h}.$$

Since $\chi=20{\cdot}15\dfrac{Zh}{k}$, R is evidently a function $\mathrm{R}\left(\dfrac{Zh}{k}, \dfrac{b}{\sigma h}\right)$ of $\dfrac{Zh}{k}$ and $\dfrac{b}{\sigma h}$. This fact is demonstrated in the figure, where the two curves represent R as a function of $\dfrac{Zh}{k}$ for constant values of $\dfrac{b}{\sigma h}$ of 1/16 and 1/8 respectively. These curves could be used in an experimental determination of σ.

In a powerful field H the divergence of the stream is slight, and (42) and (43) assume simplified forms analogous to (29) and (30) :

$$q=\mathrm{A}e^{-\frac{\lambda x'^2}{2h}}=\mathrm{A}e^{-\frac{\chi}{2}\left(\frac{x}{\sigma h}\right)^2}$$

and

$$\mathrm{R}=\mathrm{Erf}\left[\frac{b}{\sigma h}\left(\frac{\chi}{2}\right)^{\frac{1}{2}}\right].$$

This latter form of solution has been previously obtained by Bailey, who has tested it experimentally [15] and used it to study the properties of a mixed stream of electrons and ions.

(*b*) In considering the effect of a transverse field we suppose the apparatus to be of the slit design and H to be applied parallel to the slit and to the central strip of the receiving electrode.

The differential equation (14) is now replaced by the following :

$$\mathrm{K}\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial z^2}\right)=\mathrm{W}_x\frac{\partial q}{\partial x}+\mathrm{W}_z\frac{\partial q}{\partial z}, \quad \ldots\ldots \quad (44)$$

where W_z is the velocity of drift in the direction of the electric field Z and W_x the transverse velocity acquired under the action of H.

Equation (44) is now written

$$\left.\begin{aligned} &\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial z^2}=2\beta\frac{\partial q}{\partial x}+2\gamma\frac{\partial q}{\partial z}, \\ \text{where}\quad &2\beta=\frac{\mathrm{W}_x}{\mathrm{K}} \quad \text{and} \quad 2\gamma=\frac{\mathrm{W}_z}{\mathrm{K}}. \end{aligned}\right\} \quad \ldots\ldots \quad (45)$$

A quantity λ is also introduced thus :

$$\lambda^2=\beta^2+\gamma^2.$$

Make the substitution $q=e^{\beta x+\gamma z}\mathrm{V}$ in (45).

Then

$$\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial z^2}=\lambda^2\mathrm{V}. \quad \ldots\ldots \quad (46)$$

The required solution of (44) is therefore

$$q=\mathrm{A}e^{\beta x+\gamma z}\frac{z}{r}\mathrm{K}_1(\lambda r). \quad \ldots\ldots \quad (47)$$

The general solution (47) does not readily lend itself to simple discussion,

but when the stream is only moderately divergent ($\lambda h=\chi>15$) we may use the simple form which appears in (28) :

$$q=\frac{Ae^{\gamma z+\beta x-\lambda r}}{r^{3/2}}.$$

It is convenient to introduce quantities ϕ, ψ, and θ in connexion with the definitions given in (45). These quantities are defined as follows :

$$\left.\begin{aligned} &\theta=gd.\phi=\tan^{-1}\frac{x}{h},\\ \text{i. e.,}\quad &\tan\theta=\frac{x}{h}=\sinh\phi,\\ &\sec\theta=\frac{r}{h}=\cosh\phi;\\ \text{also}\quad &\sinh\psi=\frac{\beta}{\gamma}=\frac{W_x}{W_z}. \end{aligned}\right| \qquad \ldots\ldots\ldots \quad (46)$$

The current which is received by that portion of the electrode (the plane $z=h$) lying to the positive side of an abscissa x_1 is proportional to

$$\int_{x_1}^{\infty}\frac{e^{\beta x-\lambda r}}{r^{3/2}}dx=\int_{\phi_1}^{\infty}e^{-\gamma h\cosh(\phi\ \ \psi)}(\cosh\phi)^{-\frac{1}{2}}d\phi,$$

while that to the remaining portion of the electrode is proportional to

$$\int_{-\infty}^{\phi_1}e^{-\gamma h\cosh(\phi-\psi)}(\cosh\phi)^{-\frac{1}{2}}d\phi,$$

where $\sinh\phi_1=\frac{x_1}{h}$. When the stream is not greatly divergent the principal variation of the integrands is due to the exponential factor, the term $(\cosh\phi)^{-\frac{1}{2}}$ remaining close to unity over the effective range of integration.

In such cases the ratio of the currents to the two portions of the receiving electrode on either side of the abscissa x_1 may be written :

$$S=\frac{\int_{\eta_1}^{\infty}e^{-\gamma h\cosh\eta}d\eta}{\int_{-\infty}^{\eta_1}e^{-\gamma h\cosh\eta}d\eta},$$

in which $\eta=(\phi-\psi)$; $\eta_1=(\phi_1-\psi)$.

The ratio S becomes unity when $\eta_1=0$, *i. e.*, $\phi_1=\psi$.

The corresponding abscissa x_1 and direction θ, according to (46), are

$$\frac{x_1}{h}=\tan\theta_1=\sinh\psi=\frac{W_x}{W_z}.$$

The formula

$$\tan\theta_1=\frac{W_x}{W_z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

is the basis of a formula which has been used extensively by Townsend[1],[2] and his associates in investigations on the motions of electrons in gases. The present analysis shows that the formula would not be accurate when applied to a stream of electrons which diverged appreciably, and in such cases it does not appear possible to find any simple formula for the angle θ_1 which bisects the current. It would be possible, however, to estimate θ_1 graphically.

References.

(1) Townsend, J. S., ' Electricity in Gases,' Chap. V.
(2) Townsend, J. S., ' Motion of Electrons in Gases,' Oxford.
(3) Bailey, V. A., Phil. Mag. l. p. 825 (1925).
(4) M. Laporte, ' Les Phénomènes Elémentaires de la Décharge Electrique dans les Gaz,' p. 70.
(5) Pidduck, F. B., ' Treatise on Electricity,' Chap. XII, Art. 218.
(6) ' Electricity in Gases,' pp. 163 and 177.
(6) Bateman, H., ' Partial Differential Equations ' (Cambridge), pp. 343–4.
(8) Lamb, H., ' Hydrodynamics,' 6th ed., p. 498.
(9) Compare Rayleigh, ' Theory of Sound,' Vol. II. Chap. XVII.
(10) Compare Bateman, H., *loc. cit.* p. 386.
(11) MacRobert, T., ' Spherical Harmonics ' (Methuen), p. 273.
(12) Jenke-Emde, ' Tables of Functions,' 3rd ed., pp. 133 and 136.
(13) Janke-Emde, *loc. cit.* p. 236.
(14) Jeffreys, H., ' Operational Methods in Mathematical Physics ': (*a*) p. 110, (*b*) p. 100.
(15) Bailey, V. A., Phil. Mag. ix. pp. 560 and 625 (April 1930).
(16) Green, G., Phil. Mag. xii. p. 233 (1931).

XXXVII. *A Simple Application of the Laplace Transformation.*

By H. S. Carslaw *.

[Received May 10, 1940.

In the November (1939) number of this Journal (pp. 579–583) two problems in Conduction of Heat are solved by Mr. C. J. Tranter using contour integrals and an argument similar to that of Chapter XI. of my book on Conduction of Heat. I would like to point out that these and similar questions can be solved by quite elementary work, using the Laplace Transformation, and I would strongly recommend this method in preference to the earlier and more advanced.

To show how simple it is the discussion of Mr. Tranter's problems is given below on these lines. They are as follows :—

I. Two semi-infinite solids of different materials are in contact along the plane $x=0$. The initial temperature for $x<0$ is a constant different from that for $x>0$.

II. A semi-infinite slab of thickness $2a$ initially at a constant temperature different from zero is placed in an infinite solid of different material initially at zero.

To find the temperature in each case at time $t>0$.

I. With the usual notation in the first problem we required the solution of the equations :—

$$\frac{\partial v_1}{\partial t}=\kappa_1\frac{\partial^2 v_1}{\partial x^2},\ x<0 \qquad : \quad \frac{\partial v_2}{\partial t}=\kappa_2\frac{\partial^2 v_2}{\partial x^2},\ x>0 \ldots t>0. \quad (1)$$

$$v_1=\kappa_1 v_0, \text{ say, when } x<0: \quad v_2=0, \qquad x>0 \ldots t=0. \quad (2)$$

$$\left.\begin{aligned} v_1&=v_2 \\ K_1\frac{\partial v_1}{\partial x}&=K_2\frac{\partial v_2}{\partial x}\end{aligned}\right\}, \text{ when } x=0, \qquad t>0. \quad (3)$$

Multiply (1) by $e^{-pt}(p>0)$, integrate both sides with respect to t from 0 to ∞, and also integrate the left-hand sides by parts. Write $\bar{v}$ for the Laplace Transform of v ; *i. e.*,

$$\bar{v}=\int_0^\infty e^{-pt}v\,dt \quad (p>0).$$

---

* Communicated by the Author.

In place of the above system of equations we thus obtain "subsidiary equations" in $\bar{v}_1$ and $\bar{v}_2$, and new boundary conditions, from which we are able to determine $\bar{v}_1$ and $\bar{v}_2$.

Then we write down v_1 and v_2 from the values thus found for $\bar{v}_1$ and $\bar{v}_2$.

The subsidiary equations are

$$\frac{d^2\bar{v}_1}{dx^2} - q_1^2\bar{v}_1 = -v_0, \; x<0 \quad : \quad \frac{d^2\bar{v}_2}{dx^2} - q_2^2\bar{v}_2 = 0, \; x>0, \qquad (4)$$

where $$q_1^2 = p/\kappa_1 \quad \text{and} \quad q_2^2 = p/\kappa_2. \qquad (5)$$

Also $$\bar{v}_1 = \bar{v}_2$$

and $$K_1\frac{d\bar{v}_1}{dx} = K_2\frac{d\bar{v}_2}{dx} \Big\}, \text{ when } x=0. \qquad (6)$$

From (5) we have

$$\bar{v}_1 = \frac{v_0}{q_1^2} + Ae^{-q_1|x|}, \quad x<0 \Big\}, \qquad (7)$$

and $$\bar{v}_2 = Be^{-q_2x}, \quad x>0 \Big\}$$

the negative index in (7) being chosen, as $\bar{v}_1$ and $\bar{v}_2$ are not to be infinite when $|x| \to \infty$.

Using (6) these give at once

$$\bar{v}_1 = \frac{v_0\kappa_1}{p}\left(1 - \frac{\sigma}{1+\sigma}e^{-q_1|x|}\right), \quad x<0 \Big\}, \qquad (8)$$

and $$\bar{v}_2 = \frac{v_0\kappa_1}{p(1+\sigma)}e^{-q_2x}, \quad x>0 \Big\}$$

where $$\sigma = \frac{K_2q_2}{K_1q_1} = \frac{K_2\sqrt{\kappa_1}}{K_1\sqrt{\kappa_2}} \qquad (9)$$

Now it is known from elementary integrals that, when $x>0$,

$$\frac{e^{-\sqrt{p}.x}}{p} = \int_0^\infty e^{-pt}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{t}}\right)dt, \qquad (10)$$

where $$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du.$$

Also $$\frac{1}{p} = \int_0^\infty e^{-pt}dt.$$

It follows from (8) that

$$v_1 = \frac{v_0\kappa_1}{1+\sigma}\left[1 + \sigma \operatorname{erf}\left(\frac{|x|}{2\sqrt{(\kappa_1 t)}}\right)\right] \Big\}, \qquad (11)$$

and $$v_2 = \frac{v_0\kappa_1}{1+\sigma}\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{(\kappa_2 t)}}\right)\right] \Big\}$$

II. In this problem it is clear that we need only consider the region $x>0$ and there is no flow of heat across the plane $x=0$.

We have thus to solve the equations :—

$$\frac{\partial v_1}{\partial t}=\kappa_1\frac{\partial^2 v_1}{\partial x^2}, \qquad 0<x<a \quad : \quad \frac{\partial v_2}{\partial t}=\kappa_2\frac{\partial^2 v_2}{\partial x^2}, \quad x>a \ldots \quad t>0. \tag{12}$$

$$v_1=\kappa_1 v_0, \text{ say, when } 0<x<a: \qquad v_2=0, \qquad x>a \ldots \quad t=0. \tag{13}$$

$$\frac{\partial v_1}{\partial x}=0, \text{ when } x=0, \qquad t>0. \tag{14}$$

$$v_1=v_2 \quad \text{and} \quad \mathrm{K}_1\frac{\partial v_1}{\partial x}=\mathrm{K}_2\frac{\partial v_2}{\partial x}, \qquad \text{when} \quad x=a \ldots \quad t>0. \tag{15}$$

Proceeding as above, we obtain the following equations for $\bar{v}_1$ and $\bar{v}_2$:

$$\frac{d^2\bar{v}_1}{dx^2}-q_1^2\bar{v}_1=-v_0, \quad 0<x<a \quad : \quad \frac{d^2\bar{v}_2}{dx^2}-q_2^2\bar{v}_2=0, \quad x>a. \tag{16}$$

$$\frac{d\bar{v}_1}{dx}=0, \text{ when } x=0. \tag{17}$$

$$\bar{v}_1=\bar{v}_2 \quad \text{and} \quad \mathrm{K}_1\frac{d\bar{v}_1}{dx}=\mathrm{K}_2\frac{d\bar{v}_2}{dx} \quad \text{when } x=a. \tag{18}$$

From (16)
$$\left.\begin{aligned}\bar{v}_1&=\frac{v_0}{q_1^2}+\mathrm{A}e^{q_1x}+\mathrm{B}e^{-q_1x},\\ v_2&=\mathrm{C}e^{-q_2x}.\end{aligned}\right\} \tag{19}$$

Also (17) gives A=B and (18) determines A and C. We thus find without difficulty

$$\bar{v}_1=\frac{v_0\kappa_1}{p}\left[1-\frac{\sigma}{1+\sigma}\left(\frac{e^{-q_1(a-x)}+e^{-q_1(a+x)}}{1-ye^{-2q_1a}}\right)\right], \tag{20}$$

where
$$y=(1-\sigma)/(1+\sigma). \tag{21}$$

Also
$$\bar{v}_2=\frac{v_0\kappa_1}{p(1+\sigma)}\frac{(1-e^{-2q_1a})}{(1-ye^{-2q_1a})}e^{-q_2(x-a)}. \tag{22}$$

We can write (19) in the form

$$\bar{v}_1=\frac{v_0\kappa_1\sigma}{p(1+\sigma)}\sum_0^\infty y^n[(1-e^{-q_1[(2n+1)a-x]})+(1-e^{-q_1[(2n+1)a+x]})]. \tag{23}$$

Therefore, by (10),

$$v_1=\frac{v_0\kappa_1\sigma}{1+\sigma}\sum_0^\infty y^n\left[\operatorname{erf}\frac{[(2n+1)a-x]}{2\sqrt{(\kappa_1 t)}}+\operatorname{erf}\frac{[(2n+1)a+x]}{2\sqrt{(\kappa_1 t)}}\right]. \tag{24}$$

Also from (22)

$$\bar{v}_2=\frac{v_0\kappa_1}{p(1+\sigma)}\left[1-(1-y)\sum_0^\infty y^n(e^{-2q_1(n+1)a})\right]e^{-q_2(x-a)}. \tag{25}$$

Therefore

$$v_2 = \frac{v_0\kappa_1}{1+\sigma}\left[\left(1-\operatorname{erf}\frac{(x-a)}{2\sqrt{(\kappa_2 t)}}\right)-(1-y)\sum_0^\infty y^n\left(1-\operatorname{erf}\left(\frac{(x-a)}{2\sqrt{(\kappa_2 t)}}+\frac{(n+1)a}{\sqrt{(\kappa_1 t)}}\right)\right)\right]$$

$$= \frac{v_0\kappa_1}{(1+\sigma)}\left[-\operatorname{erf}\frac{(x-a)}{2\sqrt{(\kappa_2 t)}}+(1-y)\sum_0^\infty y^n \operatorname{erf}\left(\frac{(x-a)}{2\sqrt{(\kappa_2 t)}}+\frac{(n+1)a}{\sqrt{(\kappa_1 t)}}\right)\right]. \quad . \quad (26)$$

Examples of the use of the Laplace Transformation in the solution of boundary problems have been given in a former paper *, where references to the literature will be found.

It may be remarked that in obtaining the equations for $\bar{v}$ from those for v, some assumptions as to the nature of v have been made. The applied mathematician would probably pass these. The analyst's objections may best perhaps be met by verification of the results obtained.

The connexion between the Laplace Transformation method and that using contour integrals, by means of the Mellin Inversion Formula †, is well known. After much practice with both, the former seems to me much the better. Also I believe that engineers and others who use the Heaviside Operational Calculus would find their work simpler and much more convincing, if they adopted the Laplace Transformation procedure.

---

* Carslaw and Jaeger, Phil. Mag. (ser. 7) xxvi. p. 473 (Oct. 1938).

† *Loc. cit.* p. 475.

XXXVIII. *A Contribution to the Theory of Elasticity of Non-Isotropic Materials (with Applications to Problems of Bending and Torsion).*

By Eric Reissner, Ph.D.,
Massachusetts Institute of Technology *.

[Received June 13, 1939.]

This paper contains a contribution to the two-dimensional theory of elasticity of materials of rectilinear anisotropy. The theory and its applications presented are very much simpler than the corresponding developments for isotropic materials. The simplification is due to the fact that one of the two moduli of elasticity which occur in plane stress is assumed equal to zero, which means that the material is not resistant to normal stresses in a certain direction. A material for which these assumptions are very nearly true is a corrugated sheet in which pitch and depth of the corrugations are small compared with the overall dimensions of the sheet.

In I. the general theory is given, that is expressions are developed for stresses and displacements in terms of arbitrary functions which have to be determined by boundary conditions. In II. bending of a narrow rectangular beam is considered, and an exact solution—so far missing for a beam of isotropic material—is given which shows in which way the stress distribution depends on the way the beam is supported and the loading is applied. In III. applications to the theory of bending of wide-flanged box beams are indicated. In IV. torsion of box beams is treated, the end-sections of which are restrained against warping. Some simple formulæ for the relation between Bredt's shear stress and the secondary normal stresses are given.

I.

Calling σ_x, σ_y normal stress components in x- and y-direction, $\tau_{xy}=\tau_{yx}=\tau$ the shear stress components, u and v the displacements in x- and y-direction, the problem of plane stress for a material of rectilinear homogeneous anisotropy consists in finding a solution of the following system of equations :—

---

* Communicated by the Author.

1. The Equations of Equilibrium :

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau}{\partial y}&=0,\\ \frac{\partial \tau}{\partial x}+\frac{\partial \sigma_x}{\partial y}&=0.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

2. The Stress-Strain Relations :

$$\left.\begin{aligned}\epsilon_x\equiv\frac{\partial u}{\partial x}&=\frac{1}{E_x}(\sigma_x-\nu_x\sigma_y),\\ \epsilon_y\equiv\frac{\partial v}{\partial y}&=\frac{1}{E_y}(\sigma_y-\nu_y\sigma_x),\\ \gamma\equiv\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}&=\frac{1}{G}\tau.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

The elastic constants E_x, E_y, ν_x, ν_y, G are not independent of each other. It is known that the existence of a strain energy function implies that

$$E_x\nu_y=E_y\nu_x. \quad \ldots\ldots\ldots \quad (3)$$

Assuming

$$E_y=0, \quad \ldots\ldots\ldots \quad (4)$$

it follows from (3) that also

$$\nu_y=0, \quad \ldots\ldots\ldots \quad (5)$$

and the system of equations (2) becomes

$$\left.\begin{aligned}E_x\frac{\partial u}{\partial x}&=\sigma_x-\nu_x\sigma_y,\\ 0&=\sigma_y,\\ G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)&=\tau.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2a)$$

The result that $\sigma_y=0$, which follows in this case from the stress-strain relations, is physically necessary since a finite σ_y-stress would cause infinite normal strain in the y-direction. In reality there is, of course, no material for which actually $E_y=0$, and the theory has to be considered as a limiting case for materials with very small E_y/E_x.

To make such an ideal material with $E_y=0$ resistant to forces in y-direction stiffening members in this direction have to be assumed, as will be shown in the examples treated subsequently.

Putting $\sigma_y=0$ in the equations of equilibrium (1) leads to the following general expressions for the remaining stress-components :

$$\left.\begin{aligned}\tau&=f(y),\\ \sigma_x&=-xf'(y)+g(y),\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

where f and g are arbitrary functions. From the remaining two stress-strain relations (2 a) one finds the displacements u and v by integration in the following form :

$$\left.\begin{aligned} u &= \frac{1}{E_x}\left[-\frac{x^2}{2}f'(y)+xg(y)+h(y)\right], \\ v &= \frac{1}{E_x}\left[\frac{x^3}{6}f''(y)-\frac{x^2}{2}g'(y)+\frac{E_x}{G}xf(y)+k(y)\right]. \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

Here h and k are two more arbitrary functions.

Equations (6) and (7) represent the general solution of the problem. Boundary conditions along lines $x=$const. give total differential equations for the four arbitrary functions. Boundary conditions along lines $y=$const. serve to determine the constants in the solutions of the boundary differential equations.

II.

As first application of the preceding solution the following problem is considered : To determine the distribution of stresses in a beam of narrow rectangular section resisting to normal stresses only in spanwise direction (fig. 1). It will be found out how far the results of the elementary theory of beam bending are in error for a material of this sort, when the height of the beam is comparable with its length.

The stress distribution in the beam depends on the way the end is fixed and on the way the loads are applied. In order to fix the ideas it shall be assumed that the beam is a cantilever, the end built-in in such a way that both displacements vanish and that at the free end the load is applied through a stiffener, rigid in the direction of its own axis and non-resisting to bending in the plane of the flat sheet representing the beam. (These assumptions are very nearly true for ordinary stiffener sizes.) It will be noted that this same problem has not yet been solved rigorously for the beam of isotropic material, the difficulty consisting there in the satisfaction of all conditions at the built-in end. The inherent reason for this difficulty is that for the isotropic beam *à priori* assumptions concerning the stress distribution are made (semi-inverse method), whereas here the stress distribution is determined directly from the prescribed boundary conditions.

Mathematically these boundary conditions are for the case here considered

$$\left.\begin{aligned} &x=0\,; \quad u=0,\ v=0, \\ &x=l\,; \quad \sigma_x=0,\ v=\text{const.}, \\ &y=\pm h\,; \quad \tau=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The conditions at $x=0$ give with u and v from (7)

$$h(y)=0, \quad k(y)=0. \qquad (9)$$

The conditions at $x=l$ become

$$\left.\begin{aligned} -lf'()+g(y)&=0y, \\ \frac{l^3}{6}f''(y)-\frac{l^2}{2}g'(y)+l\frac{E_x}{G}f(y)&=c. \end{aligned}\right\} \qquad (10)$$

From (10) follows for f and g

$$\left.\begin{aligned} f(y)&=c_1\sinh\kappa y+c_2\cosh\kappa y-\frac{lcG}{E_x}, \\ g(y)&=lf'(y)=l\kappa[c_1\cosh\kappa y+c_2\sinh\kappa y], \end{aligned}\right\} \qquad (11)$$

where $$\kappa=\sqrt{\frac{3E_x}{G}}\frac{1}{l}.$$

The stresses are

$$\tau=f(y), \quad \sigma_x=(l-x)f'(y). \qquad (12)$$

The condition that at the upper and lower edge the shear stress vanishes gives

$$\pm c_1\sinh\kappa h+c_2\cosh\kappa h-\frac{lcG}{E_x}=0, \qquad (13)$$

and hence

$$c_1=0, \quad c=c_2\frac{E_x}{lG}\cosh\kappa h. \qquad (14)$$

The stresses are then given by

$$\left.\begin{aligned} \tau&=c_2(\cosh\kappa y-\cosh\kappa h), \\ \sigma_x&=c_2(l-x)\,\kappa\sinh\kappa y. \end{aligned}\right\} \qquad (15)$$

The value of the constant c is obtained by equating the resultant of the shear stress τ equal to the applied load P.

$$P=t\int_{-h}^{h}\tau\,dy, \quad c_2=\frac{P}{2t}\kappa\,(\sinh\kappa h-\kappa h\cosh\kappa h)^{-1}.$$

With that the stresses become, writing for brevity,

$$\left.\begin{aligned} \rho&=\sqrt{\frac{3E_x}{G}}, \\ \tau&=\frac{P}{2th}\frac{\rho(h/l)[\cosh\rho(h/l)-\cosh\rho(y/l)]}{\rho(h/l)\cosh\rho(h/l)-\sinh\rho(h/l)}, \\ \sigma_x&=\frac{P}{2th}\left(1-\frac{x}{l}\right)\frac{\rho^2(h/l)\sinh\rho(y/l)}{\rho(h/l)\cosh\rho(h/l)-\sinh\rho(h/l)}. \end{aligned}\right\} \qquad (16)$$

For small values of h/l these expressions become

$$\left.\begin{aligned} \tau &= \frac{P}{2th}\frac{3}{2}\left(1-\left(\frac{y}{h}\right)^2\right), \\ \sigma_x &= \frac{3}{2}\frac{P}{th^3}(l-xy), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (17)$$

which is in accordance with the elementary theory of beam bending. This has, of course, to be so since in the elementary theory the resistance to the normal stress σ_y is not taken into consideration. It can be seen from the expressions (16) for the stresses that an increase of the ratio h/l tends to increase the maximum normal stress compared with that

Fig. 1.

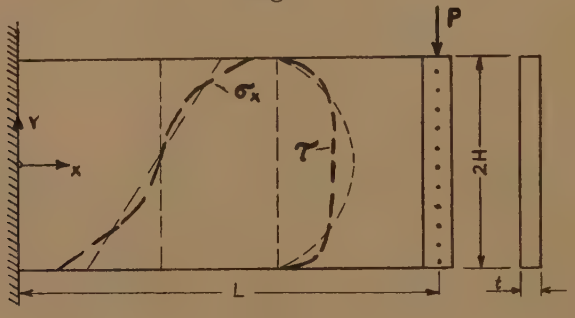
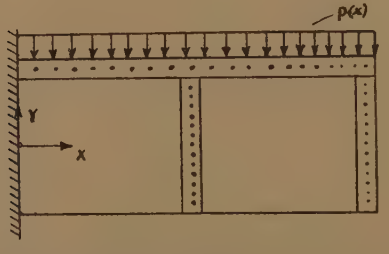


Fig 2.



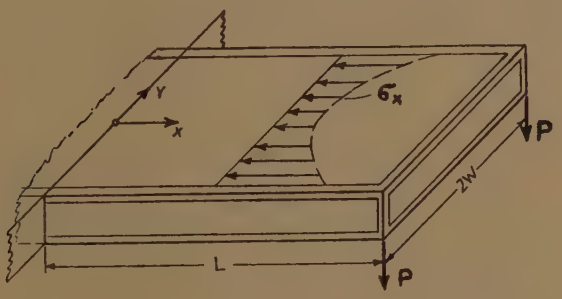
same stress resulting from the elementary theory, and to decrease the ratio of the corresponding shear stresses (fig. 1). It would be interesting to find out whether the same is true for a beam of isotropic material, the conditions of loading and support being the same as here assumed.

In an analogous manner one may treat further beam bending problems. One may take into consideration the elastic behaviour of the transverse stiffeners in more detail. Or one may take different load distributions. In this case one has always to assume that the sheet is loaded only by means of the stiffeners, and that a continuously distributed load necessitates edge stiffeners in spanwise direction which unload themselves to the transverse stiffeners which then stress the sheet material (fig. 2).

III.

A useful application of the theory is to the calculation of the efficiency of wide-flanged box-beams. For a material of the sort considered here it is possible to take into consideration the influence of spanwise as well as of transverse stiffeners quite readily. This has not been possible so far for cover-sheets of isotropic material *. It is noteworthy that such wide cover-sheets frequently are made from corrugated sheets, so that this theory applies very well.

Fig. 3.



IV.

As a further application the problem of the influence of end-restraints on the torsion of box beams will be considered †. The results of the calculations given here may be expected to hold quantitatively for box beams of corrugated material. To fix the ideas the two following simple structures will be analyzed :—

1. A box beam of length l and width $2w$, built in on one end and loaded at the free end by two equal and opposite forces applied to the side-beams (fig. 4). The transverse frame at the free end of the beam is assumed to be constructed in such a way that no resultant loads in their plane are transmitted to the horizontal webs. Four equal flange stiffeners of area A are assumed, and a transverse end stiffener rigid in the direction of its

* For references, detailed discussion of the problem, and examples worked out see E. Reissner, J. Ae. Sc. v. p. 294 (1938), H. Katz, M.I.T. Thesis, 1938, J. F. Aldridge, M.I.T. Thesis, 1939.

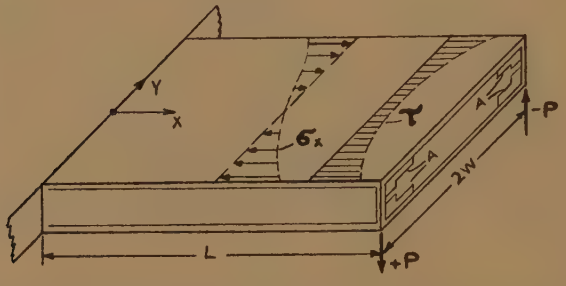
† This problem has been treated by H. Reissner, Zeitschr. f. Flugt. u. Motorluftschiff. xvii. p. 384 (1926), and also by D. Williams, British A.R.C., R. and M. 1761 (1936).

axis and not resisting to bending in the plane of the horizontal webs. The boundary conditions are then

$$\left.\begin{aligned} &x=0\,; \quad u=0, \quad v=0, \\ &x=l\,; \quad \sigma_x=0, \quad v=0, \\ &y=0\,; \quad \sigma_x=0, \\ &y=\pm w\,; \quad \pm A\frac{\partial\sigma_x}{\partial x}-t\tau=\frac{P}{h}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The significance of all but the last of eq. (18) is evident. The last condition expresses the equilibrium condition for an element of the longitudinal stiffener of area A under the influence of the shear in side webs and horizontal webs.

Fig. 4.



The conditions along $x=0$ and $x=l$ give, with the help of the general solutions (6) and (7) in I.,

$$\left.\begin{aligned} &\tau=f(y)=c_1\sinh\kappa y+c_2\cosh\kappa y, \\ &\sigma_x=(l-x)f'(y)=(l-x)\kappa(c_1\cosh\kappa y+c_2\sinh\kappa y), \\ &\kappa=\sqrt{\frac{3E_x}{G}}\frac{1}{l}. \end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

The symmetry condition along $x=0$ gives

$$c_1=0.$$

The equilibrium condition for the edge stiffener gives then

$$-A\kappa c_2\sinh\kappa w-tc_2\cosh\kappa w=\frac{P}{h}, \quad . \quad . \quad . \quad . \quad (20)$$

and with that the stresses are

$$\left.\begin{aligned} \tau&=-\frac{P}{ht}\frac{\cosh\kappa y}{\cosh\kappa w+(A/tw)\kappa w\sinh\kappa w}, \\ \sigma_x&=-\frac{P}{ht}\left(1-\frac{x}{l}\right)\frac{\kappa l\sinh\kappa y}{\cosh\kappa w+(A/tw)\kappa w\sinh\kappa w}. \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

These formulæ show that the shear stress decreases from the edges towards the middle of the sheet and that the presence of the edge-stringer reduces the shear in the horizontal webs.

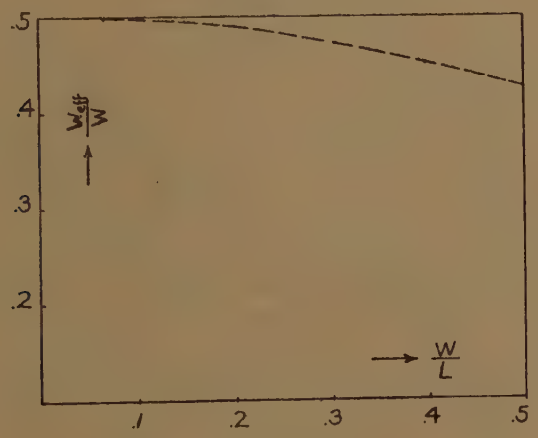
It is of interest to compute the effective width of the horizontal flanges in bending, which, according to the elementary theory of bending, would be $w_{\text{eff}}=\frac{1}{2}w$.

Here one has (fig. 5)

$$\frac{w_{\text{eff}}}{w}=\frac{\int_0^w \sigma_x\,dy}{w(\sigma_x)_{y=w}}=\frac{\cosh \kappa w-1}{\kappa w \sinh \kappa w}\approx\frac{1}{2}\left(1-\frac{1}{12}(\kappa w)^2\right). \quad . \quad . \quad (22)$$

2. As a second example a box beam is considered whose side webs are about as wide as its horizontal webs, and the load is applied through a frame rigid in its plane (fig. 6).

Fig. 5.



For the sake of simplicity no corner stiffeners will be assumed in this case.

The boundary conditions are

$$x=0\,;\quad u_1=0,\quad v_1=0,\quad u_2=0,\quad v_2=0, \quad . \quad . \quad . \quad . \quad (23)$$

$$x=l\,;\quad \sigma_{x_1}=0,\quad \sigma_{x_2}=0,\quad v_1=hc,\quad v_2=wc. \quad . \quad . \quad . \quad (24)$$

(Here the constant c is to be determined such that a torque T results.) For reason of symmetry it is

$$\left.\begin{aligned} y_1=0\,;\quad \sigma_{x_1}=0,\\ y_2=0\,;\quad \sigma_{x_2}=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the equilibrium and continuity conditions along the corners are

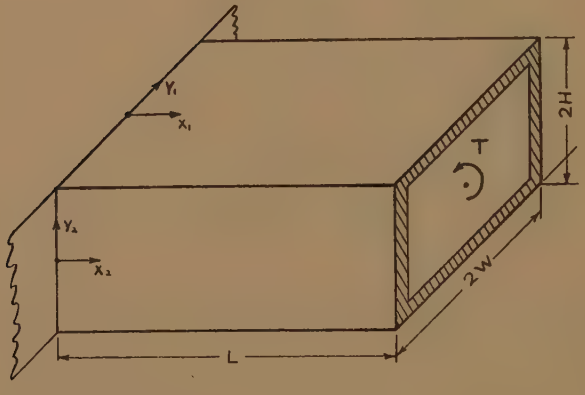
$$\left.\begin{aligned} y_1&=-w, \\ y_2&=h, \end{aligned}\right\} \quad \left.\begin{aligned} \sigma_{x_1}&=\sigma_{x_2}, \\ t_h\tau_2&=t_w\tau_1=0. \end{aligned}\right\} \quad \ldots\ldots \quad (26)$$

From (23) and (24) follows

$$\left.\begin{aligned} f_1(y_1)&=c_1 \sinh \kappa y_1+c_2 \cosh \kappa y_1+\mathrm{G}\frac{hc}{l}, \\ f_2(y_2)&=d_1 \sinh \kappa y_2+d_2 \cosh \kappa y_2+\mathrm{G}\frac{wc}{l}, \end{aligned}\right\} \quad \ldots\ldots \quad (27)$$

$$k(y)=0, \quad k(y)=0, \quad g(y)=lf'(y). \quad \ldots\ldots \quad (28)$$

Fig. 6.



The symmetry conditions (25) give

$$c_1=0, \quad d_1=0, \quad \ldots\ldots \quad (29)$$

and the remaining conditions (26) lead to

$$\left.\begin{aligned} -c_2 \sinh \kappa w&=d_2 \sinh \kappa h, \\ t_h\left(d_2 \cosh \kappa h+\mathrm{G}\frac{wc}{l}\right)&=t_w\left(c_2 \cosh \kappa w+\frac{\mathrm{G}hc}{l}\right). \end{aligned}\right\} \quad \ldots \quad (30)$$

This gives

$$\left.\begin{aligned} d_2&=\frac{\mathrm{G}c}{l}\,\frac{\sinh \kappa w(t_w h-t_h w)}{t_w \sinh \kappa h \cosh \kappa w+t_h \cosh \kappa h \sinh \kappa w}, \\ c_2&=-\frac{\mathrm{G}c}{l}\,\frac{\sinh \kappa h(t_w h-t_h w)}{t_w \sinh \kappa h \cosh \kappa w+t_h \cosh \kappa h \sinh \kappa w}. \end{aligned}\right\} \quad \ldots \quad (31)$$

With that the stresses become

$$\left.\begin{aligned}
\tau_1 = f_1(y) &= \frac{Gc}{l}\left\{h - \frac{(t_w h - t_h w)\sinh \kappa h \cosh \kappa y}{t_w \sinh \kappa h \cosh \kappa w + t_h \cosh \kappa h \sinh \kappa w}\right\},\\
\tau_2 = f_2(y) &= \frac{Gc}{l}\left\{w + \frac{(t_w h - t_h w)\sinh \kappa w \cosh \kappa y}{t_w \sinh \kappa h \cosh \kappa w + t_h \cosh \kappa h \sinh \kappa w}\right\},\\
\sigma_{x_1} = (l-x)f_1'(y) &= -\frac{Gc}{l}\kappa(l-x)\frac{(t_w h - t_h w)\sinh \kappa h \sinh \kappa y}{t_w \sinh \kappa h \cosh \kappa w + t_h \cosh \kappa h \sinh \kappa w},\\
\sigma_{x_2} = (l-x)f_2'(y) &= +\frac{Gc}{l}\kappa(l-x)\frac{(t_w h - t_h w)\sinh \kappa w \sinh \kappa y}{t_w \sinh \kappa h \cosh \kappa w + t_h \cosh \kappa h \sinh \kappa w}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

In these expressions the constant c follows from

$$2ht_w \int_w^w \tau_1 dy_1 + 2wt_h \int_h^{-h} \tau_2 dy_2 = T.$$

Equations (32) show :

1. Constant shear and zero normal stresses occur if

$$\frac{t_w}{t_h} = \frac{w}{h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

that is if the thicknesses of the sheet have the same relation as the corresponding widths *.

2. The same holds if ky is so small that one may write

$$\sinh \kappa y \approx \kappa y, \quad \cosh \kappa y \approx 1,$$

that is for beams the length l of which is large compared with the cross-sectional diameters.

Of especial interest is the relation between the shear stress and the normal stress in the sheet. The ratio of the two is especially simple computed along the edge ($y_1 = -w$, $y_2 = h$). One has

$$\frac{\sigma_{x_1}(0, w)}{\tau_1(w)} = \frac{\kappa l(t_w h - t_h w)\sinh \kappa h \sinh \kappa w}{t_h[h \cosh \kappa h \sinh \kappa w + w \sinh \kappa h \cosh \kappa w]}. \quad . \quad . \quad (34)$$

For sufficiently large values of l this becomes

$$\frac{\sigma_{x_1}(0, w)}{\tau_1(w)} \approx \frac{\kappa l(t_w h - t_h w)\kappa^2 hw}{t_h(h\kappa w + w\kappa h)} = \frac{\kappa^2 l(t_w h - t_h w)}{2t_h},$$

and with $\kappa^2 = 3E_x/Gl^2$, $E_x/G = 2{\cdot}5$,

$$\left(\frac{\sigma_x}{\tau}\right)_{\max.} \approx \frac{15}{4}\frac{w}{l}\left(1 - \frac{h}{w}\frac{t_w}{t_h}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This is a simple result for the ratio of maximum normal stress and maximum shear stress.

---

* This same result for beams of isotropic material has been found by H. Reissner, *loc. cit.*

XXXIX. *Note on the Barrier-layer Resistance of Selenium Photo-elements.*

By Arne Eld Sandström *.

[Received February 13, 1940.]

1. In a previous paper † the author published the results of some experiments concerning changes in the barrier-layer resistance of selenium photo-elements after exposure either to "visible" light or to X-rays. This resistance was found to be a function of the time which had elapsed since the element was restored to darkness. However, it escaped the author that this change of resistance had been observed by Lange in 1931 ‡. Lange mentions this phenomenon in a short paragraph only, supposing the "tiring" of photo-elements to be due to this change of resistance. In all his measurements on photo-elements the author until recently had failed to observe any "tiring." However, a closer examination revealed the fact that the elements really did show a "tiring" effect, if the external resistance was high enough. As most of the measurements were carried out with as low external resistances as possible, it is easily understood that no "tiring" was observed, especially as it was found that comparatively high external resistances were needed for the effect to appear.

A research by Elvegård, Lindroth, and Larsson § revealed the fact that the "tiring" of at least some types || of selenium photo-elements depends on the wave-length of the light. For the wave-lengths 5250 and 5400 A. they found an *increase* in the initial photo-current. This apparently indicates that there is no *simple* relation between the "tiring" effect and the change in barrier-layer resistance, as independently of the wave-length the resistance of an element is considerably lower during exposure than in darkness.

2. It is possible to increase the resistance of a selenium barrier-layer element by means of electric shocks ¶. In this way the author contrived

---

* Communicated by Professor A. E. Lindh.
† Arne Eld Sandström, Phil. Mag. xxviii. p. 642 (1939).
‡ Bruno Lange, *Phys. Zeit.* xxxii. p. 850 (1931).
§ E. Elvegård, S. Lindroth and E. Larsson, J. O. S. A. xxviii. p. 33 (1938).
|| Zierold-elements.
¶ Hans Schweickert, *Zeits. f. Phys.* cix. p. 413 (1938).

to change the resistance of an element from 6700 to 11,800 ohms (when kept in darkness). The change of the barrier-layer resistance after the exposure of the element to " visible " light was then determined once more. The time of exposure was so long (20 min.) that the resistance certainly had reached its minimum value. It was found that the general shape of the curve representing the increase of the resistance of the element when restored to darkness is the same in the case of the higher resistance as in the case of the lower.

As mentioned in a previous paper, an element without any apparent barrier-layer did not show a lower resistance after exposure to light than before. This same element was also treated by electric shocks. A marked increase of resistance was observed and at the same time the barrier-layer began to develop. Then, after exposure to light the resistance was found to be lower than when the element had been in darkness for some time. However, in the case of this element the return of the resistance to the higher value was more swift than in the case of the other element.

3. During the exposure the resistance of a selenium photo-element is very low. The aforementioned element having a resistance of 11,800 ohms while in darkness was exposed to a very intense flood of light. Its resistance was then found to be 150 ohms approximately. In earlier experiments the resistance had invariably been found to change for very low voltages in a rectilinear way *. Thus, for the moments following close upon an exposure it was possible to calculate the changes of the barrier-layer resistance from the variations of the current for a given voltage (some millivolts) between the electrodes of the element. 15 seconds after the end of the exposure the resistance was 4500 ohms. Thus, there must be a nearly immediate increase of the resistance when the element is restored to darkness.

Laboratory of Physics,
University of Uppsala.

---

* Arne Eld Sandström, *l. c.* fig. 4, p. 643 (1939).

XL. OBITUARY.

Sir J. J. THOMSON, O.M., M.A., D.Sc., Sc.D., LL.D., F.R.S.

Born Dec. 18, 1856. Died Aug. 30, 1940.

Few of the great Victorians are left to contemplate the changed aspect of a world which in their heyday seemed fitted to survive all the stresses that fate might call on it to endure. None the less, as we see it now, that Victorian society carried within itself the seeds of change and was already moving, slowly, but with ever-increasing speed, along the road that led to the catastrophe of a second world-war.

Ethical implications apart, physical science, during the last fifty or sixty years, has witnessed a similar revolution. To the physicist of the eighteen seventies or eighties, the foundations of his science seemed to be enduringly laid. Lord Kelvin could say that of nothing was he more firmly convinced than of the reality and substantiality of the luminiferous ether ; and more than one eminent physicist took the view that future advances would depend on a discussion of the reasons for the small differences obtained in the accurate measurements of physical constants. This, within a few years of the discovery of Röntgen radiation, of radio-activity, of the publication of the first of Planck's great papers, and, perhaps most fundamental of all, of the enunciation of that concept which was the first step towards the correlation of so many perplexing phenomena—the concept of the electron.

In this great revolution, the genius of J. J. Thomson played a dominating part.

Thomson was born at Cheetham, on the outskirts of Manchester on December 18th, 1856. He was intended for the profession of an engineer, and to that end was sent, by a fortunate chance, to Owens College, then housed in Cobden's former residence in Deansgate. The College had just turned the corner and was struggling to success. Conditions were terribly cramped, but an institution which could reckon among its teachers such outstanding figures as those of Roscoe, Balfour Stewart, Osborne Reynolds, Stanley Jevons, A. W. Ward, and Bryce could be trusted to rise above mere straitness of material circumstance. And a college housed in a city which was the scene of the labours of Dalton and of Joule, which had fostered the growth of the greatest of our provincial philosophical societies, was not likely to be lacking in devotion to the physical sciences.

Barker's mathematical teaching, Balfour Stewart's physical researches, and Osborne Reynolds's intense originality of outlook—" he was one of the most original and independent of men and never did anything or expressed himself like anybody else "—all played their share in shaping Thomson's mind, and so helping to determine the pace and direction of the future advances of physical science.

Thomson entered Trinity College, Cambridge, as a scholar in 1876, and kept every term from that date until his death on August 30th, 1940—an almost unexampled record. He coached with Routh, the most famous of all the

famous mathematical teachers that Cambridge has bred. He took the Tripos in January 1880, and came out as Second Wrangler, with Larmor as Senior.

There was no organised laboratory teaching of undergraduates in Thomson's undergraduate days, and after taking his degree he began his work at the Cavendish Laboratory by making some experiments suggested by certain aspects of Maxwell's theory, and then undertaking a now well-known determination of "*v*." About this period, too, inspired by Crookes's experiments on cathode rays, he began mathematical investigations dealing with the properties of moving electric charges. In 1884, when the late Lord Rayleigh resigned the Cavendish Chair, Thomson was chosen as his successor ("Things in the University have come to a pretty pass," said a well-known college tutor, "when mere boys are made Professors"), and entered on that brilliant career which has so signally and beneficially influenced the progress of physical science.

There is small need to tell again the story of those years during which J. J. *was* the Cavendish Laboratory; they have left their mark on the world's history, and the tasks of those years, and the affection of all those who knew him and who laboured with him, are a monument to his memory more enduring than brass. The discovery of the electron and the first measurements of the electronic charge; the positive ray spectrum and the discovery of isotopes; the admission in 1895 of research students from other universities and the resulting stream of ancillary and independent researches; the application of Lagrangian methods to problems of physics and chemistry; the great treatise on the conduction of electricity in gases; the work of the war years of 1914–18; all this immense output, inspired and guided by the genius of the Cavendish Professor, gave to the Laboratory a standing second to none in the great laboratories of the world.

Deserved honours fell to his share. He was President of the Royal Society from 1915 to 1920; he was President of the British Association in 1909; he was a member of the Order of Merit, and a Nobel Laureate; in 1918 he was chosen as Master of Trinity.

But Thomson was greater than the greatest of his achievements. To know him was a liberal education, and no man held more securely the affection of his students. His work will live as long as physical science endures, and his personal memory and influence will continue to leave its mark on humanity through years as yet untold. For, to those of us who hold the belief that the human race, despite discouragements and failures, is still striving towards the light, and will emerge victorious from that struggle, the example and effect of a life such as that of J. J. Thomson, simple, direct, kindly and upright, will ever serve as a sign that the enduring things are good, and that evil, though it may have its day, will one day cease to be.

XLI. *Notices respecting New Books.*

The Physical Examination of Metals.—Vol. I. *Optical Methods.* By Bruce Chalmers. [Pp. 181+viii.] (London: Edward Arnold & Co. Price 14*s.* net.)

So much work has been carried out by physicists during the last few years relating to the structure and properties of metals that the title of the present book leads us to expect something quite different from its actual contents. The preface, however, soon corrects any misunderstanding regarding the aim; it is to explain to metallurgists "in as simple language as possible" certain more or less standard methods of physical investigation which are now being increasingly applied to the study of metals. The present volume, which treats only optical methods, is concerned mainly with interference, diffraction, and polarization, and the various kinds of metallurgical problems which can be investigated by means of these phenomena. Physicists will find little which is new to them apart from the applications, while metallurgists need have no fear that the language is not simple enough, for anyone who has passed an intermediate science examination in Physics should be able to read it easily.

The present volume deals only with methods which a metallurgist with reasonable laboratory facilities might himself utilze; in a second volume we are promised an account of other physical methods, presumably X-ray and electron diffraction methods among others, which require special experimental technique.

For the most part the treatment of optical phenomena is entirely orthodox and contains practically nothing which cannot easily be found in numerous standard textbooks. It is presented, however, in a particularly clear and lucid manner and avoids all mathematical complexities. The treatment of several points is open to criticism. In a discussion of interference fringes formed by multiple reflexion between parallel plates the resultant amplitude is expressed in a form which, if not incorrect, is certainly misleading; the formula in question (p. 45) gives the resultant amplitude as the sum of a series of cosine terms equivalent to $A=\sum_n a_n \cos \phi_n$ whereas, of course, in general

$$A^2=[\sum_n a_n \cos \phi_n)^2+(\sum_n a_n \sin \phi_n)^2.$$

In the ensuing argument, special cases mainly are considered in which the phase angles are zero or multiples of π; the sine series is then zero and the simpler formula leads to correct results. In the chapter on diffraction a diagram is given (p. 102) illustrating the formation of fringes at a straight edge; the observant reader would infer that the fringes are all of equal maximum intensity, which is also the intensity at points far from the edge. This conclusion would then be corrected by a more detailed diagram on p. 104. Turning next to the chapter on polarization, we find on p. 120 a diagram of a nicol prism showing the behaviour of the ordinary and extraordinary rays. Why, however, does the author show the extraordinary ray passing almost undeviated through the prism while the ordinary ray is so largely deviated that the angle of refraction is almost zero? Admittedly the figure is intended to be diagrammatic, but it would be just as easy and much more satisfactory to give a diagram nearer the

true state of affairs. These appear to the reviewer to be small defects which could easily have been avoided and which mar somewhat a book which in other respects is excellent.

To conclude, the book should prove particularly useful to metallurgists, for whom it is primarily written, and to physicists concerned with metallurgical and allied problems ; in a less specialized way it will be of interest to physicists generally who are not oblivious to the applied side of their subject. G. W. B.

The Physical Treatises of Pascal. Translated by I. H. B. and A. G. H. Spiers ; Introduction and Notes by Frederick Barry. [Pp. xxviii+181.] (New York : Columbia University Press, 1937 ; London : Oxford University Press. Price 16*s*.)

When Bacon withheld " sense " from inanimate things, but credited them with " subtile perceptions," physics had not been wholly mechanised. When " reciprocal perception " preceded the " operation " of one body upon another, physical things still had their inanimate analogues of animate natures. Pascal, in one respect, reflected the seventeenth-century passage from such hylozoistic, or quasi-animistic, notions to a purely physical concept of the inanimate world.

Pascal substituted for " nature's abhorrence of a vacuum " a purely mechanical cause in " the weight of the air." In the Baconian physics, bodies, " delighting as it were " in mutual contact, prevented vacua by " motion of connection." Though nature seemed to Pascal to be unable to abhor, since she was " not at all animated or sensible," he dared not lightly discard the maxim that nature did abhor vacua, and clung to the " ancient principle " till he was compelled to abandon it. In true seventeenth-century vein he abandoned it because of " decisive experiment."

After a Foreword by Barry and Perier's Preface the book prints Pascal's ' A Treatise on the Equilibrium of Liquids.' The ' Treatise on the Weight of the Mass of the Air ' follows it appropriately, for Pascal argued by deductive analogy from hydrostatic pressures to actions by " the weight of the air." Very broadly speaking, before the seventeenth century laboratory practice relied on deduction for experimental success. Pascal participated in the seventeenth-century institution of the experimental method proper by the check on deduction through experimental *verification*.

The great value of the book under review, as a contribution to the history of science, is enhanced by a series of relevant notes. One records Boyle's suspicions that Pascal was still deductively-minded enough to present some deductions as ostensible experiments. An ordinary siphon delivers oil from a higher to a lower vessel because, as Pascal argued, the oil is heavier than the air. The siphoning would be reversed, Pascal also argued, if the vessels and siphon were immersed in a " tank of water," because water is heavier than oil. This may be *mathematically* sound, but it is *experimentally* impossible, for the oil in the vessels would rise through the water. Boyle noted that one of Pascal's *experimenters* sat in a tank with his head well below water-level : the accompanying diagram, which included the experimenter himself, left no doubt on the point.

Pascal, however, did also depend on genuine experimental tests. Boyle used an air-pump to vary the pressure on the mercury in the Torricellian tube ; Pascal used different levels on the Puy de Dôme mountain. As is well known, Pascal's brother-in-law, Perier, assisted by others, performed " the usual vacuum experiment," on Pascal's behalf, at different points on the Puy de

Dôme in 1648. Pascal himself turned from the "physical sciences" to the "science of ethics," and Perier records how he repudiated even the celebrated experiments on the Puy de Dôme as vain and empty. Fortunately, the authors of this book did not share this prejudice when they included the relevant documents. Barry notes that Perier's own experimental skill contributed to the success of the experiments. Pascal himself confirmed Perier's results on a much more modest scale— obtaining, in one instance, a "difference of half a line" between the heights of a Torricellian mercury column at the bottom and top of "a private house ninety steps high."

The contents include two "Fragments" found among Pascal's papers. They also include records of barometric heights made at various places by different observers. Relevantly to Pascal's work on the "vacuum" the contents also include some of Boyle's experiments and, as appendices, some remarks by Galileo and some of Torricelli's letters. Extracts from Stevin constitute appendix i., and the book ends with a Bibliographical Note.

The history of science is indebted to all concerned in this excellent production. The index is as good as the book. JOSHUA C. GREGORY.

Stephen Timoshenko: *60th Anniversary Volume*. [Pp. viii+277.] (Macmillan & Co., Ltd., London. Price 22*s*.)

THIS book is a collection of 28 original papers by eminent authors from U.S.A., England, Holland, Turkey, and Germany on problems in pure elasticity, applied elasticity, and theory of structures. The book has been written by the many friends, pupils, and associates of Professor Timoshenko to celebrate his sixtieth birthday.

Prof. Timoshenko has published numerous papers of fundamental importance on stress analysis and vibration problems in engineering and on the development of the field of applied mechanics. It is therefore natural that the papers in this volume should deal in some form with many of the problems that have been investigated by him. The papers are given in alphabetical order of the names of the authors, and many are accompanied by a comprehensive bibliography. The volume is admirably produced and may be warmly commended to all post-graduate students, research workers, and mathematical technologists who are interested in the wide selection of questions discussed. These include problems connected with buckling and friction phenomena, fatigue failures, plastic strain and creep, torsion vibration eliminators and torsional vibration in branched systems, transient torques in induction motors, loaded clamped square plates, and Castigliano's principle of minimum strain-energy and grillage beams. R. H. E.

Cambridge Physical Tracts: *Electron Optics*. By the Research Staff of Electrical and Musical Industries Limited. Compiled and written by OTTO KLEMPERER. [Pp. x+107.] (Cambridge University Press. Price 6*s*.)

THIS tract succeeds in giving a concise account of electron optics, intelligible to the physicist who is unspecialized even in ordinary optics. Electrostatic lenses are discussed with particular reference to the field-plotting method of deriving equipotential surfaces; magnetic lenses are discussed less fully. In both cases an elementary theoretical treatment is given. Finally, lens aberrations and practical applications are briefly described, and there is a useful bibliography. The derivation of the thin magnetic lens formula is open to objection (p. 53); statements about the rotation produced by a magnetic

lens are incorrect (pp. 58–59); and there are two references to the laws of thermodynamics as having a bearing on geometrical optics, which are very obscure (pp. 7, 69). Nevertheless, the tract should prove useful to physicsts wishing to acquire a working knowledge of electron optics. D. H. P.

Mellor's Modern Inorganic Chemistry. Revised and edited by Dr. G. D. Parkes in collaboration with Dr. J. W. Mellor. [Pp. xix+915.] (Longmans, Green & Co., London, 1939. Price 12*s.* 6*d.*)

The merits of Dr. Mellor's Inorganic Chemistry, of which this edition is the ninth, are too well known to need emphasizing. It is pertinent, however, to remark that, in its present form, this edition has been practically remodelled and brought up to date. The most obvious, and indeed a most welcome, change is the grouping together, in the first sixteen chapters, of the general and theoretical parts of the subject. Then follows a group of seven chapters dealing with the properties of the elements hydrogen, oxygen, nitrogen, carbon, sulphur and the halogens. In the remainder of the book the properties of the remaining elements and of their compounds are discussed strictly according to the Periodic Table.

The veteran has not been provided with a few patches on his battered harness —he is equipped with a completely new and efficient suit of armour. In his new guise, he may well be assured of a repetition of his earlier, and well-merited, successes. A. F.

Advances and Applications of Mathematical Biology. By N. Rashevsky. [Pp. xi+214.] (U.S.A.: University of Chicago Press. Great Britain and Ireland: Cambridge University Press. Price 12*s.* net.)

Quantitative methods are increasingly applied to biological phenomena, and with this increase comes the necessity for the use of approximate methods which can be applied to biological systems too complex to come under the simple symmetrical cases which are usually the subject of accurate analysis. Mr. Rashevsky discusses, in the matter of diffusion, for example, approximate solutions which are obtained by considering cell shapes of a given ratio of length to breadth, and by suggesting reasonable mean values for the varying concentrations. He thus arrives at results which, tested by simple particular cases, give reasonably close approximations.

In the present work he discusses various problems of cell growth and movement, protoplasmic streaming, excitation in peripheral nerves, the mathematical biophysics of simple neurological structures and of psychophysical discrimination, the discrimination of intensities and the perception of visual patterns.

The discussions are clear and stimulating, and the author is to be congratulated on having produced a work that will be read with interest by all physiologists whose studies have a physical bias.

British Association Mathematical Tables.—Vol. VIII. *Number-Divisor Tables.* Revised and in part prepared by J. W. L. Glaisher. Extended and edited by the Committee for the Calculation of Mathematical Tables. [Pp. x+100.] (Cambridge University Press, 1940. Price 15*s.*)

Dr. Wishart, who has undertaken the editorial work in connexion with the revision and extension of the tables partly prepared by the late J. W. L. Glaisher, is to be heartily congratulated on this, the latest volume to be added

to the list of the volumes of tables published under the direction of the British Association Tables Committee.

The tables in the new volume are divided into four sets. Table I. gives (for integral positive values of n not exceeding 10,000) the factors of n in powers of primes, Euler's ϕ-function $\phi(n)$, the number $\nu(n)$ of divisors of n, and the sum $\sigma(n)$ of these divisors. Table II. gives, in effect, all solutions x of the equation $\phi(x)=m$, where $m<2500$. Table III. gives solutions of the equation $\nu(x)=m$, and Table IV. solutions of $\sigma(x)=m$.

The value of the tables is obvious ; their history is given in the Introduction ; it only remains to add that the volume is printed and produced at the high standard which we have learned to look for in the publications of the Cambridge University Press.

Elementary Mathematics from an Advanced Standpoint. By Professor Felix Klein. [Pp. ix+214.] (London : Macmillan & Co., 1939. Price 15*s*. net.)

This welcome addition to the literature of mathematics is the second part of a work designed to appeal specially to teachers of the subject. The first part dealt with Arithmetic, Algebra and Analysis ; the second part is divided into three sections, the first dealing with the simplest geometric manifolds, the second with geometric transformations, the third with a systematic discussion of geometry and its foundations.

The exposition is in lecture form, and the reader is led, easily and gradually, to concepts which are on the frontiers of mathematical knowledge, and which, in hands other than those of Professor Klein, would be regarded as quite outside the field of view of any but a highly specialized student. It may well be that some of the author's views are to be regarded as outmoded and that, were he writing to-day, the development of his ideas would take a different course. This does not, however, in any way detract from the great value of the book, which will be a source of pleasure and profit to many generations of teachers and of learners.

The translation reads well ; there are one or two unimportant slips which may easily be corrected in a later edition.

The Raman Jubilee Volume. Proceedings of the Indian Academy of Sciences [Vol. 8, No. 5, pp. 243–565.] (Price 10*s*.)

This volume, commemorating the fiftieth birthday of Sir C. V. Raman and the completion of ten years' research on the Raman effect, takes an appropriate form. Opening with an excellent portrait and biography, it comprises thirty-eight original contributions, running to a total of some three hundred pages. The list of authors alone is a testimony to the extent of Raman's influence and stimulus. The twenty-six papers dealing with the Raman effect serve well to illustrate the fruitfulness both in achievement and promise of that great discovery in the investigation of the problems of crystal forces, molecular vibrations, and chemical binding.

This publication is the effective expression of a happy thought. It will not only please Raman's friends and admirers—it cannot fail to stimulate further advances. J. E.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

CONTENTS OF No. 203.

Page

XLII. The Ignition of Inflammable Gases by Hot Moving Particles.—II. By STEWART PATERSON, M.A., B.Sc., formerly Metcalfe Research Fellow and Thomson Experimental Scholar, Glasgow University 437

XLIII. Solutions of Mathieu's Equation.—I. By J. G. BRAINERD, Sc.D., and C. N. WEYGANDT, M.S., Assistant Professor and Instructor respectively in the Moore School of Electrical Engineering of the University of Pennsylvania 458

XLIV. Einstein's Distant Parallelism and Dirac's Equation. By H. W. HASKEY 478

XLV. On the Inelastic Scattering of a Beam of Particles by Hydrogen and Helium. By Prof. K. C. KAR, D.Sc. 487

XLVI. The Dielectric Constants of some Metallic Sulphates containing various amounts of Water of Crystallization. By C. K. CHENG, Ph.D., Assistant Demonstrator, Department of Physics, University of Toronto... 505

XLVII. A Table of Complete Elliptic Integrals. By ALAN FLETCHER, M.A., Ph.D., University of Liverpool..... 516

XLVIII. Notices respecting New Books :—
Mr. GERHARD HERZBERG's Molecular Spectra and Molecular Structure.—I. Diatomic Molecules 520

Index..... 521

With Title-page, Contents, &c.

*** It is requested that all Communications for this Work may be addressed. post-paid, to the care of Messrs. TAYLOR AND FRANCIS, LTD., Printing Office, Red Lion Court, Fleet Street, London, E.C. 4.

PRINTED IN GREAT BRITAIN BY TAYLOR AND FRANCIS, LTD.,
RED LION COURT, FLEET STREET, LONDON, E.C.4.

XLII. *The Ignition of Inflammable Gases by Hot Moving Particles.*—II.

By Stewart Paterson, M.A., B.Sc.,
formerly Metcalfe Research Fellow and Thomson Experimental Scholar, Glasgow University *.

[Received April 1, 1940.]

In this paper further experiments are described on the ignition of inflammable gases by hot moving particles.

1. *Introduction.*

Previous work † on the ignition of coal-gas-air, hydrogen-air, and pentane vapour-air mixtures by platinum and quartz spheres had shown the existence of a definite minimum temperature to which any given sphere must be raised before it was capable, under the experimental conditions, of igniting a given mixture. This temperature depended upon the nature and strength of the mixture, and upon the diameter of the sphere. The observation ‡ that it depended also, to a marked degree, upon the speed of relative motion of gas and sphere, increasing apparently without limit as the speed was raised, cast doubt on the original conception § of an "instantaneous ignition temperature" (*i. e.*, one corresponding to infinite speed of flight), and focused attention rather upon low relative speed. Experiments have therefore been made with particles moving more slowly (at 1·2 metre/sec.) than heretofore, and the observed ignition temperatures, which correspond in every case to the above speed, form the subject of this report. They are lower, and in some cases considerably lower, than those previously determined for higher speeds.

2. *Experimental Arrangement.*

Really low and easily controlled and measured speeds might be secured by suspending the chosen sphere at rest, heating it inductively, and

---

* Communicated by Professor E. Taylor Jones.

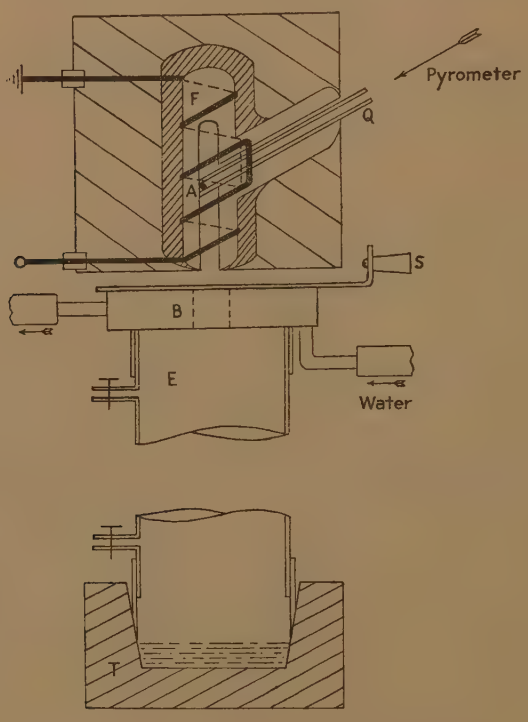
† R. S. Silver, Phil. Mag. (7) xxiii. p. 633 (1937). S. Paterson, *ib.* (7) xxviii. p. 1 (1939).

‡ Paterson, *loc. cit.*

§ McDavid, Journ. Chem. Soc. cxi. p. 1004 (1917); Silver, *loc. cit.*, p. 636.

passing a current of explosive gas over it. Such a method, which recalls the work of various authors on hot wire ignition, has many advantages, but the disadvantages are also pronounced. Apart from obvious problems of suspension and rapid replacement of one sphere by another at the same temperature, inductive heating is restricted to conductors, and the cases of moving gas and moving sphere are equivalent only when elaborate precautions are taken to ensure uniformity of the gas stream.

Fig. 1.



Semi-diagrammatic view of apparatus.

The technique used consisted simply in heating the sphere in a vertical tube furnace and dropping it through a shutter into the explosive mixture ; by abbreviating the drop, one could command speeds as low as 1 metre/sec. Details will be gathered from fig. 1. The furnace F was a small blind T-tube of alundum, wound with " Kanthal Al " strip and lagged with alundum and asbestos wool. (The tube was moulded on a waxed former, and the method of winding is shown in the figure.) The experimental sphere A was supported against the vertical furnace wall within a quartz

tube Q of appropriate bore which passed loosely through the side-arm of the " T." Since this arm inclined at about 20° to the horizon, the sphere automatically took up the required position when inserted in Q, and could then be dropped clear by withdrawing Q a fraction of an inch. This simple method allows constant observation with a pyrometer through Q ; and it eliminates speed as a variable factor, since the speed of entry into the gas depends only on the (fixed) distance AB. In all our experiments this distance was 7·3 cm., and the speed of entry therefore 1·2 metre/sec., which probably represents about the lower limit of speed available in the present method, since it does not appear possible to make a high-temperature vertical tube furnace *much* shorter than 10 cm.

The explosion vessel E was a vertical pyrex tube (3·5×40 cm.), fitted, as in the former experiments,* with brass sheaths, wooden target T, perforated cover B, and rotary shutter S. Asbestos wool was at first used in T to catch the spheres, but water was ultimately substituted, for reasons to be explained. The cover cap B and shutter S became very hot from their inevitable proximity to the furnace ; it was suspected that this caused a lowering of the ignition temperature, and a water-cooled cap was constructed, dependent on natural convection to a large tank : this worked admirably, the shutter never becoming more than slightly warm to the hand.

The experimental procedure is sufficiently obvious : when the chosen sphere had been heated to the required temperature, S was opened and Q slightly withdrawn. After a little practice this could be accomplished very quickly, and it was not thought necessary to " mechanise " the shutter. In the event of ignition occurring (there was never any ambiguity) the condensed water was evaporated from E by a current of air before the next trial.

3. *Experimental Results.*

As in the previous work, most of the experiments were made with spheres of platinum and quartz, though alumina, porcelain, alundum, slate, titanium oxide, and nickel, as well as several complex slags were also used. Except in the cases of platinum and nickel, these spheres were fashioned in a carbon arc ; and only a limited number of substances lend themselves to this treatment. The explosive mixtures were coal-gas-air, hydrogen-air, methane-air, and pentane-air

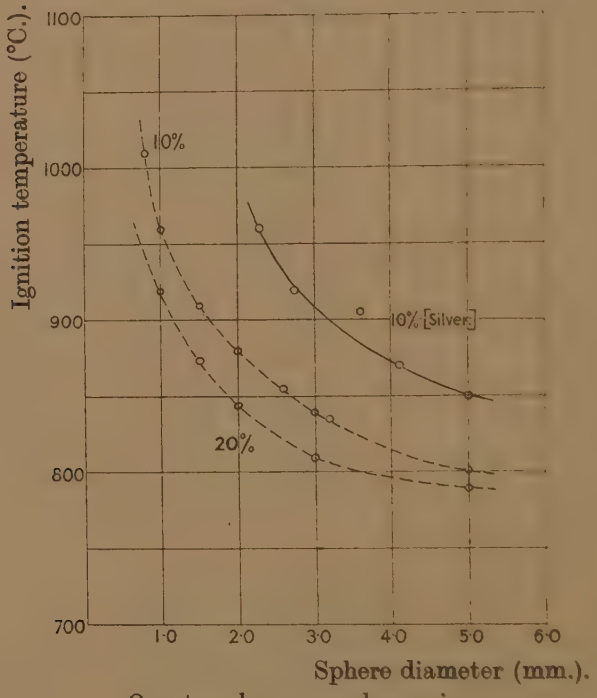
A. *Coal-gas-air : variation of ignition temperature with diameter of sphere : quartz spheres.*

As soon as it was found that quartz spheres moving through 10 per cent. coal-gas-air at 1·2 metre/sec. gave lower ignition temperatures than those

* Paterson, *loc. cit.*

previously recorded, the relation between these temperatures and the diameters of the spheres was reinvestigated. Fig. 2 enables the results to be compared with those of Silver for the same mixture (10 per cent.) but higher speed (4 metre/sec.). The general shape of the curve is unchanged, but the temperatures are considerably lower; and this fall is greatest for the smaller spheres. As the efficiency of quartz has been found to decrease slightly with use (see par. I of this section), it must be stated that the curve refers to new and highly polished spheres.

Fig. 2.



Quartz spheres: coal-gas-air.

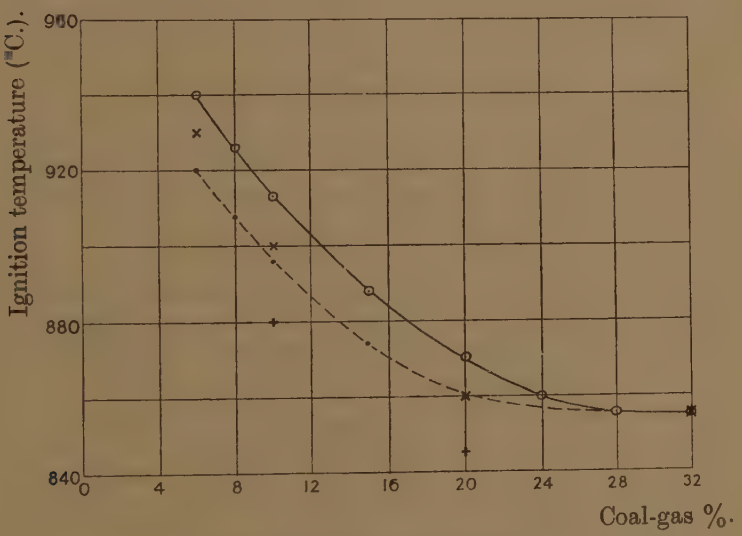
The curve for 20 per cent. coal-gas-air under present conditions is also reproduced in fig. 2. It is similar to the 10 per cent. characteristic, but does not coincide with it. This raises the question of variation with mixture strength.

B. *Coal-gas-air: variation of ignition temperature with percentage gas in mixture: quartz, alundum, and platinum spheres.*

The effect of mixture strength, in the case of 2 mm. spheres, is illustrated in fig. 3. A regular decrease of ignition temperature occurs as the

percentage varies from its lower explosive limit to the value for maximum flame-speed. Thereafter, the decrease is small and may not be significant; but there certainly appears to be no increase (*cf.* the writer, *loc. cit.* p. 8). Reference may here be made to the writer's previous work in which a series of curves * was derived connecting the ignition temperature and speed of platinum and quartz spheres for various strengths of mixture. These curves appeared to converge at low speeds to a single point on the temperature axis; it was suggested, in other words, that the "minimum" ignition temperature did not depend on mixture strength. However, the lowest speeds then used were of the order of 10 metre/sec., and the apparatus was not designed

Fig. 3.



2 mm. spheres: coal-gas-air.

for use even at this level. The curves referred to, instead of "tailing off" at 1000° C., require, in fact, to be extended downwards, and at the low speed now in use a pronounced variation of ignition temperature with mixture strength is still apparent.

In connexion with the curves of fig. 3 two remarks must be made. The platinum curve refers to a sphere which had been used for a long time with the asbestos target (see section 2 above), and was found, in fact, to have acquired a coating of fused asbestos. Under these conditions the sphere behaved with great consistency; but its performance became more

* *Loc. cit.* p. 7, fig. 4.

erratic when the coating was removed. The behaviour of *clean* platinum spheres has indeed several remarkable features, which will be described later (par. H). The quartz curve was also obtained with an "old" sphere. In this case (as in all others described, except the above) a water target had been substituted for the asbestos, so that no coating was formed; but the igniting power of quartz spheres, as already remarked, decreases slightly (10–20°) with use, before reaching the stable values of fig. 3. Thus, the two isolated points, which represent the lowest ignition temperatures recorded, refer to freshly made spheres. Further details of this effect are again given in par. I.

It is perhaps also worth recording that the ignition temperature of a coal-gas jet, with which the explosion tube was replaced, corresponded with the lowest point on the appropriate curve of fig. 3. There is apparently no particular virtue in the fact that our explosive mixture was normally enclosed in a tube.

C. *Hydrogen-air : variation of ignition temperature with sphere diameter : quartz spheres.*

Results for 10 per cent. hydrogen-air corresponding to those of fig. 2 are displayed in fig. 4. The relationship between sphere diameter and ignition temperature is of the same general type. Silver's curve * at 4 metre/sec. in a 20 per cent. mixture is again included for comparison. If the ignition temperature varies with mixture strength in the same way as for coal-gas-air—which is probable,—so that a 20 per cent. mixture is more easily ignited than a 10 per cent., the effect of speed must be even greater than fig. 4 would suggest. In view of the violence of explosion in richer hydrogen-air mixtures, however, a percentage-variation curve has not been sought with the present apparatus.

D. *Methane-air : quartz and platinum spheres.*

In addition to coal-gas-air and hydrogen-air, mixtures of air with methane and pentane vapour were examined by Silver. Methane-air he showed to have very high ignition temperatures : the 8 per cent. mixture could not be ignited by a 6·5 mm. platinum sphere below 1200° C. † The above work on coal-gas and hydrogen gave hope that this figure might be reduced under the present conditions, and a 10 per cent. mixture was chosen in the belief that it would be, if anything, more easily ignited than the 8 per cent. ‡ Failures, however, were recorded up to 1100° C. with a 5·5 mm. platinum sphere, and up to 1115° C. with a 5·2 mm.

---

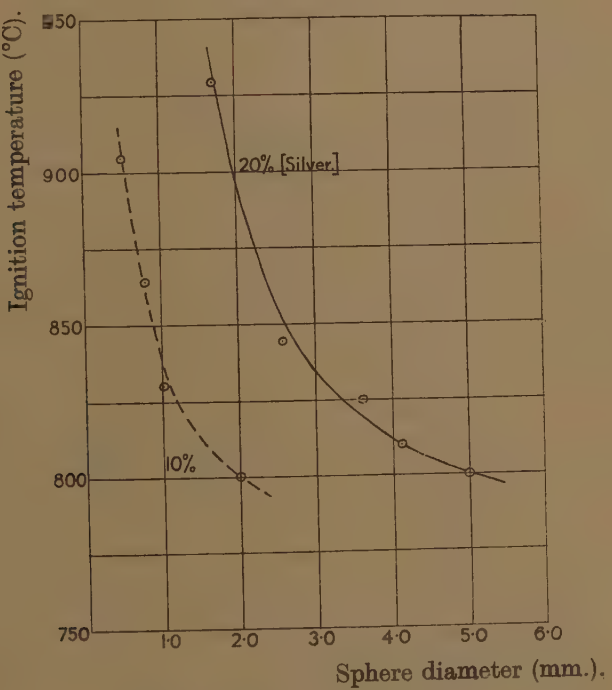
* *Loc. cit.* p. 647, fig. 5.
† *Loc. cit.* p. 644.
‡ See, however, p. 454.

quartz. The small vertical furnace is relatively inefficient, and 1120° C. represents its safe limit. The actual ignition temperatures were therefore not determined ; but it seems that they are still conspicuously high.

E. *Pentane vapour-air : variation of ignition temperature with sphere diameter : platinum spheres.*

The relation between ignition temperature and sphere diameter for a 3 per cent. pentane-air mixture is recorded, along with Silver's results at 4 metre/sec., in fig. 5, which recalls figs. 2 and 4 : the new curve is again

Fig. 4.



Quartz spheres : hydrogen-air.

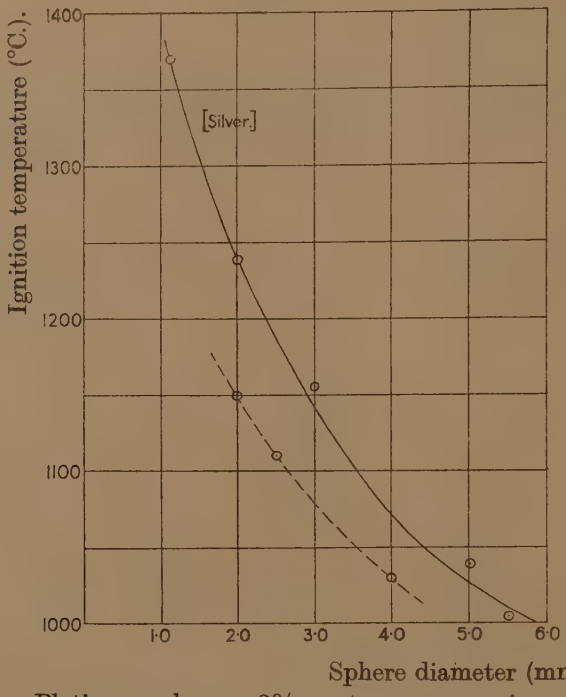
similar to the old but beneath it, and the difference is again greatest for the smaller spheres.

F. *Pentane-air : variation of ignition temperature with mixture strength : platinum, quartz, and nickel spheres.*

4 mm. spheres were used in this case, in order to work at a temperature suitable for the furnace. Mixtures containing $1\frac{1}{2}$, 2, 3, 4, $4\frac{1}{2}$, 5 per cent. pentane were examined, but no appreciable variation of ignition temperature could be detected. The results are best summarized in a table

(Table I.). It will be seen that the overall variation for any one sphere substance is about 10°, which scarcely exceeds the error of measurement.

Fig. 5.



Platinum spheres: 3% pentane vapour-air.

TABLE I.

Per cent. C_5H_{12}.	4·0 mm. Pt (old).	4·0 mm. Ni.	4·0 mm. quartz.
1½.....	1030° C.	1025° C.	—
2.....	1035	1030	1035° C.
3.....	1030	1020	1020
4.....	1035	1030	1030
4½.....	1030	1030	1030
5.....	1030	1025	—

We must conclude that the ignition temperature under present conditions does not depend upon mixture strength. The effect of sphere substance will be discussed directly.

G. *Coal-gas-air, hydrogen-air and pentane-air: effect of sphere substance upon ignition temperature.*

Silver's results * for these mixtures referred to spheres of quartz and platinum at 4 metre/sec., and the effect of sphere substance under such conditions was relatively very small. He suggested, however, that this might be due to coincidence.

TABLE II.

Material of sphere.	10 per cent. hydrogen.	10 per cent. coal-gas.
Platinum (old, asbestos-coated)	810° C. (12 per cent.)	915° C.
Platinum (old, clean)	820	915
Platinum (polished)	>875	>1070
Nickel	815	900
Quartz (new) [SiO_2]	800	880
Alumina [Al_2O_3]	—	880
Alundum	810 (12 per cent.)	900
Porcelain	810	870
Slate	825	895
Titanium oxide [TiO_2] 2·1 mm.	825	890
Tephroite [$2MnO.SiO_2$]	825	900
Fayalite [$2FeO.SiO_2$] 1·8 mm.	835	860
$FeO.Al_2O_3$ 2·1 mm.	825	890
[$35MnO$, $25SiO_2$, $40FeO$] 1·9 mm.	830	895

Sphere diameter (unless otherwise stated), 2 mm.

It will be seen from fig. 3 that, although quartz spheres at 1 metre/sec. in coal-gas-air are unquestionably more efficient than asbestos-coated platinum, the agreement is again relatively close; and it extends to alundum, as the curves show. The pentane results of Table I., which cover the explosive range, cast doubt in this case also on the existence of any important variation with sphere substance.

The question has been examined for 10 per cent. coal-gas-air and 10 per cent. hydrogen-air over a more extensive range of materials. Spheres of approximately 2 mm. diameter were made in the arc from

* *Loc. cit.*

four slags of suitable melting-point and surface-tension: tephroite ($2MnO.SiO_2$), fayalite ($2FeO.SiO_2$), $FeO.Al_2O_3$, and a ternary mixture containing 35MnO, $25SiO_2$, 40FeO *; titanium oxide (TiO_2), glazed porcelain, and alumina were also used, and slate gave excellent spheres with a high polish; a 2 mm. nickel sphere was turned on the lathe; and results were of course already known for quartz and alundum, and also for platinum with and without asbestos coating. Table II. summarizes the ignition temperatures of 10 per cent. hydrogen-air and 10 per cent. coal-gas-air by this fairly representative collection, and will give some idea of the relative importance or unimportance of sphere substance.

The surfaces were examined microscopically before use, and exhibited wide differences in condition; some (*e. g.*, alundum) were crystalline, some rough but apparently amorphous, some so highly polished (quartz, porcelain) that it was almost impossible to focus on the surface itself; the nickel sphere rapidly acquired a thin and adherent oxide film. The significance (if any) of these differences may also be gauged from Table II. It will be seen that platinum forms an exceptional case, and the behaviour of platinum spheres, which is unusually interesting, deserves further mention.

H. "*Ageing*" *of platinum spheres.*

It was noticed in previous work † that the igniting power of platinum spheres in coal-gas-air increased with use. This effect reappeared early in the present experiments, but on a more impressive scale. Thus, though an old 2 mm. sphere ("2·0 D") gave consistent ignition at 920° C. in 10 per cent. coal-gas-air, a new specimen (2·0 F) failed repeatedly at temperatures up to 1070° C., and could probably be taken much higher without effect; for a larger sphere (2·5 A), also new, failed in each of 28 successive shots at temperatures ranging from 935° C. to 1070° C. After repeated use, however, 2·0 F became as efficient as 2·0 D; and 2·5 A, as would be expected, surpassed them both.

It was thought at first that an explanation could be found in the action of the asbestos target. As the experiment proceeded, various remarkable colour changes took place in the platinum surface, and it was established beyond doubt that these were due to the target. Asbestos consists of a number of complex silicates, some of which (*e. g.*, crocidolite) are readily fusible; minute fragments adhere to the glowing platinum surface.

---

* The ferrous slags became magnetic during treatment, so that they were presumably reduced in the arc.

† Paterson, *loc. cit.* p. 10. *Cf.* also Thornton, Phil. Mag. (6) xxxviii. p. 624 (1919). White and Price, Journ. Chem. Soc. cxv. p. 1263 (1919). Shepherd and Wheeler, Safety in Mines Research Board Paper, xxxvi. p. 19 (1927).

and spread out upon it when the sphere is replaced in the furnace. As this process results in the formation of a substantial asbestos coat, it will not, perhaps, appear surprising that a new and clean platinum sphere should behave differently from one which has been long in use.

The effect, however, cannot be explained in this way. To begin with, if the improvement is due solely to asbestos deposition, it should disappear when the deposit is removed. The platinum spheres were cleaned easily and thoroughly by washing in hydrofluoric acid, and fresh accretion was prevented by substituting a water target for the asbestos wool. Beyond a loss of consistency, the results with 2·0 D and 2·0 F (in 10 per cent. coal-gas-air) showed little change: the spheres were still capable, on occasion, of producing ignition at as low a temperature as before, and never failed to do so above 950° C.; there was no suggestion of a return to the 1070° C. level. As a matter of fact, it is now known that changes occur in the platinum surface, and an improvement takes place simultaneously in its igniting power when no asbestos is present and no deposit whatever is formed. The surface, which originally presented the appearance, even under considerable magnification, of a compact and brightly reflecting sheet of metal with a conspicuous "high-light," becomes dull and milky to the eye and reflects very diffusely. When magnified it now shows an extremely fine pitting or etching: the general effect could be described as tinselled. This change is not due either to the sudden cooling when the sphere is dropped, or, as might be suggested, to the reaction at its surface during ignition; it can be produced simply by heating the sphere for a few hours in a quartz tube at 1000° C. On the other hand, a sphere whose polish has been destroyed in this or any other way, may be restored to the original condition, with its lustre even enhanced, by heating for a similar period *at a bright orange* in the tip of a capillary coal-gas flame; lower temperatures are, however, ineffective.

It seems possible to correlate these surface changes directly with the changes in igniting power, which, like them, continued to occur after the asbestos deposition had been eliminated. Thus a new and brightly polished sphere (2·0 G) failed to ignite 10 per cent. coal-gas-air at 940° C., 965° C., 985° C., and 1000° C. Three days later, it succeeded at 970° C., and finally at 940° C.; the surface was now unmistakably tinselled, though still bright. Again, 2·0 F, whose asbestos coat had been removed revealing a tinselled surface, gave consistent ignition (of the same mixture) at 920–940° C. It was heated for some time to a low red in the flame-tip, without, however, recovering its polish; nor had its igniting power decreased. After two hours at a bright orange, the surface, while rather granular in appearance, was highly lustrous and entirely without pitting. Failures were now recorded at 1000° C., 1000° C., 1060° C., and success only at 1095° C.; the sphere failed repeatedly at 960° C., but in the end

produced ignition about 990° C. It was found to be still bright and lustrous, but with finer markings in evidence. After a fresh two hours' orange heat in the flame-tip (which removed the markings), 2·0 F gave immediate ignition at 1095° C., which may be regarded as a probable upper limit for even a "new" sphere; but it failed consistently at 1020° C.–1050° C. It was now steeped at 1020° C. in the furnace for 4½ hours, whereupon it succeeded at once at 1015° C. and 970° C., and was found to have resumed its familiar tinselling. The entire cycle of ageing and rejuvenation was repeated once more, and could apparently be repeated without limit.

This effect has not been studied so closely in the case of hydrogen-air, but there is no doubt that it exists here too. A new and polished sphere (2·0 H) succeeded in igniting 10 per cent. hydrogen-air at 890° C., but failed at 850° C. After a very few trials, however, ignition was produced as low as 820° C. At this stage the surface was still apparently polished, but fine division could be seen under the microscope; a successful shot at 935° C. in 10 per cent. coal-gas-air showed that the ageing applied to this mixture also. 2·0 H was now heated for two hours to bright orange in the flame-tip, examined and retested in 10 per cent. coal-gas-air. The surface had lost its tinselling and acquired the "granular" lustrous appearance. It failed at 1015° C. and 1010° C. to ignite the coal-gas mixture, and at 875° C. and 860° C. in 10 per cent. hydrogen-air. Success was then recorded at 890° C., 860° C., and later at 820° C., the original ignition temperature for old spheres.

The effect is well marked in pentane vapour-air. A 3 per cent. mixture was used, and could be ignited very consistently at temperatures above 1030° C. by 4·0 A, an old platinum sphere with characteristically tinselled surface. 4·0 B, which was new and polished, failed at 1065° C. and 1110° C.; it succeeded at 1155° C., but failed again twice at 1120° C. Next day, however, ignition was immediately produced at 1040° C., and the usual tinselling could be detected. (4·0 B was still somewhat inferior to 4·0 A in 10 per cent. coal-gas-air.) After four hours' intense heating in the flame-tip, 4·0 B, whose surface was now lustrous and "fused" in appearance and without a trace of tinselling, failed repeatedly at 1120° C. to ignite the pentane-air mixture. The cycle is therefore complete.

Failure was also recorded up to 1040° C. in 10 per cent. coal-gas-air; this may be compared with the figure of 840° C. for "old" spheres, again exemplifying the effect in coal-gas.

I. "*Ageing*" *of quartz spheres.*

Changes in igniting power of the kind described appear to be peculiar to platinum: no improvement with use was detected in any of the other

materials. Thus, a 2 mm. alundum sphere, which ignited 10 per cent. coal-gas-air at 905° C. on its first trial, still failed consistently at 900° C. after its fiftieth. The surface change, which seemed to amount chiefly to roughening of the crystal faces by fracture, had had no effect upon performance. In the case of quartz, which was perhaps the most consistent and efficient igniting substance examined, not only could no improvement be detected, but a definite though slight deterioration took place. Fresh quartz spheres of 2 mm. diameter may be relied upon, under present conditions, to ignite 10 per cent. coal-gas-air at 880° C.; after prolonged use they can be trusted not to do so, and will perhaps fail at 895° C. or 900° C. This seems, however, to be the limit of the effect.

A simultaneous microscopic change occurs in the quartz surface, which loses its almost perfect initial polish and is gradually fractured and roughened until it ends by resembling sandpaper.

4. *Discussion.*

Since this paper is intended to record the above observations rather than to explain them, they have been described as far as possible without comment. Theoretical approach to the problems of initiation and propagation of flame is difficult, and the suggestions which follow must therefore be regarded as more than usually tentative.

An approximate thermal theory of ignition by heated spheres was put forward by Silver *, and found to agree, as far as could be judged, with his experiments on diameter-temperature variation. The modifications introduced later † to allow for speed of motion did not affect this agreement. It is interesting to see how far the present results, to which the modified theory is directly applicable, may be correlated with those of Silver.

According to the modified theory, a linear relation should exist between $Y \equiv \log_e \left[\frac{T_s - T_\infty}{a^{3/2}} \right]$ and $X \equiv 1/T_s$, where T_s and a denote temperature and radius of the sphere, and T_∞ initial temperature of the explosive gas; in fact,

$$Y = f(V) - \frac{E}{2} \cdot X, \qquad \ldots\ldots\ldots \quad (1)$$

V being the speed of motion and E an activation energy. The results contained in figs. 2, 4, 5 do satisfy a linear relation of this type; and they determine values of E, which, while consistently higher, are in very reasonable agreement (Table III.) with those derived from Silver's work on the

* *Loc. cit.* p. 648.

† Paterson, *loc. cit.* p. 16.

same combustible gases. Moreover, the activation energies for 10 per cent. and 20 per cent. coal-gas-air coincide, as presumably they ought to do; and the displacements parallel to Y of the (X, Y) curves constructed from Silver's and the present figures respectively agree, not only in sign but also in order of magnitude, with those predicted by (1). For, since $f(V)$ is of the form: Const.$-1/2 \log_e V$, and the ratio of speeds used by Silver and the writer was $\simeq 4$, the (constant) difference in ordinate of the two curves for a single explosive mixture should be of the order of one unit; which, in fact, it is.

So far, then, as concerns our present results, and those of Silver at comparable speeds, the theory is satisfactory. It was further shown * to be capable of explaining the writer's experimental relation between speed and ignition temperature for a number of coal-gas-air mixtures. Unfortunately, however, the value of E required was much smaller

TABLE III.

Explosive mixture.	Sphere.	E (Silver).	E (Paterson).
10 per cent coal-gas-air . . .	2·0 mm. quartz.	35,000	41,000
20 per cent. coal-gas-air . .	2·0 mm. quartz.	—	41,000
10 per cent. hydrogen-air .	2·0 mm. quartz.	—	49,000
20 per cent. hydrogen-air .	2·0 mm. quartz.	39,000	—
3 per cent. pentane-air . . .	2·0 mm. platinum.	33,000	35,000

Values of E (cal.) derived from: $\log_e \frac{T_s - T_\infty}{a^{3/2}} = f(v) - \frac{1}{2} E/T_s$.

(about 10,000 cal.) than those of Table III. This difficulty was noticed at the time. It might, perhaps, be explained in terms of the asbestos coating, which must have existed in the speed-temperature experiments; but we have seen that asbestos-coated, and indeed all except new or "rejuvenated" platinum surfaces, behave at low speeds very much like quartz. Now the quartz results of fig. 3 required an activation energy of 41,000 cal. It is, therefore, reasonable to expect that the present method would lead to a similarly high value for coated platinum.

Three major criticisms of the theory can be made. It regards ignition as determined by a balance of conduction and reaction in the boundary layer, though conduction alone must provide a balance when there is no reaction; it focuses attention on the sphere surface and ignores the hot

* Paterson, *loc. cit.*

wake, whereas one might easily argue that the relative importance lies in just the opposite quarter ; and it neglects the agency of diffusion of active particles compared with that of heat transfer. Now the tendency of recent research has been to lay greater and greater stress on diffusion of active particles and to suggest that it may even outweigh thermal conduction in controlling flame.

It is possible, following Landau *, to develop an alternative theory in which both diffusion of chain-carriers and conduction of heat are taken into account, and in which the hot wake, regarded as a cylindrical source of each, is considered to be responsible for ignition. This theory does not, apparently, succeed, any more than the former, in reconciling all our experimental facts ; but if ignition does originate in the hot wake, there is ample reason for failure ; for the wake behind a sphere changes its character completely within a very moderate range of Reynolds number. According to Möller †, who examined photographically the wake behind a sphere of radius r moving through fluid in a vessel of cross-section 41^2, "perfectly smooth laminar flow occurs (when $r/1=1/20$) up to $Re=300$ ‡. If Re is gradually increased to 450, the laminar flow, though it still persists, is disturbed, at first occasionally and later regularly, by the formation of eddies. Beyond $Re \doteqdot 450$ ($r/1=1/20$), these eddies commence to detach themselves as vortex rings. They separate unsymmetrically, and so give the impression of being linked in a kind of vortex chain. Above $Re \doteqdot 1000$ ($r/1=1/8$) there appear from time to time discrete eddies, which, as Re is further increased, progressively supersede the chain vortices, until at about $Re=1500$ the fluid contains only discrete eddies forming a "vortex street." The vortex street does not prove stable : the vortices soon unite in cloudy balls, which again show a definite periodicity." Now the results described in this paper refer to values of Re not greater than 30 ; those of Silver to the range 30–150 ; while in the speed-temperature work Re varied between 190 and 1500. If the hot wake plays a determining rôle in ignition, this rôle will almost certainly change its character with that of the wake ; and it will not then be surprising that a theory which presupposes a simple cylindrical wake should fail to reconcile experiments made under such widely divergent conditions of flow.

The resultant ignition criterion, which is derived in the Appendix, is of the same general form, viz.,

$$T_s^x V^y \leqslant A a^z . e^{-E'/T_s}, \qquad (2)$$

---

* Chem. Rev. xxi. p. 245 (1937) ; J. Chem. Phys. vii. p. 112 (1939).

† *Phys. ZS.* xxxix. p. 57 (1938).

‡ $Re \equiv 2Vr/\nu$ (Reynolds number), where ν is the kinematic viscosity.

as its predecessor, though the small positive numbers x, y, z have different values. These values depend to a considerable extent on the assumptions made; our present criterion runs

$$T_s^{10}V^3 \leqslant Aa^7 . e^{-E'/T_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A is, of course, independent of T_s, V, a; and E′ is a sum of various activation energies. Without discussing this somewhat precarious theory at greater length, we may remark that it is capable of explaining Silver's and the writer's diameter-temperature results to the extent that our original theory explained them. With one reservation *, however, as much can be said of any theory which leads to the form (2). Values of E′ derived from (3) for the diameter-temperature work are shown in Table IV.; the coal-gas-air speed-temperature experiments, as before, require a relatively smaller figure, around 30,000 cal. This is not surprising, for it seems improbable that the derivation of (3) should hold beyond $Re \simeq 400$: the wake changes too profoundly. We need not assume, of course, that the wake is unimportant at higher Reynolds numbers; nevertheless, its influence, which was supposed to arise from quiescence and a high concentration of heat and initial centres, may be thought likely to decrease as it becomes turbulent and scattered. In this case the sphere surface may be directly responsible for ignition, and some such theory as was previously put forward, modified to take account of chain-carrier diffusion, will be appropriate.

TABLE IV.

Explosive mixture.	Sphere.	E′ (Silver).	E′ (Paterson).
10 per cent. coal-gas-air . .	2·0 mm. quartz.	87,500	98,000
20 per cent. coal-gas-air . .	2·0 mm. quartz.	—	97,500
10 per cent. hydrogen-air .	2·0 mm. quartz.	—	120,000
20 per cent. hydrogen-air .	2·0 mm. quartz.	105,000	—
3 per cent. pentane-air . . .	2·0 mm. platinum.	78,000	86,000

Values of E′ (cal.) derived from $\log_e \frac{T_s^{10}}{a^7} = \phi(V) - E'/T_s$.

In connexion with the effect of sphere substance upon ignition, and the "ageing" of platinum and quartz spheres, it is necessary to explain why every substance used, including platinum, gave roughly the same

* The Y-displacement explanation (p. 450) will apply only if $x/y \simeq 2$; this is fulfilled by (3).

minimum ignition temperature, but in the case of platinum ignition could be prevented, even at a temperature several hundreds of degrees higher, by suitable pre-treatment of the sphere. It is true that the surface roughening which was observed to accompany improvement with the platinum spheres implies a larger increase in surface reaction than in heat transfer, and therefore, according to our original theory, a fall of ignition temperature. But simple explanations in terms of roughness or smoothness do not appear possible, since quartz spheres which became as rough as the platinum (and were originally far smoother) showed no improvement—in fact, a slight deterioration. Again, artificial roughening was found to have no effect upon the ignition temperature.

The factor in Landau's theory which may be supposed to express the effect of sphere substance and surface condition is the active particle density n_0. It can readily be verified from the Appendix, if it is not obvious *a priori*, that an isothermal increase in n_0 must reduce the ignition temperature, and *vice versa*. Now, apart from changes in the activation energy E_2 of the surface reaction, n_0 will evidently depend upon the freedom with which active products of that reaction are allowed to escape from the surface, in other words, upon the degree of adsorption of such products on the surface itself. If, then, we assume that the rate of reaction was approximately equal for all the surfaces used, and that adsorption was, except for platinum, negligible, we can understand why the ignition temperature did not vary widely with sphere substance. On this view, the unusual behaviour of new or " rejuvenated " platinum must be taken to indicate a high degree of adsorption of active particles ; and the process of " ageing," during which the platinum ignition temperature fell to the level common to other substances, must be interpreted as a slow poisoning of the adsorptive surface. This poisoning can apparently be hastened by moderate heating in a quartz tube, and removed by continued exposure at bright orange to a coal-gas-air flame ; it coincides with a small-scale fracture or tinselling of the surface.

These suggestions seem to fall in line with the well-known exceptional catalytic and adsorptive properties of platinum *. In particular, it may be claimed that they offer a simpler and more convincing explanation of the improvement in igniting power than that put forward by Thornton and others †, according to which fresh platinum would fail to cause ignition because the reaction on its surface was too rapid to be capable of spreading. The exact nature of the chain-carriers which are adsorbed

---

* See, *e. g.*, Mellor, ' Inorganic Chemistry,' vol. xvi.

† Thornton, Phil. Mag. (6) xxxviii. p. 618 (1919). White and Price, J. Chem. Soc. cxv. p. 1257 (1919). Mason and Wheeler, *ib.* cxxi. p. 2083 (1922) ; cxxv. p. 1869 (1924). Coward and Guest, J. Amer. Chem. Soc. xlix. p. 2479 (1927).

is uncertain; however, the fact that the surface becomes so rough, and that it can be aged by heating in an air-filled silica tube, and rejuvenated by intense heating in a coal-gas-air flame (during which the temperature attained appears greatly increased by surface combustion) may suggest that they include oxygen atoms *.

One further point requires comment,—the variation of ignition temperature with percentage combustible gas in mixture. Here, very little can be said. The records of such variation, when ignition was produced by passage through hot tubes †, or into hot vessels ‡, by sudden compression §, by various kinds of electric discharge ||, and by hot wires, &c.¶, do not lead to any general conclusion. In some cases ignition occurred most readily with the weakest mixture, in others with the strongest, in others again there was a minimum ignition temperature at an intermediate strength. Our observations on coal-gas-air and pentane-air have already been described, but it does not seem easy to put any interesting construction upon them at the moment.

5. *Summary.*

1. Further experiments are described on the ignition of gases by hot moving spheres. A dropping technique enabled lower speeds (1·2 metre/sec.) to be used than heretofore.

2. The results are as follows :—

(*a*) Ignition temperatures of 10 per cent. and 20 per cent. coal-gas-air and 10 per cent. hydrogen-air by quartz spheres, and of 3 per cent. pentane vapour-air by platinum spheres vary with sphere diameter in a manner similar to that previously recorded. The actual temperatures are, however, lower, especially for the smaller sizes of sphere.

---

* *Cf.* Hall, J. Amer. Chem. Soc. xxii. p. 494 (1900). Roberts, Phil. Mag. (6) xxv. p. 270 (1913).

† Mitscherlich, *ZS. Anorg. Chem.* cxxi. p. 65 (1922), &c.

‡ Mason and Wheeler, J. Chem. Soc. cxxi. p. 2081 (1922); cxxv. p. 1871 ff. (1924). Mardles, Trans, Far. Soc. xxvii. p. 681 (1931). Naylor and Wheeler, J. Chem. Soc. p. 2460 ff. (1931); p. 1240 (1933); &c.

§ Falk, J. Amer. Chem. Soc. xxviii. p. 1527 (1906); xxix. p. 1548 ff. (1907). Dixon, J. Chem. Soc. xcvii. p. 661 (1910). Dixon and Crofts, *ib.* cv. p. 2036 (1914). Dixon, Harwood and Higgins, Trans. Far. Soc. xxii. p. 267 (1926).

|| Coward, Cooper, and Jacobs, J. Chem. Soc. cv. p. 1069 (1914). Thornton, Proc. Roy. Soc. (A) xc. p. 272 (1914); xci. p. 17 (1914); xcii. pp. 9, 381 (1915). Wheeler, J. Chem. Soc. cxvii. p. 903 (1920). Coward and Meiter, J. Amer. Chem. Soc. xlix. p. 396 (1927), &c.

¶ Thornton, Phil. Mag. (6) xxxviii. p. 624 (1919). Coward and Guest, J. Amer. Chem. Soc. xlix. p. 2479 (1927).

(*b*) In the case of coal-gas-air, a continuous decrease of ignition temperature occurs as the mixture strength is raised from its lower to upper explosive limit. This applies to spheres of alundum, quartz, and asbestos-coated platinum. No variation with mixture strength of the ignition temperature of pentane-air by platinum, quartz, or nickel spheres could be observed.

(*c*) The high level of the ignition temperature of methane-air was confirmed; but detailed records are still lacking.

(*d*) The effect of sphere substance on the ignition temperatures of 10 per cent. coal-gas-air and 10 per cent. hydrogen-air was examined for an extensive range of materials; except in the case of platinum, this effect appears to be of secondary importance.

(*e*) The ignition temperatures of coal-gas-air, hydrogen-air, and pentane-air mixtures by fresh platinum spheres are very much higher than by most other substances; platinum, however, improves with use, until this difference has practically disappeared. Its surface becomes simultaneously "tinselled." This process of "ageing" may be accelerated by soaking at red heat in a quartz tube; conversely, the sphere is "rejuvenated" and its tinselling removed by prolonged exposure at a bright orange to a coal-gas-air flame.

(*f*) Comparable changes do not take place with the other substances used; in particular, a slight deterioration is observed in quartz.

3. As far as can be judged, the theory already offered is capable of correlating the results summarized in 2 (*a*) with work by Silver at somewhat greater speeds. Difficulties connected with its simultaneous application to the writer's experiments at high speeds lead to the formulation of an alternative theory, based on Landau's analysis and expressly restricted to low Reynolds numbers. It is shown that this theory is also capable of the above correlation, as well as offering a reasonable account of 2 (*d*), (*e*), (*f*). Simultaneous application to high-speed work is now, of course, out of the question, and for such work a parallel theory, perhaps of the previous type, would be required.

I wish again to thank Professor E. Taylor Jones for constant encouragement and advice; also Messrs. Imperial Chemical Industries, Ltd., who extended my research grant and gave permission to publish. The experimental work was done in the Research Laboratories of the Natural Philosophy Department at Glasgow University, and I am greatly indebted to Mr. William Reid for many helpful suggestions both in these and in the previous experiments.

Appendix.

*Application of Landau's Theory * to ignition by Hot Wakes.*

We regard the heat which has passed from the sphere by conduction into its boundary layer † as canalized in a narrow cylindrical wake of indefinite length but constant radius r_0. The cylindrical wake is thereby raised to a mean temperature T_1. It is also supposed to be enriched with active particles, the product of surface reaction within the boundary layer: these are present in concentration n_0. The cylindrical wake therefore constitutes a source of the type proposed by Landau. The active particles now diffuse outwards; each generates Q units of heat in unit time, and they increase in number at a rate proportional to their density; while the temperature is determined through conduction of heat and its generation by the active particles. Under these conditions, it can be shown by a linear transformation ‡ that Landau's mathematical statement of the problem must lead to non-dimensional solutions for relative temperature and concentration of the form

$$\left(\frac{n}{n_0}, \frac{T-T_\infty}{T_1-T_\infty}\right)=f_{1,2}\left[\frac{r}{r_0}, \frac{D}{r_0^2}t; \frac{\alpha r_0^2}{D}; \frac{\kappa}{D}; \frac{Qr_0^2n_0}{k(T_1-T_\infty)}\right], \quad . \quad (1)$$

where r_0 is the radius of the source, α, D the branching and diffusion coefficients of the active particles, and k, κ the thermal and thermometric conductivities of the gas; $f_{1,2}$ being definite, absolute functions. It follows that Landau's ignition criterion, viz., that the axial temperature shall never fall, will define a critical relation of the form

$$\frac{k(T_1-T_\infty)}{Qr_0^2n_0}=\phi\left[\frac{\alpha r_0^2}{D}; \frac{\kappa}{D}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which ϕ is a third absolute function.

Landau's analysis, which assumes further that $D=\kappa$, leads, in fact, to a relation between the dimensionless quantities $\frac{k(T_1-T_\infty)}{Qr_0^2r_0}$ and $\frac{\alpha r_0^2}{D}$, which may be written, over a wide range,

$$\frac{k(T_1-T_\infty)}{Qr_0^2n_0} \leqslant K\frac{\alpha r_0^2}{D}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

K being an absolute constant.

Neglecting factors which do not vary with a, V, T_s, we may reasonably replace $Q\alpha$ by an expression of the type $T_1^{-1}e^{-E_1/RT_1}$, and if we suppose

---

* *Loc. cit.*

† Thermal and hydrodynamical boundary layers may be assumed to coincide, since Prandtl's number $\simeq 1$ for gases.

‡ *Cf., e. g.*, Davis, Phil. Mag. (6) xliv. p. 920 (1922), for method.

the rate of creation of active particles by the surface to be independent of the speed of motion V, n_0 will take some such form as $\frac{a^2}{Vr_0^2}T_s^{-2}e^{-E_s/RT_s}$.

The main difficulty is to identify r_0 and T_1. From our initial assumption that all the heat transferred by convection is employed in raising a cylindrical wake of radius r_0 to a mean temperature T_1, we can deduce, when T_∞ is negligible, that

$$r_0^2T_1 \simeq a^2T_sRe^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If this picture is completed by making all the gas contained in the boundary layer flow into the same cylinder, and if the temperature and velocity profiles within the layer are assumed linear and coincident (which is legitimate), it can also be shown that

$$r_0^2T_s \simeq a^2T_1Re^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Consequently,

$$T_1 \simeq T_s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$r_0 \simeq aRe^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Inserting (6) and (7) in (3), and substituting for $Qn_0\alpha$ the forms proposed above, we obtain a limiting relation which, when due allowance is made for the temperature variation of k, D, and ν, reduces to

$$T_s^{10}V^3 = Aa^7e^{-E/T_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A is a constant, independent of a, V, T_s.

XLIII. *Solutions of Mathieu's Equation.*—I.

By J. G. Brainerd, Sc.D., and C. N. Weygandt, M.S., Assistant Professor and Instructor respectively in the Moore School of Electrical Engineering of the University of Pennsylvania *.

[Received May 18, 1939.]

From a technical point of view Mathieu's Equation, equation (1) below, has been steadily increasing in importance in recent years. Concurrently a large amount of work has been devoted to analysing the equation and tabulating solutions in the special case in which the solutions are periodic. Unfortunately, in many if not most applications the periodic solutions are special cases of the more general solution of the technical problem. It is proposed in this paper to add to the groundwork necessary or desirable for the tabulation of non-periodic solutions of Mathieu's equation. Various properties of certain solutions of the equation are given ; a few of the solutions themselves are presented in graphical form ; a limited table of the characteristic exponent associated with solutions of the equation is given ; and a few properties and curves of the periodic functions associated with the non-periodic solutions of the equation are shown. In order to limit the problem to reasonable size, the range $0 \leqslant \epsilon \leqslant 10$ and $0 \leqslant k \leqslant 1$, where ϵ and k are two parametric coefficients in the Mathieu equation, has been chosen. Except for general mathematical deductions, all results offered here should be assumed to be limited to stable solutions of Mathieu's equation within this range, in which many technical problems fall.

The work is divided as follows :—

I. Introduction.
II. Floquet's Theorem—Introduction of the u Functions.
III. Properties of the g and h Solutions and of the ϕ_1, ϕ_2, u_1, and u_2 Functions.
IV. Sample Curves of the g and h Solutions and of the u_1 and u_2 Functions
V. Calculation of the Characteristic Exponent.
VI. Relationships between the Derivatives of the g and h Solutions.
VII. Values of the g and h Derivatives at certain Points.
VIII. Series for, and Zeros of, the g and h Solutions.
IX. Summary and Acknowledgements.

* Communicated by Dr. W. F. G. Swann.

I. *Introduction.*

Mathieu's equation (the canonical form of which will be taken as

$$\frac{d^2y}{dt^2}+\epsilon(1+k\cos t)y=0, \quad \ldots\ldots\ldots \quad (1)$$

where ϵ and k are constants, y the dependent and t the independent variable) appears in the analysis of a simple series electric circuit containing a variable electric parameter [1], in the theory of super-regeneration [2], in electric machinery problems [3], and in many problems [4], [5] in other fields. Mathieu presented the equation in connexion with the study of the vibration of an elliptic membrane [6], in which it was necessary that $y(t+\pi)=y(t)$ in order that boundary conditions be satisfied. As the solution of (1) is periodic only for special values of ϵ and k, the problem posed by Mathieu was less that of solving (1) in general than that of determining values of (ϵ, k) which would yield periodic solutions of (1). Such solutions (period π or 2π) are called Mathieu Functions. A large amount of effort has been devoted [7] to determining Mathieu Functions and to the adjunct problem of determining values of ϵ and k which will yield periodic solutions of (1). The non-periodic solutions, comprising all solutions except those for the special cases in which (ϵ, k) fall on certain not very numerous lines in the ϵ, k plane, remain nameless. In virtually all the important technical applications of (1) in those fields with which the authors are concerned, the parameters ϵ and k are given, and the non-periodic solutions will usually be indicated *.

II. *Floquet's Theorem—Introduction of the u Functions.*

Consider equation (1), which is a second order linear differential equation with one variable coefficient, two irreducible constants, and no singularities except at infinity. Let $g(t)$ and $h(t)$ be solutions of (1) such that

$$g(0)=1,\ g'(0)=0,\ h(0)=0, \text{ and } h'(0)=1. \quad \ldots\ldots \quad (2)$$

(Throughout this paper derivatives will be denoted by primes.) Then any solution of (1) can be written

$$y(t)=Ag_t+Bh_t, \quad \ldots\ldots\ldots \quad (3)$$

---

* The influence which Mathieu's problem has had in directing the course of studies of (1) is illustrated by the following used by Whittaker and Watson [8] to introduce their discussion of Floquet's theory :

"We shall now discuss the nature of the solution of Mathieu's equation when the parameter [ϵ] is no longer restricted to give rise to periodic solutions; this is the case which is of importance in astronomical problems, as distinguished from other physical applications of the theory and elsewhere."

"We have seen that in physical (as distinguished from astronomical) problems the constant [ϵ] in Mathieu's equation has to be chosen to be such a function of [k] that the equation possesses a periodic solution."

where A and B are arbitrary constants and g_t and h_t have been written for $g(t)$ and $h(t)$ respectively. This is the complete solution of (1), and states the well-known fact that knowledge of the g and h functions is sufficient to obtain any solution [9].

By Floquet's theorem [10],

$$y(t+2\pi)=\sigma y(t) \text{ where } \sigma \text{ is determined by} \quad . \quad . \quad . \quad . \quad (4)$$

$$\sigma^2-\sigma(g_{2\pi}+h'_{2\pi})+(g_{2\pi}h'_{2\pi}-h_{2\pi}g'_{2\pi})=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is shown later [III (*d*)] that $g_{2\pi}=h'_{2\pi}=b$, and it follows from (1) and the definitions of g and h that $g_{2\pi}h'_{2\pi}-h_{2\pi}g'_{2\pi}=1$. Hence

$$\sigma_1, \sigma_2=b\pm\sqrt{b^2-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Floquet's theorem further states that

$$y=C_1e^{j\mu t}\phi_1+C_2e^{-j\mu t}\phi_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where C_1 and C_2 are constants, ϕ_1 and ϕ_2 are periodic functions of t, period 2π, and the characteristic exponent μ is defined by ($j=\sqrt{-1}$)

$$\sigma_1=e^{j2\pi\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (1) is unchanged if t is replaced by $-t$, hence $\phi_2(t)=\phi_1(-t)$.

The function g is a solution of (1), consequently by (7) it can be written

$$g_t=C_1e^{j\mu t}\phi_1(t)+C_2e^{-j\mu t}\phi_2(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$g_{t+2\pi}=C_1e^{j\mu(t+2\pi)}\phi_1(t)+C_2e^{-j\mu(t+2\pi)}\phi_2(t), \quad . \quad . \quad . \quad . \quad (9\,a)$$

whence, setting $t=0$ in (9) and (9 *a*), and using (6) and the definition of b,

$$C_1\phi_1(0)=C_2\phi_2(0).$$

Therefore

$$g_t=\frac{e^{j\mu t}}{2}\frac{\phi_1(t)}{\phi_1(0)}+\frac{e^{-j\mu t}}{2}\frac{\phi_2(t)}{\phi_2(0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By the same method it can be shown that

$$H_t\equiv\frac{\sqrt{1-b^2}}{h_{2\pi}}h_t=\frac{e^{j\mu t}}{2j}\frac{\phi_1(t)}{\phi_1(0)}-\frac{e^{-j\mu t}}{2j}\frac{\phi_2(t)}{\phi_2(0)} \quad . \quad . \quad . \quad . \quad (10\,a)$$

Since g is real in all cases of present interest, when μ is real it is necessary that the coefficients of the exponentials in (10) be conjugates. Let

$$\frac{\phi_1(t)}{\phi_2(0)}\equiv u_1+ju_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\frac{\phi_2(t)}{\phi_2/0)}\equiv u_1-ju_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11\,b)$$

The functions u_1 and u_2 are periodic with period 2π. Substituting in (10) and (10 *a*) :

$$g=u_1 \cos \mu t - u_2 \sin \mu t \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,a)$$

and

$$H=u_1 \sin \mu t + u_2 \cos \mu t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,b)$$

These are basic equations giving g and h in terms of the periodic functions u_1 and u_2, which are real when μ is real.

Solved for u_1 and u_2, equations (12) yield

$$u_1=H_t \sin \mu t + g_t \cos \mu t \quad . \quad . \quad . \quad . \quad . \quad . \quad (13\,a)$$

$$u_2=H_t \cos \mu t - g_t \sin \mu t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13\,b)$$

which allow the periodic u_1 and u_2 to be determined from the non-periodic g and h.

The periodic functions u_1 and u_2 satisfy

$$u_1''+[\epsilon(1+k \cos t)-\mu^2]u_1=2\mu u_2'$$

and

$$u_2''+[\epsilon(1+k \cos t)-\mu^2]u_2=-2\mu u_1'.$$

By substituting $jU_2 \equiv u_2$ these become

$$u_1''+[\epsilon(1+k \cos t)-\mu^2]u_1=j2\mu U_2'$$

and

$$U_2''+[\epsilon(1+k \cos t)-\mu^2]U_2=j2\mu u_1',$$

which are the equations for the charges u_1 and U_2 in a simple coupled circuit, coupling $j2\mu$, each mesh of which contains a capacitance varying as $1/(\epsilon-\mu^2+\epsilon k \cos t)$ and equal inductances.

It is of technical interest to note that g and h may be considered a sine wave of varying but periodic magnitude and phase angle :

$$g=\sqrt{u_1^2+u_2^2} \cos \left[\mu t+\tan^{-1}\left(\frac{u_2}{u_1}\right)\right]$$

and

$$H=\sqrt{u_1^2+u_2^2} \sin \left[\mu t+\tan^{-1}\left(\frac{u_2}{u_1}\right)\right].$$

In the theory of modulation [11] it is customary to consider several cases, two of which are : (1) amplitude modulation, represented in its simplest form by a sine wave of varying and periodic magnitude and constant phase angle ; (2) phase modulation, represented in its simplest form by a sine wave of varying and periodic phase angle and constant magnitude. It is seen that g and H may each be interpreted as combined amplitude and phase modulated waves.

III. *Properties of the g, h, ϕ_1, ϕ_2, u_1, and u_2 Functions.*

In this section it is shown that (*a*) g is even and h is odd about $t=0$; (*b*) u_1 is even and u_2 is odd about $t=0$; (*c*) u_1 is even and u_2 is odd about $t=\pi$; (*d*) $h'_{2\pi}=g_{2\pi}$; and there are discussed (*e*) the amount of in-

formation required to obtain a complete solution of Mathieu's equation, and (f) possible methods of presenting a complete solution.

(a) *Proof that g is even and h is odd about $t=0$.*—Equation (1) is unchanged when $2\pi+t$ is substituted for t, hence $g_{2\pi+t}$ is a solution, and

$$g_{2\pi+t}=g_{2\pi}g_t+g'_{2\pi}h_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Furthermore, the equation is unchanged if t is replaced by $-t$, hence if $F(t)$ is a solution, $F(-t)$ is also. Therefore $g_{2\pi-t}$ is a solution, which by (14) is

$$g_{2\pi-t}=g_{2\pi}g_{-t}+g'_{2\pi}h_{-t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\,a)$$

But since $g_{2\pi-t}$ is a solution, it can be written $g_{2\pi-t}=Ag_t+Bh_t$ or, determining the constants A and B by setting $t=0$ in $g_{2\pi-t}$ and in the derivative of $g_{2\pi-t}$ with respect to t,

$$g_{2\pi-t}=g_{2\pi}g_t-g'_{2\pi}h_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\,b)$$

Comparing (15 a) and (15 b), each of which holds for all values of t, it follows that

$$g(t)\equiv g(-t) \text{ and } h(t)\equiv -h(-t). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Hence g is even and h is odd about $t=0$.

(b) *Proof that u_1 is even and u_2 is odd about $t=0$.*—That u_1 is even and u_2 odd about $t=0$ follows immediately from (13 a) and (13 b).

For $$u_1(-t)= \quad H_t \sin\mu t+g_t\cos\mu t$$

and $$u_2(-t)\equiv -H_t\cos\mu t+g_t\sin\mu t\,;$$

hence $$u_2(t)\equiv u_1(-t) \text{ and } u_2(t)\equiv -u_2(-t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

whence u_1 is even and u_2 odd about $t=0$. From (11) it follows that $\phi_2(t)=\phi_1(-t)$.

(c) *Proof that u_1 is even and u_2 is odd about $t=\pi$.*—To avoid lengthy algebra a proof that u_1 is even about the point $t=\pi$ will be outlined, with details omitted. By (13)

$$u_1(\pi+t)-u_1(\pi-t)=H_{\pi+t}\sin\mu(\pi+t)+g_{\pi+t}\cos\mu(\pi+t)$$
$$-H_{\pi-t}\sin\mu(\pi-t)+g_{\pi-t}\cos\mu(\pi-t).$$

Expand the sine and cosine terms, and the H terms in terms of $g_{\pi+t}$ and $g_{\pi-t}$ (note that $g'_{2\pi}h_{\pi-t}=h'_{2\pi}g_{\pi-t}-g_{\pi+t}$, etc.). The coefficients of $\sin\mu t$ and $\cos\mu t$ in the resulting expression can be shown to be zero, whence $u_1(\pi+t)=u_1(\pi-t)$ and u_1 is even about $t=\pi$.

Likewise it can be shown that $u_2(\pi+t)\equiv -u_2(\pi-t)$, that is, u_2 is odd about $t=\pi$.

(d) *Proof that* $h'_{2\pi}=g_{2\pi}$.—From $g_{t+2\pi}=g_{2\pi}g_t+g'_{2\pi}h_t$ and a similar expansion of $h_{t+2\pi}$ it follows that

$$g_t=g_{\pi+t}h'_{2\pi+t}-h_{2\pi+t}g'_{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\,a)$$

and

$$h_t=g_{2\pi}h_{2\pi+t}-h_{2\pi}g_{2\pi+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\,b)$$

Likewise from similar expansions for $g_{t-2\pi}$ and $h_{t-2\pi}$,

$$g_t=g_{-2\pi}g_{2\pi+t}+g'_{-2\pi}h_{2\pi+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\,a)$$

and

$$h_t=h_{-2\pi}g_{2\pi+t}+h'_{-2\pi}h_{2\pi+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\,b)$$

Comparing (18) and (19), it is seen that

$$h'_{2\pi}=g_{-2\pi},\ g'_{2\pi}=-g'_{2\pi},\ g_{2\pi}=h'_{-2\pi},\ \text{and}\ h_{2\pi}=-h_{-2\pi}.$$

Because g is even and h is odd about $t=0$, these are redundant, and the only new information yielded is

$$g_{2\pi}=h'_{2\pi}\equiv b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

As a result of this relationship (a similar one holds for g and h' at any point where t is an integer multiple of 2π) the characteristic exponent μ can be easily calculated when *either* g or h' at $t=2\pi$ is available : from (6) and (8)

$$\cos 2\pi\mu=b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

(*e*) *Quantity of Information needed for a Complete Solution of Mathieu's Equation.*—According to (3) both g and h are needed to give a general solution of (1) ; Floquet's theorem reduces the required information to μ and two periodic functions u_1 and u_2, all of which can be obtained if g and h are given for an interval of 2π of t. The proof above that u_1 is even and u_2 is odd about $t=\pi$ reduces the required knowledge to g and h for an interval π of t, and means of calculating μ. There follow the necessary proofs that the complete solution of Mathieu's equation is known when any of the following are available (time intervals are taken beginning at $t=0$ for convenience, other starting-points may be used) : (1) g and h for $0\leqslant t\leqslant\pi$; (2) g for $0\leqslant t\leqslant 2\pi$; (3) h for $0\leqslant t\leqslant 2\pi$.

(1) When g and h are given for $0\leqslant t\leqslant\pi$, the functions u_1 and u_2 can be found (and the complete solution of (1) thus established) as soon as μ is known. A sufficient formula is $g_\pi h'_\pi=\cos^2\mu\pi$, the derivation of which is given in VII.

(2) If g is given for $0\leqslant t\leqslant 2\pi$, the characteristic exponent μ follows from (21) and h, although linearly independent of g, can be calculated from $h_t=(bg_t-g_{2\pi-t})/g'_{2\pi}$ or from any one of numerous other relationships. Hence g and h are known for $0\leqslant t\leqslant 2\pi$, μ can be calculated and u_1 and u_2 found. The combination μ, u_1, and u_2 is sufficient to give the general solution of (1).

(3) When h is given for $0 \leqslant t \leqslant 2\pi$, the equation (21) yields μ at once, and g can be found by $g_t = (bh_t + h_{2\pi - t})/h_{2\pi}$. Hence μ and g and h for $0 \leqslant t \leqslant 2\pi$ are known, and the complete solution of (1) can be obtained.

(f) *Possible Methods of presenting the complete Solution of Mathieu's Equation.*—There are numerous possible methods of presenting functions which enable the complete solution of Mathieu's equation to be obtained, some of which are : (1) g and h can be tabulated for an interval $0 \leqslant t \leqslant \pi$; (2) either g or h can be tabulated for $0 \leqslant t \leqslant 2\pi$; (3) u_1 and u_2 can be tabulated for $0 \leqslant t \leqslant \pi$ (a table of values of μ for various (ϵ, k) combinations must be supplied) ; (4) ϕ_1 and ϕ_2 can be plotted in the complex plane, so that vectors from the origin will give ϕ_1 and ϕ_2 for $0 \leqslant t \leqslant \pi$ (a table of μ values must be supplied).

Of (1)–(4), it seems probable that (3) would be best adapted to technical needs, and indeed to most purposes, since the periodicity of u_1 and u_2 gives a start to the analysis of any particular solution. However, a tabulation of the u functions must be accompanied by a table of μ to present the solution of (1), and to calculate μ requires, according to the method being developed in this paper, either $g_{2\pi}$ or $h'_{2\pi}$ or some combination which will yield b. If this requires the plotting of either the g or h solution, the amount of work required to give the u tabulation is appreciably greater than that for g or h. A limited table of values of μ for various (ϵ, k) is given in V.

IV. *Sample Curves of the g and h Solutions and of the u_1 and u_2 Functions.*

Figs. 1 and 2 show g and g' and h and h' respectively in the interval $0 \leqslant t \leqslant 4\pi$ for the case $\epsilon = 9$, $k = 0{\cdot}9$. For this particular set of parameters a value of μ is $0{\cdot}212$. Fig. 3 shows the corresponding u functions. Figs. 4 and 5 show g and g', and h and h', respectively, for the interval $0 \leqslant t \leqslant -2\pi$.

V. *The Calculation of the Characteristic Exponent.*

The characteristic exponent * μ can always be taken as a real or a pure imaginary, depending on (ϵ, k) ; it need not be taken as a complex with both real and imaginary parts not zero. When μ is real, the solution is

---

* Whittaker and Watson (p. 413) state : " . . . . it might be expected (from Floquet's theory) that the determination of an explicit expression for the solution . . . . of Mathieu's . . . . equation . . would be a comparatively easy matter ; this however is not the case . . . . The crux of the problem is to determine $[\mu]$; when this is done, the determination of $[\phi_1]$ presents comparatively little difficulty." In the present case the determination of μ is not difficult. See also Barrow, Proc. I. R. E. xxi. no. 8, p. 1182 (1933).

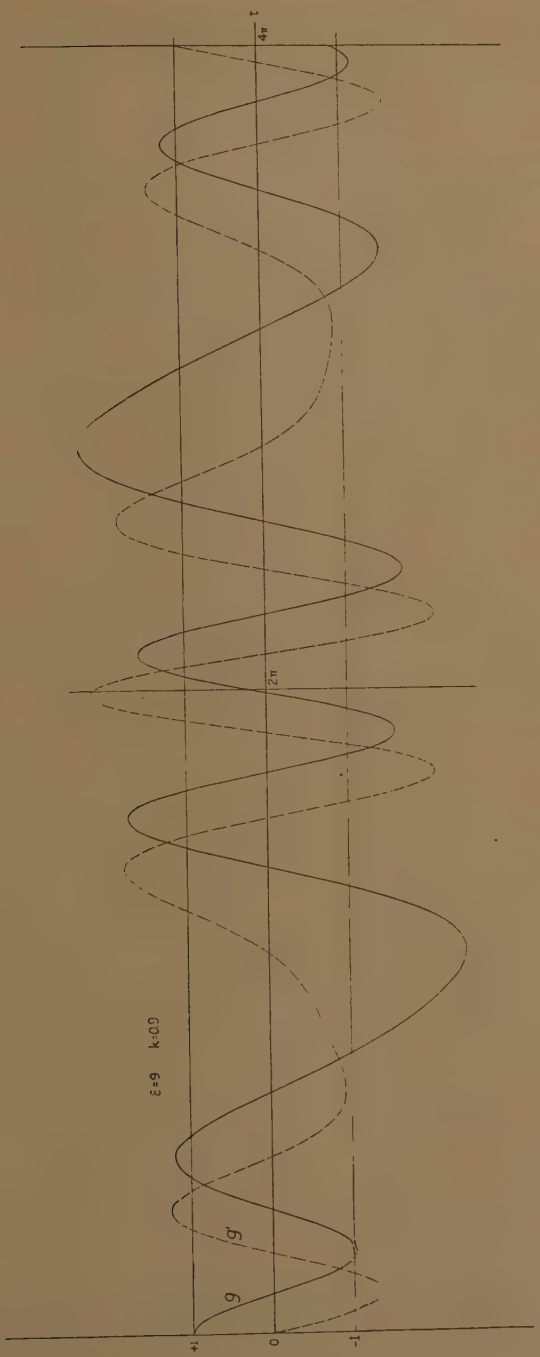


Fig. 1.

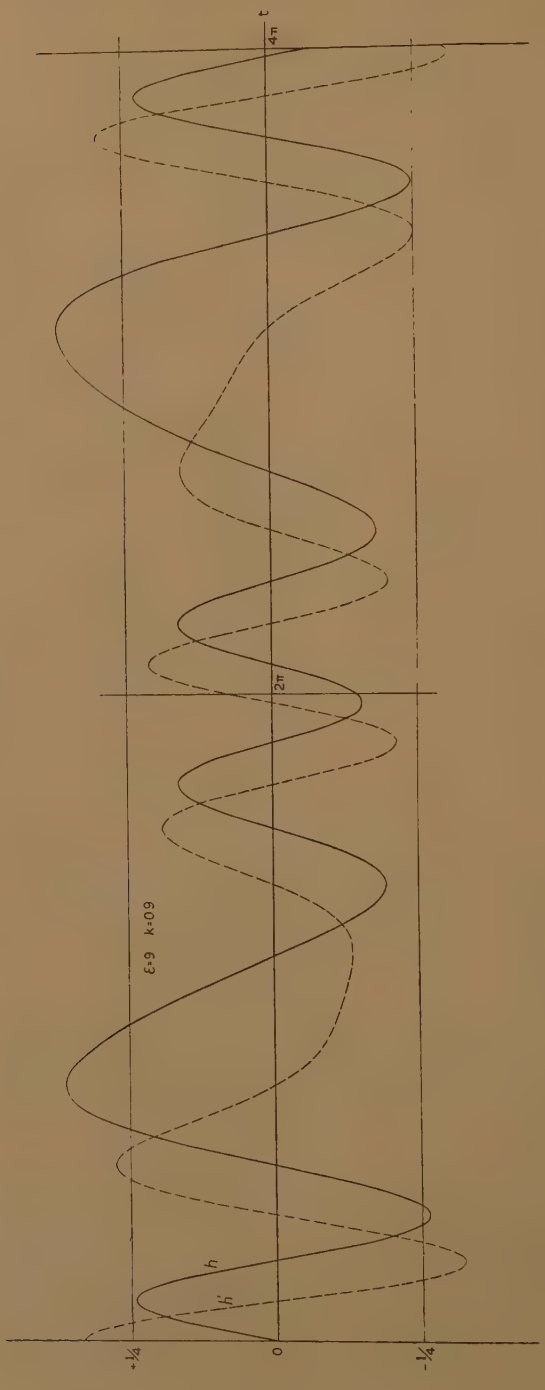


Fig. 2.

Fig. 3.

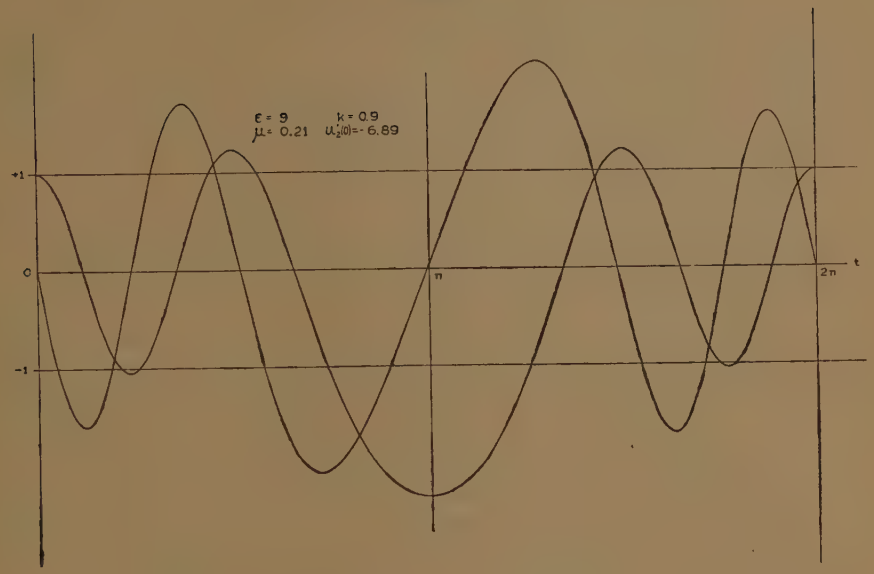
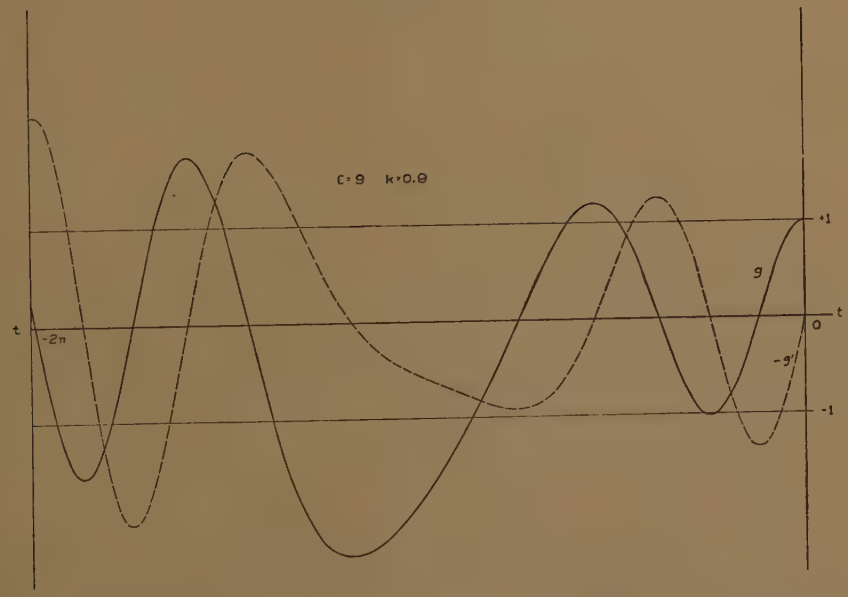


Fig. 4.



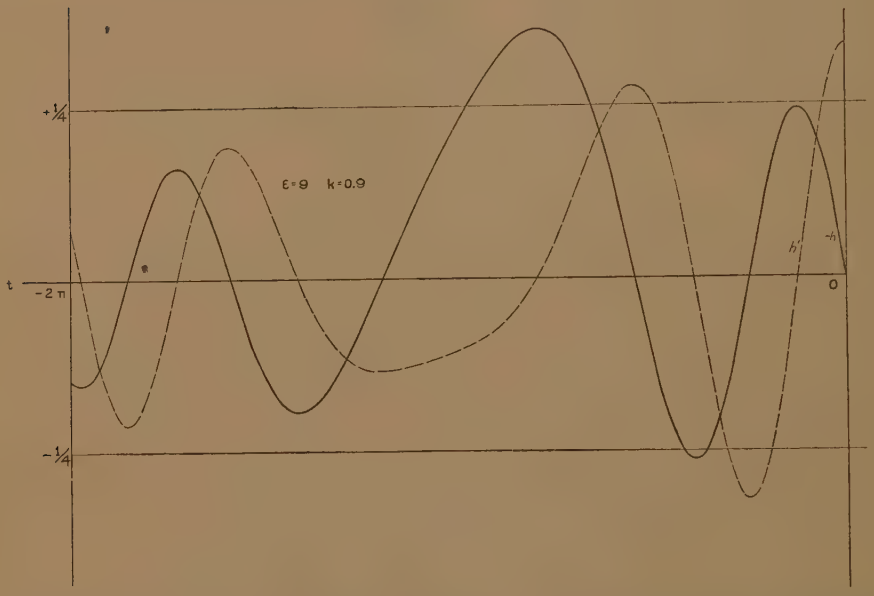
defined as stable ; when μ is purely imaginary the converse holds. Between an area in the ϵ, k plane in which μ is real, and an adjacent area in which μ is a pure imaginary, there is a line along which μ is such that the solution

of Mathieu's equation is periodic (neither stable nor unstable). From equations (12) it follows that μ must be an integer or an integer divided by 2 along such a line in the ϵ, k plane.

The characteristic exponent μ can be determined from the extremely simple equation $\cos 2\pi\mu = b$, and consequently a simple criterion of stability is

$$|b| = |g_{2\pi}| = |h'_{2\pi}| < 1 \text{ for } \mu \text{ real},$$
$$|b| = |g_{2\pi}| = |h'_{2\pi}| > 1 \text{ for } \mu \text{ a pure imaginary}.$$

Fig. 5.



Other expressions for μ may be derived. For example, using equations of the type of (10), (18), and (19), $g_{N2\pi} = h'_{N2\pi} = \cos N2\pi\mu$ where N is any integer. Likewise, it is shown in section VII. that $\cos^2 \mu\pi = g_\pi h'_\pi$.

In Table I. are given some values of $b \equiv g_{2\pi} = h'_{2\pi}$ for various ϵ and k, obtained mechanically. No attempt was made to obtain great accuracy, and to be conservative the accuracy should be considered * about 1 per cent.

* Actually $g_{2\pi}$ and $h'_{2\pi}$ were obtained separately to four significant figures, and in most cases the two values agreed to within 0·2 per cent. However, when the check was poorer, but was within 1 per cent., the results were averaged to give the value in the table. The latter should be taken as a preliminary presentation ; an appreciably larger table will be needed for practical use. The omission of data for $k = 0{\cdot}7$ and for $3\sqrt{\epsilon} = 7$ is due to temporary mechanical difficulties. The omission is of no significance as far as b is concerned.

TABLE I.

Values of b for various ϵ, k.

$k\rightarrow$ $3\sqrt{\epsilon}\downarrow$	0·1.	0·2.	0·3.	0·4.	0·5.	0·6.	0·8.	0·9.	1·0.
1.....	−0·502	0·505	−0·510	−0·516	−0·524	−0·534	−0·560	−0·575	−0·592
2.....	−0·502	−0·517	−0·543	−0·578	−0·625	−0·681	−0·821	−0·906	−0·999
3.....	1·001	1·002	1·003	1·006	1·012	1·024	1·072	1·112	1·172
4.....	−0·501	−0·484	−0·456	−0·410	−0·353	−0·270	−0·030	0·128	0·329
5.....	−0·501	−0·522	−0·549	−0·596	−0·657	−0·738	−0·977	−1·150	−1·381
6.....	1·004	1·000	0·998	0·991	0·978	0·949	0·819	0·693	0·514
8.....	−0·506	−0·535	−0·572	−0·636	−0·718	−0·813	−1·018	−1·108	−1·177
9.....	1·002	1·001	0·997	0·983	0·951	0·890	0·575	0·232	−0·347

In the ϵ, k plane, areas corresponding to stable solutions are separated from non-stable areas by a curve along which the solution of (1) is periodic. From the data available and the general uniformity, *i. e.*, lack of singularities, in (1) it appears that the unstable areas may be classified into two groups, in one of which b is always negative and $-b>1$ and in the other of which b is always positive and $b>1$. Furthermore, these two appear to alternate—in passing across the ϵ, k plane parallel to the ϵ axis, an unstable region in which b is positive appears to be between two unstable regions in which b is negative, and *vice versa*. (This disregards the stable region which separates any two consecutive unstable areas.)

VI. *Relations between the Derivatives of the g and h Solutions.*

It is possible to set up a number of relations between the derivatives of g and h, and also some equivalent relations involving certain integrals. A few of these are listed below. Equation (23) is a fundamental relation which holds for any two solutions of (1); equation (22) follows immediately * upon integration of (23) and use of the values of the g and h solutions at $t=0$; the other equations may be derived by differentiation and appropriate substitutions. All the derivatives of g and h at a given point can be found when g, h, g', and h' are known at that point. Equation (22) reduces the information required to any three of these.

Examples of relationships between g and h and their derivatives :

$$gh' - hg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$gh'' - hg'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$gh''' - hg''' = -\epsilon(1+k\cos t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$gh^{\mathrm{IV}} - hg^{\mathrm{IV}} = 2\epsilon k \sin t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$gh^{\mathrm{V}} - hg^{\mathrm{V}} = 3\epsilon k\cos t + \epsilon^2(1+k\cos t)^2 . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Examples of relationships between g' and h' and derivatives of g and h :

$$g'h - h'g = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$g'h'' - h'g'' = \epsilon(1+k\cos t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$g'h''' - h'g''' = -\epsilon k \sin t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$g'h^{\mathrm{IV}} - h'g^{\mathrm{IV}} = -\epsilon k\cos t - \epsilon^2(1+k\cos t)^2 \quad . \quad . \quad . \quad . \quad (30)$$

$$g'h^{\mathrm{V}} - h'g^{\mathrm{V}} = \epsilon k \sin t[1+4\epsilon(1+k\cos t)] . \quad . \quad . \quad . \quad . \quad (31)$$

Equation (22) may be written

$$\frac{d}{dt}\left(\frac{h}{g}\right) = \frac{1}{g^2} \text{ or } \frac{d}{dt}\left(\frac{g}{h}\right) = -\frac{1}{h^2},$$

* Equation (22) can be derived directly from a theorem due to Abel and used by Whittaker and Watson [4], p. 419.

whence*

$$h=g\int\frac{1}{g^2}\,dt \quad \ldots\ldots\ldots \quad (32)$$

$$h'=g'\int\frac{1}{g^2}\,dt+\frac{1}{g} \quad \ldots\ldots\ldots \quad (33)$$

$$h''=g''\int\frac{1}{g^2}\,dt \text{ etc.} \quad \ldots\ldots\ldots \quad (34)$$

VII. *Values of the Derivatives of the g and h Solutions at certain Points.*

Below are listed some values of, and relationships between, certain of the derivatives of g and h at $t=0,\ \pi,\ 2\pi$.

At $t=0$, all odd derivatives of g and even derivatives of h are zero, and

$$g=1, \qquad h'=1,$$
$$g''=-\epsilon(1+k), \qquad h'''=-\epsilon(1+k),$$
$$g^{\mathrm{IV}}=\epsilon^2(1+k)^2+\epsilon k, \qquad h^{\mathrm{V}}=\epsilon^2(1+k)^2+3\epsilon k.$$
$$g^{(\mathrm{N})}=h^{(\mathrm{N}+1)}, \quad \mathrm{N}=0,\ 1,\ 2,\ 3.$$

At $t=2\pi$, $h'=g$ and $h''=g''$ as at $t=0$, and further at $t=2\pi$

$$g=b, \qquad h=a,$$
$$g'=c, \qquad h'=b,$$
$$g''=\frac{b}{a}h'', \qquad h''=-\epsilon(1+k)a,$$
$$g'''=\frac{c}{b}h''', \qquad h'''=-\epsilon(1+k)b,$$
$$g^{\mathrm{IV}}=\frac{b}{a}h^{\mathrm{IV}}, \qquad h^{\mathrm{IV}}=[\epsilon^2(1+k)^2+\epsilon k]a,$$
$$g^{\mathrm{V}}=\frac{c}{b}h^{\mathrm{V}}, \qquad h^{\mathrm{V}}=[\epsilon^2(1+k)^2+3\epsilon k]b.$$
$$g^{(\mathrm{N})}=\frac{b}{a}h^{(\mathrm{N})}, \quad \mathrm{N}=0,\ 2,\ 4,$$
$$g^{(\mathrm{N})}=\frac{c}{b}h^{(\mathrm{N})}, \quad \mathrm{N}=1,\ 3,\ 5,$$
$$g_{2\pi}^{(\mathrm{N})}=bg_0^{(\mathrm{N})}, \quad \mathrm{N}=0,\ 2,\ 4,$$
$$h_{2\pi}^{(\mathrm{N})}=bh_0^{(\mathrm{N})}, \quad \mathrm{N}=1,\ 3,\ 5.$$

* Equation (32) is similar to a result given by Whittaker and Watson [4], p. 427.

The relations at $t=2\pi$ are of the same form as those for t any integer multiple of 2π (including zero) provided a, b, and c are taken as h, $h'=g$, and g' respectively for the particular value of t used. Thus at $t=0$, a and c should be replaced by zero and b by 1, which reduces the above relationships to those given for $t=0$, It should be borne in mind that b, a, and c are not independent, since $b^2-ac=1$.

At $t=\pi$, the derivatives can be expressed in terms of b and a ($=h'_{2\pi}$ and $h_{2\pi}$ respectively) and one quantity determined at $t=\pi$. Taking the latter to be h_π,

$$g=\frac{b+1}{a}h_\pi, \qquad h=h_\pi,$$

$$g'=\frac{b-1}{2h_\pi}, \qquad h'=\frac{a}{2h_\pi},$$

$$g''=-\epsilon(1+k)\frac{b+1}{a}h_\pi, \qquad h''=-\epsilon(1-k)h_\pi,$$

$$g'''=-\epsilon(1-k)\frac{b-1}{2h_\pi}, \qquad h'''=-\epsilon(1-k)\frac{a}{2h_\pi},$$

$$g^{IV}=[\epsilon^2(1-k)^2-\epsilon k]\frac{b+1}{a}h_\pi, \qquad h^{IV}=[\epsilon^2(1-k)^2-\epsilon k]h_\pi.$$

$$\frac{g^{(N)}}{h^{(N)}}=\frac{b+1}{a}=\frac{c}{b-1}, \quad N=0, 2, 4, \ldots$$

$$\frac{g^{(N)}}{h^{(N)}}=\frac{b-1}{a}=\frac{c}{b+1}, \quad N=1, 3, 5, \ldots$$

$$\frac{g''}{g}=\frac{g'''}{g'}=-\epsilon(1-k)=\frac{h''}{h}=\frac{h'''}{h'},$$

It may be noticed (1) that all derivatives at $t=2\pi$ are known when g, g', h and h', are known at $t=\pi$, but the converse has not been shown, and (2) that $2h_\pi h'_\pi=a$, $2g_\pi g'_\pi=c$, and $2g'_\pi h_\pi+1=b=2g_\pi h'_\pi-1$. From the latter

$$g_\pi h'_\pi=\frac{b+1}{2}=\cos^2\mu\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (35\,a)$$

and

$$g'_\pi h_\pi=\frac{b-1}{2}=-\sin^2\mu\pi, \quad . \quad . \quad . \quad . \quad . \quad (35\,b)$$

which give μ in terms of measurements at $t=\pi$.

The relationships

$$2h_\pi h'_\pi=a,\ 2g_\pi g'_\pi=c \quad \text{and} \quad 2g'_\pi h_\pi+1=b=2g_\pi h'_\pi-1 \quad . \quad . \quad (36)$$

yield b, a, and c, which also can be obtained from other relations of similar form :

$$2h'_\pi h''_\pi=-\epsilon(1-k)\,a \text{ and } 2g'_\pi g''_\pi=-\epsilon(1-k)c$$

$$\frac{2g_\pi' h_\pi''}{-\epsilon(1-k)} + 1 = b = \frac{2g_\pi'' h_\pi}{-\epsilon(1-k)} - 1.$$

$$2h_\pi'' h_\pi''' = \epsilon^2(1-k)^2 a \quad \text{and} \quad 2g_\pi'' g_\pi''' = \epsilon^2(1-k)^2 c.$$

VIII. *Series for, and Zeros of, the g and h Solutions.*

The discussion of the series and zeros of the g and h solutions, to which this section is devoted, will be limited. It is hoped to present a more detailed report on this aspect of the study at some future time.

Probably the most important result of the proof in Section III. that a knowledge of g and h for $0 \leqslant t \leqslant \pi$ is sufficient to give the complete solution of Mathieu's equation is that simple series can be used to determine g and h in the required interval, whereas such series present difficulties when the interval $0 \leqslant t \leqslant 2\pi$ (to which Floquet's theorem had reduced the problem) must be covered.

Both g and g' are specified at $t=0$, hence all derivatives of g can be found at that point by differentiating $g''=xg$ where $x=-\epsilon(1+k\cos t)$. Since xg is a product, the higher derivatives of g can be written down by Liebniz's rule. The calculation of the successive derivatives is thus a simple, straightforward procedure, although tedious. The number of terms required in a Taylor's series for an accuracy of, say, four decimal places appears to be large, but not great, in the range to which this paper is limited ($\epsilon \leqslant 10$, $k \leqslant 1$). Specifically, for $\epsilon=9$ and $k=0{\cdot}9$, g has been calculated from the Taylor's series. Eleven non-zero terms were required to give g correct to one part in 10,000 at $t=0{\cdot}3\pi$. Thereafter the curve was continued to $t=\pi$ by the Kutta-Runge [12] method, the application of which is quite simple in the current case.

Assuming that the series converges more rapidly for smaller values of ϵ and k (this appears likely), the g solution can be obtained for any given (ϵ, k) in the range to which this paper is limited by an appreciable, but not an extreme, amount of work.

From the table of values of derivatives of g at $t=0$, given in the preceding section,

$$g = 1 - \epsilon(1+k)\frac{t^2}{2!} + [\epsilon^2(1+k)^2 + \epsilon k]\frac{t^4}{2!} + \cdots$$

For sufficiently small values of t, g can be closely determined by the first three terms on the right-hand side. In this range the first zero of g (zero nearest $t=0$) may be fixed if the roots for $g=0$ are real :

$$0 = 1 - \epsilon(1+k)\frac{t^2}{2} + [\epsilon^2(1+k)^2 + \epsilon k]\frac{t^4}{24},$$

or

$$t^2 = \frac{6\beta \pm 3{\cdot}464\sqrt{\sigma^2 - 2\epsilon k}}{\sigma^2 + \epsilon k}, \quad \ldots\ldots\ldots \quad (37)$$

TABLE II.
First Zeros of g.

$k \rightarrow$.	0·1	0·2.	0·3.	0·4.	0·5.	0·6.	0·8.	0·9.	1·0.
For $\epsilon=4$:									
t (scaled from curves) .	0·750	0·719	0·693	0·667	0·646	0·625	0·589	0·573	0·557
t (calculated)	0·762	0·732	0·704	0·679	0·657	0·637	0·601	0·585	0·570
For $\epsilon=9$:									
t (scaled from curves) .	0·500	0·479	0·458	0·443	0·427	0·411	0·391	0·380	0·370
t (calculated)	0·507	0·486	0·467	0·451	0·435	0·422	0·398	0·387	0·377

where $\beta=\epsilon(1+k)$. Practically, this formula gives a close approximation for many values of ϵ, k. Table II. shows a comparison of some values calculated from (37) and those obtained from the mechanically-plotted g curves. The usefulness of (37) disappears as ϵ becomes small; for $\epsilon=1$ and $0\leqslant k\leqslant 1$, the formula gives results which in extreme cases are incorrect by 0·5 radian.

The Taylor's series about $t=0$ for h' is

$$h'=1-\epsilon(1+k)\frac{t^2}{2!}+[\epsilon^2(1+k)^2+3\epsilon k]\frac{t^4}{4!}+\ldots,$$

and the first three terms bear a close resemblance to the corresponding series for g. For t sufficiently small so that h' can be approximated by the first three terms on the right-hand side, h' will be zero when

$$0=1-\epsilon(1+k)\frac{t^2}{2}+[\epsilon^2(1+k)^2+3\epsilon k]\frac{t^4}{24},$$

or

$$t^2=\frac{6\beta\pm 3\cdot 464\sqrt{\beta^2-6\epsilon k}}{\sigma^2+3\epsilon k} \quad \ldots\ldots\ldots \quad (38)$$

provided the root is not complex. When β^2 is appreciably greater than $6\epsilon k$, the first zero of g will fall near the first zero of h'. For example, for $\epsilon=9$ and $k=0\cdot 9$, the first zero of a g and h' fall as shown in Table III.

TABLE III.

	t from curve.	t from formula.
$g=0$	0·380	0·387
$h=0$	0·385	0·391

In Table IV. are given all the zeros of g, g', h, h' for $0\leqslant t\leqslant 2\pi$ and $\epsilon=9$, $k=0\cdot 9$; these results are given in units (sixths of an inch) used in scaling from the graphs and can be converted to t by using the fact that 2π units of $t=120\cdot 7$ sixths of an inch on the graphs.

It is seen from the table that not only do the first zeros of g and h', and the first zeros after $t=0$ of g' and h, but also all other zeros of each pair fall more or less close to one another. This illustrates a result which has been found to be correct within rather gross limits (compare g at 45·0 and h' at 47·5 in Table IV.) for cases in which a check has been made.

It will be noticed that g, h, g', h' each have six zeros in $0\leqslant t\leqslant 2\pi$. This illustrates another rule found to hold in all cases studied, viz., the number of zeros approximates $2\sqrt{\epsilon}$. This is indicated by (1), since when $k=0$ the number of zeros will be exactly $2\sqrt{\epsilon}$ in $0\leqslant t\leqslant 2\pi$, and it is not

unreasonable in view of the uniformity of the variation with ϵ of solutions of Mathie equation.

IX. *Summary and Acknowledgments.*

To summarize : Within the parameter range considered here ($0 \leqslant \epsilon \leqslant 10$, $0 \leqslant k \leqslant 1$) stable solutions of Mathieu's equation, which is becoming increasingly important technically, may be obtained by determining the specific solutions g and h in the interval $0 \leqslant t \leqslant \pi$ or either g or h in $0 \leqslant t \leqslant 2\pi$. That only this amount of information is needed depends on the proofs given above that u_1 and u_2 are even and odd respectively about $t=\pi$. The determination of g and h for $0 \leqslant t \leqslant \pi$ can be carried out by numerical methods, as indicated. The u functions follow from

Table IV.

Zeros of g, g', h, h' ; for units used see text.
$\epsilon=9$, $k=0{\cdot}9$.

g.	h'.	g'.	h.
		0	0
7·3	7·4		
		15·0	14·9
23·1	23·4		
		32·0	32·6
45·0	47·5		
		71·2	71·8
87·2	85·1		
		96·5	95·6
105·0	104·0		
		110·8	111·7
120·0	119·1		

(13) and the characteristic exponent from one of the very simple formulas for μ developed above. The general solution for all t is then known. The first zeros of g and h can be located to good approximation by formulas derived in the body of the paper. The table of values of μ given should be helpful in immediately fixing this important quantity once ϵ and k are specified, but a much more extensive table is desirable. (The authors have available solutions of Mathieu's equation for all ϵ, k pairs for which μ values are given in Table I.)

It is a pleasure to acknowledge the friendly aid and criticism of our colleagues Dean Harold Pender, Professor James A. Shohat, and Mr. Stanley B. Hare.

Bibliography.

(1) Barrow, Smith, and Bauman, "Further Study of Oscillatory Circuits having Periodically Varying Parameters," Jl. Franklin Inst. ccxxi. nos. 3 and 4, pp. 403 and 509 (1936). This paper contains references to others, some of which have bibliographies.

(2) Carson, "Notes on the Theory of Modulation," Proc. I. R. E. x. no. 1, p. 57 (1932).

Ataka, "On Superregeneration of an Ultra-Short-Wave Receiver," Proc. I. R. E. xxiii. no. 8, p. 841 (1935).

(3) Crary, "Two-Reaction Theory of Synchronous Machines," 'Electrical Engineering,' lvi. no. 1, p. 27 (1937); Wagner, "Unsymmetrical Short Circuits on Water-Wheel Generators Under Capacitive Loading," 'Electrical Engineering,' lvi. no. 11, p. 1385 (1937). These two papers do not contain Mathieu's equation explicitly, but by reasonable approximations it can be shown that the results there given reduce to Mathieu's equation for certain special cases.

(4) Whittaker and Watson, 'Modern Analysis,' 1935 edition, p. 406; Van der Pol and Strutt, "On the Stability of the Solutions of Mathieu's Equation," Phil. Mag. v. p. 18 (1928). This paper lists numerous applications with references to papers in which the various physical problems are discussed.

(5) Strutt, 'Lamesche-Mathieusche und Verwandte Funktionen etc.' Berlin, 1932. (Contains extensive bibliography.)

(6) Whittaker and Watson, *loc. cit.* p. 404.

(7) Ince, "Researches into the Characteristic Numbers of the Mathieu Equation," Proc. Royal Soc. of Edin. xlvi. pp. 20 and 316 (1925); xlii. p. 294 (1927); lii. p. 355 (1931). New tables of Mathieu functions are being prepared by Professor P. M. Morse of the Massachusetts Institute of Technology.

(8) Whittaker and Watson, *op. cit.* p. 412.

(9) Ince, 'Differential Equations,' London, 1927, chapter vii.

(10) Whittaker and Watson, *loc. cit.* p. 412.

(11) Roder, "Amplitude, Phase, and Frequency Modulation," Proc. I. R. E. xix. no. 12, p. 2145 (1931).

(12) Runge and Konig, 'Numerisches Rechnen,' Berlin, 1924, p. 311.

XLIV. *Einstein's Distant Parallelism and Dirac's Equation.*

By H. W. Haskey †.

[Received January 26, 1940.]

§ 1. *Einstein's Distant Parallelism.*

Einstein [1] introduced a new metric which adds to the Riemannian geometrical scheme a definition of parallelism at separate points. A system of orthogonal unit vectors is set up at each point of Riemannian space, and at first there is no relation between the orthogonal system at one point P and that at another point Q. If the components of a vector at P along the orthogonal system at P are equal to the corresponding components of a vector B at Q along the orthogonal system at Q, the vectors are defined as parallel. But they are also equal in magnitude, so that the definition makes it possible to equate vectors at the two points. The vectors A and B are referred to Galilean axes, and their components are denoted by A_a, B_a, $a=1, 2, 3, 4, 5$. Each orthogonal component ϵ_a has components, h_a^μ, along the Riemannian axes x^μ, $\mu=1 \ldots 5$, which are, in general, curved lines. The h_a^μ with a fixed are components of a Riemannian vector. It follows that the h_a^μ link the Galilean and Riemannian continua.

Corresponding to the Galilean vector A there is a Riemannian one, of the same length, with components A^μ given by

$$A^\mu = h_a^\mu A_a, \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$A_a = h_a^\mu A_\mu. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Associated with the h_a^μ are the quantities h_μ^a, the cofactor of h_a^μ in the determinant $|h_a^\mu| = H$ divided by H. This gives

$$A_\mu = h_{\mu a} A_a = h_{\mu a} A_a. \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$A_a = h_{a\mu} A^\mu. \quad \ldots\ldots\ldots\ldots \quad (4)$$

From the equality of length of A in both systems

$$A^2 = A_a A_a = h_a^\mu h_a^\nu A_\mu A_\nu$$
$$= \gamma^{\mu\nu} A_\mu A_\nu$$

for any vector A,

$$\therefore \quad \gamma^{\mu\nu} = h_a^\mu h_a^\nu. \quad \ldots\ldots\ldots\ldots \quad (5)$$

---

† Communicated by Dr. H. T. Flint.

Similarly, $$\gamma_{\mu\nu}=h_{\mu a}h_{\nu a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and $$\delta^{\nu}_{\mu}=h_{\mu a}h^{\nu}_{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

with $$\delta^{a}_{b}=h^{a}_{\mu}h^{\mu}_{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As a special case of the foregoing we have

$$\iota^{\mu}=h^{\mu}_{a}\epsilon_{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

$$\epsilon_{a}=h^{\mu}_{a}\iota_{\mu}, \text{ etc.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

where ι^{μ} is the unit curvilinear vector along the μth axis and ϵ_{a} is the unit orthogonal vector along the ath Galilean axis at and near a given point. We shall later introduce equations analogous to (1.1) and (2.1) in which the matrices of Dirac and Tetrode occur.

There is nothing to fix the orientation of the Galilean axes at any point with respect to those at another point if the Riemannian system is fixed. Hence if physical phenomena are described by equations in terms of the h^{μ}_{a} they must be invariant for rotations of the h^{μ}_{a}. Einstein shows that if

$$\mathrm{A}^{*}_{b}=d_{ab}\mathrm{A}_{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

then the $h_{\mu a}$ give rise to $$h^{*}_{\mu a}=d_{ab}h_{\mu b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and that $$d_{ca}d_{cb}=\delta_{ab}=d_{ac}d_{bc}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the d_{ab} being orthogonal and independent of the Riemannian coordinates.

§ 2. *The Application of Distant Parallelism to Physics.*

The quantity

$$\mathrm{A}^{\alpha}_{\mu\alpha}=h^{\alpha}_{a}\left(\frac{\partial h^{a}_{\mu}}{\partial x^{\alpha}}-\frac{\partial h^{a}_{\alpha}}{\partial x^{\mu}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

turns out to be the component of a Riemannian vector, although it arises only in the study of distant parallelism. Flint [2] used this metric in four dimensions identifying (12) with $\frac{2\pi\iota}{h}\frac{\partial \log \psi}{\partial x^{\mu}}$, where h is Planck's constant and ψ the Schroedinger wave function. By considering the expression div (grad ψ) in a form appropriate to the new metric he found that Schroedinger's second order wave equation without spin was

$$\text{div (grad } \psi)=0.$$

§ 3. *Dirac's Equation in Galilean and Riemannian Spaces.*

The appearance of Dirac's first order equation with the matrices it introduced proved disconcerting to relativists whose geometrical apparatus—the tensor calculus—did not deal with matrices. It was left to Tetrode to introduce matrices γ^{μ} with four rows and four columns (like Dirac's) which satisfied

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\gamma^{\mu\nu}.1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The γ^μ for $\mu=1 \ldots 4$ were components of a Riemannian vector, and they were analogous to Dirac's matrices E^a which satisfied

$$E_a E_b + E_b E_a = 2\delta_{ab}, \quad \ldots\ldots\ldots \quad (14)$$

the square of any one E_a being unity. It was then possible to express Dirac's equation in tensor form, but the equation was not a direct consequence of the Riemannian structure as was Einstein's law of gravitation, $G_{\mu\nu}=0$. Flint [3] realized that the return to geometric ideas was essential for a derivation of Dirac's equation and, noting that $(\gamma^\mu A_\mu)^2$ is the square of the vector A, he defined the matrix-length of A to be $\gamma^\mu A_\mu$. Regarding the wave function ψ as a scale factor, he showed that the change of $\gamma^\mu A_\mu \psi$ on parallel displacement (in a Riemannian five-dimensional continuum gave

$$\left(\gamma^\mu \frac{\partial \psi}{\partial x^\nu} + K^\mu_\nu \psi\right) A_\mu dx^\nu, \quad \ldots\ldots\ldots \quad (15)$$

wherein $K^\mu_\nu = \gamma^\mu{}_{:\nu}$, the Riemannian covariant derivative of γ^μ, which Schroedinger [4] showed might be taken as $\gamma^\mu \Gamma_\nu - \Gamma_\nu \gamma^\mu$ with Γ_ν a matrix.

If it were asserted that the expression (15) vanished for all vectors and all displacements, ψ reduced to a constant. The situation was similar to that in Einstein's gravitational theory, viz., $B_{\mu\nu\sigma\rho} \neq 0$, but the contracted form vanished, giving $G_{\mu\nu}=0$. Accordingly the geometric structure was described by

$$K^\mu{}_{:\mu} = 0, \quad \ldots\ldots\ldots \quad (16)$$

or

$$\gamma^\mu \Gamma_\mu = \Gamma_\mu \gamma^\mu, \quad \ldots\ldots\ldots \quad (16.1)$$

and this gave Dirac's equation in the form

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0, \quad \ldots\ldots\ldots \quad (17)$$

provided that the Riemannian five world satisfied the Kaluza-Klein relations

$$\gamma_{mn} = g_{mn} + \frac{\gamma_{m5}\gamma_{n5}}{\gamma_{55}}, \quad \ldots\ldots\ldots \quad (18)$$

$m, n = 1 \ldots 4.$

$$\gamma_{m5} = k\phi_m, \quad \ldots\ldots\ldots \quad (19)$$

wherein

$$k = \frac{e}{m_0 c^2}, \quad \ldots\ldots\ldots \quad (19.1)$$

ϕ_m the e.m. potential component

$$\gamma_{55} = 1. \quad \ldots\ldots\ldots \quad (20)$$

$$\gamma^{m5} = -k\phi^m. \quad \ldots\ldots\ldots \quad (21)$$

$$\gamma^{55} = 1 + k^2 \phi^m \phi_m. \quad \ldots\ldots\ldots \quad (21.1)$$

$$\gamma^{mn} = g^{mn}, \quad \ldots\ldots\ldots \quad (21.2)$$

and x^5 is a cyclic coordinate so that

$$\frac{\partial\psi}{\partial x^5}=\frac{2\pi\imath m_0 c}{h}\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

while

$$\gamma^5=\alpha_0-k\phi_m\alpha^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or

$$\alpha_0=\frac{\gamma_{\mu 5}\gamma^\mu}{\sqrt{\gamma_{55}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and $\alpha_0^2=1$ with

$$\alpha^m\alpha_0+\alpha_0\alpha^m=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.1)$$

the α^m being Tetrode's matrices in four dimensions. Hence (17) becomes

$$\gamma^m\frac{\partial\psi}{\partial x^m}+\frac{2\pi\imath m_0 c}{h}(\alpha_0-k\phi_m\alpha^m)\psi=0. \quad . \quad . \quad . \quad . \quad (17.1)$$

§ 4. *The Dirac-Eddington Matrices.*

Simultaneously with these investigations Eddington [5] developed the calculus of wave-tensors and added a fifth matrix $E_5\equiv E_1E_2E_3E_4$ to the four appearing in Dirac's original equation. It follows that (14) holds for $a,\ b=1\ldots5$ and that $(E_5)^2=1$. He also showed that there were other pentads like $(E_1, E_2, E_3, E_4, E_5)$ *e. g.* $(E_5E_1, E_5E_2E_5E_3, E_5, E_5E_4)$. Thus Dirac's own expression of his equation

$$\left(E_1\frac{\partial}{\partial z^1}+E_2\frac{\partial}{\partial z^2}+E_3\frac{\partial}{\partial z^3}-\frac{2}{c\,\partial t}+E_4\frac{\partial\pi\imath m_0 c}{h}\right)\psi=0. \quad . \quad . \quad (25)$$

could be written as

$$\left(E_5E_1\frac{\partial}{\partial z^1}+E_5E_2\frac{\partial}{\partial z^2}+E_5E_3\frac{\partial}{\partial z^3}+E_5\frac{\partial}{\partial z^4}+E_3E_4\right)\psi=0,$$

or as

$$E_a\frac{\partial\psi}{\partial z^a}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where E_a, with $a=1\ldots5$ is a pentad and z^5 is a cyclic coordinate. Any two members of a pentad anticommute, and the square of each is unity. This being the case, E_a in (26) might be generalized to VE_aV^{-1}, where V is any matrix. Corresponding to α_0 in (24) in a Galilean world we have

$$\frac{\gamma_{55}E_5}{\sqrt{\gamma_{55}}}=E_5,$$

since for a Galilean world we can always take $\gamma_{\mu 5}=0$ unless $\mu=5$.

§ 5. *Object of this Investigation.*

The object of this paper is to show that Einstein's geometry of distant parallelism forms the natural link between the Tetrode and Dirac matrices,

and that these matrices bear the same relationship to each other as do the unit vectors ι^μ and ϵ_a in (1.1) and (2.1). It will also be shown that

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} \equiv \mathrm{E} \frac{\partial \psi}{\partial z^a},$$

so that Dirac's equation for a Galilean five-world is of precisely the same form as Flint's in a Riemannian five-world over any given small region.

§ 6. *The Connexion between the Dirac and Tetrode Matrices and Distant Parallelism.*

If E_a, $a=1 \ldots 5$ is a pentad or generalized pentad and γ^μ are the Tetrode matrices it is suggested that

$$\gamma^\mu = h_a^\mu \mathrm{E}_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

This equation corresponds to (1), and in (27) we shall show that the h_a^μ are the Einstein " beins."

On substituting (27) into Tetrode's relation (13) and using (14)

$$2\gamma^{\mu\nu} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = (\mathrm{E}_a \mathrm{E}_b + \mathrm{E}_b \mathrm{E}_a) h_a^\mu h_b^\nu = 2\delta_{ab} h_a^\mu h_b^\nu = 2h_a^\mu h^{\nu a}. \quad . \quad (28)$$

Comparing (28) with (5) it is seen that the h_a^μ in (27) are the Einstein " beins." It follows that

$$\mathrm{E}_a = h_{\mu a} \gamma^\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and the results (7) and (8) follow from (27) and (29).

It remains to be shown that the E_a in (27) and (29) possess some invariant property for rotation of the " beins," $h_{\mu a}$, when the Riemannian γ^μ are held fixed. Let

$$h^*_{\mu a} = d_{ac} h_{\mu c},$$

and as a result of the rotation E_a transforms into E^*_a, where

$$\gamma^\mu = h_a^\mu \mathrm{E}_a = h^*_{\mu a} \mathrm{E}^*_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Equation (30) gives

$$\mathrm{E}^*_a = \gamma^\mu h^*_{\mu a} = \gamma^\mu d_{ab} h_\mu^b = d_{ab} \mathrm{E}_b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Hence

$$\begin{aligned} \mathrm{E}^*_a \mathrm{E}^*_b + \mathrm{E}^*_b \mathrm{E}^*_a &= d_{ac} d_{bf} (\mathrm{E}_c \mathrm{E}_f + \mathrm{E}_f \mathrm{E}) \\ &= d_{ac} d_{bf} \cdot 2\delta_{cf} \text{ by (14)} \\ &= 2\delta_{ab} \text{ by (11)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32) \end{aligned}$$

Hence different members of the pentad E^*_a anticommute and their squares are unity. Thus the E_a preserve their pentadic properties when the $h_{\mu a}$ are rotated, but they need not be Eddington four-point matrices before or after rotation. In effect we have shown that Eddington's rotation theory is accounted for by Einstein's invariance of h_a^μ. Equations (27) and (29) are the matrix counterparts of (1.1) and (2.1).

§ 7. *The Application of the Foregoing to Dirac's Equation.*

If z_a are the coordinates in a Galilean five-world at and near a given point P in a Riemannian five-world connected with the former by distant parallelism, we have in the infinitesimal surroundings of P

$$dx^\mu = h_{\mu a}\, dz_a, \quad \ldots\ldots\ldots \quad (33)$$

and

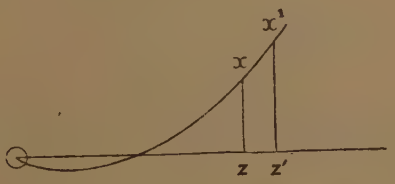
$$dz_a = h_{\mu a}\, dx^\cdot. \quad \ldots\ldots\ldots \quad (34)$$

But note that

$$x^\mu \neq h^\mu_a z_a,$$

since x^μ is not a component of a "local" vector at P.

Equations (33) and (34) show what is understood by equivalent displacements in the flat and curved spaces. Since there is nothing to fix the orientation of the flat space axes with respect to the curved space axes at any given point O, there can be no question of x^μ being a given function of the z_a even near O. It must be supposed that the x's correspond to the z's as in a projection wherein the flat and curved spaces have the same origin of coordinates, viz. O.



Hence x transfers to z, i. e., $x^1 \to z_1$, etc., and a scalar function of the coordinates $f(x^1, x^2 \ldots x^5)$ transfers to the same function of the z_a. Thus

$$f(x^1 x^2 \ldots x^5) \to f(z_1 z_2 \ldots z_5).$$

Now ψ, the matrix wave function, is a scalar differentiable function of the coordinates, so that

$$\psi(x^1 x^2 \ldots x^5) \to \psi(z_1 z_2 \ldots z_5). \quad \ldots\ldots \quad (36)$$

Further, x^1 is a scalar function of the coordinates

$$\therefore \qquad x^1 \to z_1,$$

and neighbouring points $(x^\mu + dx^\mu)$ of the curved space correspond to neighbouring points $(z_a + dz_a)$ of the flat space, provided that the components of the displacement satisfy (33) and (34).

If ψ at $(x^\mu + dx^\mu)$ is replaced by $\psi(z_a + dz_a)$ at $(z_a + dz_a)$, *i. e.*,

$$\psi(x^\mu + dx^\mu) \to \psi(z_a + dz_a). \quad \ldots\ldots\ldots \quad (37)$$

Then by (36)

$$\frac{\partial \psi}{\partial x^\mu} dx^\tau \to \frac{\partial \psi}{\partial z_a} dz_a.$$

Since by (33)

$$dx^\mu = h^\mu_a\, dz_a,$$

it follows that $$h_a^\mu \frac{\partial\psi}{\partial x^\mu} \to \frac{\partial\psi}{\partial z_a}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

This is to be compared with

$$h_a^\mu A_\mu = A_a,$$

the law for vector components.

From (38), $$E_a \frac{\partial\psi}{\partial z_a} \to h_a^\mu E_a \frac{\partial\psi}{\partial x^\mu},$$

or $$\gamma^\mu \frac{\partial\psi}{\partial x^\mu} \to E_a \frac{\partial\psi}{\partial z_a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

so that Dirac's equation $\gamma^\mu \frac{\partial\psi}{\partial x^\mu} = 0$ becomes

$$E_a \frac{\partial\psi}{\partial z_a} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39.1)$$

The function ψ in curved space is cyclic in x^5; that is, the change in ψ caused by changes in x^5 is such that

$$\psi = \psi_0(x^1x^2x^3x^4)\, e^{\frac{2\pi i m_0 c x^5}{h}},$$

or $$\frac{\partial\psi}{\partial z^5} = \frac{2\pi i m_0 c}{h}\psi.$$

In these circumstances transference to flat space yields

$$\psi = \psi_0(z_1z_2z_3z_4)\, e^{\frac{2\pi i m_0 c z_5}{h}},$$

or $$\frac{\partial\psi}{\partial z^5} = \frac{2\pi i m_0 c}{h}\psi = \frac{\partial\psi}{\partial x^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The result (39.1) arises from the change from curved to flat space, and it holds over a small region round any selected point O, and from this result Dirac's equation in a Galilean world (equation (26)) is obtained by transference of equation (17), where it applies in the presence of an electromagnetic field. In the latter circumstances an electron moves on a null geodesic of the Riemannian five-world. We shall show, however, that

$$E_a \frac{\partial\psi}{\partial z_a} = 0,$$

$a = 1 \ldots . 5$ can be re-expressed to cover electromagnetism *without adding terms like* $-\frac{e}{c}\phi_a$ *to the* $\frac{\partial}{\partial z_a}$.

Beginning with (24) we have

$$\alpha_0 = \gamma_{\mu 5}\gamma^\mu \text{ as } \gamma_{55} = 1,$$
$$= h_{\mu a}h_{5a}\gamma^\mu, \quad a = 1 \ldots . 5,$$

$$=\mathrm{E}_a h_{5a} \text{ by (29).}$$

$$\therefore \quad \mathrm{E}_5 = \frac{\alpha_0 - h_{5a}\mathrm{E}_a}{h_{55}},$$

where from now until the end of § 7 $a=1 \ldots . 4$, hence

$$\mathrm{E}_5 \frac{\partial \psi}{\partial z_5} = \frac{2\pi\imath m_0 c}{h}\left(\frac{\alpha_0}{h_{55}} - \frac{h_{5a}}{h_{55}}\mathrm{E}_a\right)\psi.$$

$$\therefore \ \mathrm{E}_1\frac{\partial\psi}{\partial z_1} + \mathrm{E}_2\frac{\partial\psi}{\partial z_2} + \ldots + \mathrm{E}_5\frac{\partial\psi}{\partial z_5} = \frac{2\pi\imath}{h}\left\{\mathrm{E}_a\left(\frac{h}{2\pi\imath}\frac{\partial}{\partial z_a} - \frac{m_0 c h_{5a}}{h_{55}}\right) + \frac{m_0 c \alpha_0}{h_{55}}\right\}\psi. \quad \ldots \quad (41)$$

Writing

$$\frac{m_0 c h_{5a}}{h_{55}} = \frac{e}{c}\Phi_a, \quad \ldots \quad (42)$$

where Φ_a will be identified as a potential component in flat space, and

$$\frac{\alpha_0}{h_{55}} = \beta_0. \quad \ldots \quad (43)$$

$$\mathrm{E}_1\frac{\partial\psi}{\partial z_1} + \ldots . + \mathrm{E}_5\frac{\partial\psi}{\partial z_5}\psi = 0$$

becomes

$$\mathrm{E}_a\left(\frac{\partial}{\partial z_a} - \frac{2\pi\imath e}{hc}\Phi_a\right)\psi + \frac{2\pi\imath m_0 c \alpha_0}{h \,.\, h_{55}}\psi = 0 \quad \ldots \quad (44)$$

This is Dirac's equation in the presence of an electromagnetic field for flat space and is analogous to (17.1) for curved space.

Returning to the quantity Φ_a introduced above we have, from (42) and (19.1),

$$h_{5a} = h_{55} k \Phi_a, \quad \ldots \quad (45)$$

$a=1 \ldots . 4$.

This gives, on comparison with (19), viz.,

$$\gamma_{5m} = k\phi_m = h_{5a}h_{ma} + h_{55}h_{m5},$$

the result

$$\phi_m = h_{ma}(h_{55}\Phi_a) + h_{m5}\left(h_{55}\cdot\frac{1}{k}\right). \quad \ldots \quad (46)$$

If for convenience we write

$$\Phi_5 = \frac{1}{k}, \quad \ldots \quad (47)$$

then

$$\begin{aligned}\phi_m &= h_{ma}(h_{55}\Phi_a) + h_{m5}(h_{55}\Phi_5) \\ &= h_{ma}(h_{55}\Phi_a), \quad \ldots \quad (47)\end{aligned}$$

$a=1 \ldots . 5$.

Hence the flat 5-vector with components $h_{55}\Phi_a$ corresponds to ϕ_m. The latter quantities with $m=1, 2, 3, 4$ have not been treated as the first four components of a Riemannian five-vector, but the foregoing suggests that ϕ_5 exists and that its value is given by

$$\phi_5 = h_{5a}h_{55}\Phi_a$$

$$= h_{55}\left(h_{5a}\Phi_a + h_{55}\cdot\frac{1}{k}\right)$$

$$= h_{55}\left(kh_{55}\Phi_a^2 + h_{55}\frac{1}{k}\right)$$

$$= \frac{h_{55}^2}{k}(1+k^2\Phi_a^2). \quad \ldots\ldots\ldots\ldots \quad (48)$$

And since $\gamma_{5m} = k\phi_m$ and $\gamma_{55} = 1$, we might have taken

$$\phi_5 = \frac{1}{k} \quad \ldots\ldots\ldots\ldots \quad (49)$$

in place of (47).

§ 8. *An Alternative Proof of the Invariance of Dirac's Equation in a Galilean Continuum.*

If Dirac's equation in Galilean five-space holds at a given point P then $\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0$ at P.

But this is a tensor equation, and thus holds at all points of the Riemannian continuum. At a point Q situated at a finite distance from P, Galilean axes can be set up with respect to which $E_a \frac{\partial \psi}{\partial z_a} = 0$ is again the "local" quantum equation.

Thus the equation holds over small regions at each of these finitely separated points where Galilean axes are set up arbitrarily oriented with respect to one another. This indicates the invariance of Dirac's equation in a Galilean continuum.

§ 9. *Conclusion.*

This discussion suggests that the geometric background of the quantum theory is provided by Riemannian space. In the discussion of the last section the small Galilean regions are not parts of one and the same Galilean space. They are regions tangential to one and the same Riemannian space at different points. If Dirac's equation in Galilean coordinates can be applied to a finite region, it is because the Riemannian curvature in this region is small.

A geometric description of this equation is that at all points of a Riemannian continuum the vector with components (γ^μ) lies tangential to the family of surfaces $\psi = \text{constant}$.

The writer wishes to thank Dr. H. T. Flint for valuable discussion and for help with § 7.

XLV. *On the Inelastic Scattering of a Beam of Particles by Hydrogen and Helium.*

By Prof. K. C. Kar, D.Sc.*

[Received May 30, 1939.]

In previous papers recently published † I have discussed the theory of elastic scattering by hydrogen and helium. In the present article the theory is further extended to the case of inelastic scattering causing excitation.

As before let us suppose that the incident beam of bombarding particles is in the direction of the x-axis and moving with velocity v_0. We have then for the incident wave outside the potential field

$$\Delta\chi_0+k_0^2\chi_0=0, \quad \ldots\ldots\ldots \quad (1)$$

where $$k_0=\frac{2\pi mv_0}{h}=\sqrt{\frac{8\pi^2 m\mathrm{E}_0}{h^2}}.$$

On coming within the potential field V of the atom, the equation becomes

$$\Delta\chi+\frac{8\pi^2 m}{h^2}(\mathrm{E}+\epsilon_{12}-\mathrm{V})\chi=0, \quad \ldots\ldots \quad (2)$$

where ϵ_{12} is the energy of excitation and E the energy of the scattered particle; so that $\mathrm{E}_0=\mathrm{E}+\epsilon_{12}$. Let us suppose that

$$\chi=\chi_0+\lambda_1\chi_1+\lambda_2\chi_2+\ldots\ldots\ldots \quad (3)$$

On substituting (3) in (2) we have for small perturbing potential

$$\left(k=\frac{2\pi mv}{h}=\sqrt{\frac{8\pi^2 m\mathrm{E}}{h^2}}\right),$$

$$\left.\begin{aligned}
&\Delta\chi_0+k_0^2\chi_0=0 \text{ (incident)},\\
&\lambda_1(\Delta\chi_1+k^2\chi_1)=\frac{8\pi^2 m}{h^2}\mathrm{V}\chi_0 \text{ (1st order scattering)},\\
&\lambda_2(\Delta\chi_2+k^2\chi_2)=\lambda_1\frac{8\pi^2 m}{h^2}(\mathrm{V}-\epsilon_{12})\chi_1 \text{ (2nd order scattering)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (4)$$

and so on.

* Communicated by the Author.

† K. C. Kar, M. Ghosh, and K. K. Mukherjee, Phil. Mag. xxiv. p. 964 (1937); K. C. Kar, Phil. Mag. xxiv. p. 971 (1937).

On solving we have for the first order scattering function, since

$$\chi_0 = \frac{1}{\sqrt{v_0}} e^{ik_0 x},$$

$$\lambda_1\chi_1(r_2, \theta_2) = -\frac{2\pi^2 m}{\sqrt{v_0}h^2}\int \mathrm{V} e^{ik_0 x_1}\cdot\frac{e^{ikr_{12}}}{r_{12}}d\tau_1. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the solution is bounded, we have

$$\frac{e^{ikr_{12}}}{r_{12}} \sim \frac{\sin kr_{12}}{r_{12}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Therefore, using the well-known Gegenbauer formula, viz.,

$$\frac{\sin kr_{12}}{r_{12}} = \pi \sum_{n}^{0,\infty}(n+\tfrac{1}{2})\frac{\mathrm{J}_{n+\frac{1}{2}}(kr_1)}{\sqrt{r_1}}\cdot\frac{\mathrm{J}_{n+\frac{1}{2}}(kr_2)}{\sqrt{r_2}}\mathrm{P}_n(\cos\theta_1)\,\mathrm{P}_n(\cos\theta_2)\,. \quad (7)$$

and expanding $e^{ik_0 x_1}$ in a series of Bessel functions, we have from (5)

$$\lambda_1\chi_1(r_2, \theta_2) = -\frac{2\pi m}{\sqrt{v_0}h^2}\int \mathrm{V}\left\{\sum_s (2s+1) i^s \sqrt{\frac{\pi}{2k_0 r_1}}\mathrm{J}_{s+\frac{1}{2}}(k_0 r_1)\mathrm{P}_s(\cos\theta_1)\right\}$$
$$\times k\left\{\sum_n (2n+1)\sqrt{\frac{\pi}{2kr_1}}\mathrm{J}_{n+\frac{1}{2}}(kr_1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa r_2)\right.$$
$$\left.\times \mathrm{P}_n(\cos\theta_1)\mathrm{P}_n(\cos\theta_2)\right\} d\tau_1. \quad (8)$$

Before transforming the above any further we must decide whether or not the perturbing potential V is spherically symmetrical with respect to the centre of the scattering atom. We shall discuss both cases separately.

Case I.—Spherically Symmetrical Potential.

Since V is a function of r only, we have from (8) on integrating with respect to θ and ϕ, remembering the orthogonal property of the Legendre function,

$$\lambda_1\chi_1(r_2, \theta_2) = -\frac{\pi}{2}\cdot\frac{8\pi^2 m}{\sqrt{v_0}h^2}\sqrt{\frac{k}{k_0}}\sum_s (2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2) i^s \mathrm{P}_s(\cos\theta_2)$$
$$\times \int_{r_0}^{\infty} \mathrm{V}(r_1)\mathrm{J}_{s+\frac{1}{2}}(k_0 r_1)\mathrm{J}_{s+\frac{1}{2}}(kr_1) r_1 dr_1 \quad (9)$$

$$= \frac{1}{\sqrt{v_0}}\sum_s (2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2) i^s \mathrm{P}_s(\cos\theta_2).\eta_s, \quad . \quad . \quad . \quad (9.1)$$

where $$\eta_s = -\frac{\pi}{2}\cdot\frac{8\pi^2 m}{h^2}\sqrt{\frac{k}{k_0}}\int_{r_0}^{\infty}\mathrm{V}(r_1)\mathrm{J}_{s+\frac{1}{2}}(k_0 r_1)\mathrm{J}_{s+\frac{1}{2}}(kr_1) r_1\, dr_1. \quad . \quad (9.2)$$

Proceeding as above, we have for the second order scattering function

$$\lambda_2\chi_2(r_2,\theta_2)=\frac{1}{\sqrt{v_0}}\Sigma(2s+1)\sqrt{\frac{\pi}{2kr_2}}.\mathrm{J}_{s+\frac{1}{2}}(kr_2)i^{-s}\mathrm{P}_s(\cos\theta_2)\eta_s\eta_s', \quad (9.3)$$

where
$$\eta_s'=-\frac{\pi}{2}\cdot\frac{8\pi^2m}{h^2}\int_{r_0}^{\infty}(\mathrm{V}-\epsilon_{12})\mathrm{J}^2_{s+\frac{1}{2}}(kr_1)r_1\,dr_1. \quad . \quad . \quad . \quad (9.4)$$

Similarly, the higher order scattering functions may be derived. Hence we have for the total scattering,

$$\chi(r_2,\theta_2)=\lambda_1\chi_1+\lambda_2\chi_2+\ldots \quad \ldots$$

$$=\frac{1}{\sqrt{v_0}}\Sigma(2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2)i^s\mathrm{P}_s(\cos\theta_2)\{\eta_s+\eta_s\eta_s'+\eta_s\eta_s'^2+\ldots\}$$

and if $\eta_s'\ll 1$

$$=\frac{1}{\sqrt{v_0}}\Sigma(2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2)i^s\mathrm{P}_s(\cos\theta_2)\cdot\frac{\eta_s}{1-\eta_s'} \quad . \quad . \quad . \quad (9.5)$$

It may be noted that for elastic scattering $k=k_0$ and $\epsilon_{12}=1$, so that $\eta_s'=\eta_s$ and so (9.5) reduces to the corresponding formula for elastic scattering (*l. c.*).

Let us now go back to the first order scattering. As the asymptotic value of

$$\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2)\sim\frac{\sin(kr_0-\frac{1}{2}s\pi)}{kr_2}\sim\frac{e^{ikr_2}}{kr_2}.i^{-s},$$

we have from (9.1),

$$\lambda_1\chi_1(r_2,\theta_2)=\frac{1}{\sqrt{v_0}}\frac{e^{ikr_2}}{kr_2}\Sigma(2s+1)\mathrm{P}_s(\cos\theta_2).\eta_s. \quad . \quad . \quad . \quad (10)$$

The series in (10) may be evaluated with the help of Gegenbauer formula (7). Take $\theta_1=0$ and

$$r_2k=r_1k_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $k<k_0$, and so $r_2>r_1$.

Thus
$$r_{12}=\{r_1^2+r_2^2-2r_1r_2\cos\theta_2\}^{\frac{1}{2}}$$

$$=r_1\left\{\left(1-\frac{k_0}{k}\right)^2-\frac{4k_0}{k}\sin^2\frac{\theta_2}{2}\right\}^{\frac{1}{2}}=\frac{k_0}{k}\frac{r_1}{\mathrm{B}}, \quad . \quad . \quad . \quad (11.1)$$

where
$$\mathrm{B}=\frac{k_0/k}{\left\{\left(1-\frac{k_0}{k}\right)^2+\frac{4k_0}{k}\sin^2\frac{\theta_2}{2}\right\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (11.2)$$

and since
$$\frac{k_0}{k}=\frac{v_0}{v}$$

$$=\frac{v_0/v}{\left\{\left(1-\frac{v_0}{v}\right)^2+\frac{4v_0}{v}\sin^2\frac{\theta_2}{2}\right\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (11.3)$$

We have now from (7)

$$B \sin\left(\frac{k_0 r_1}{B}\right) = \frac{\pi}{2}\sqrt{\frac{k_0}{k}} \sum_{s=0}^{\infty} (2s+1) J_{s+\frac{1}{2}}(kr_1) J_{s+\frac{1}{2}}(k_0 r_1) P_s(\cos\theta_2). \quad (12)$$

On combining (10) and (12) and remembering (9.2) we obtain without difficulty, for the first order scattering,

$$\lambda_1 \chi_1(r_2, \theta_2) = -\frac{1}{k_0\sqrt{v_0}} \cdot \frac{8\pi^2 m}{h^2} \cdot \frac{e^{ikr_2}}{r_2} \cdot B \int_{r_0}^{\infty} V(r_1) \sin\frac{k_0 r_1}{B} r_1 dr_1. \quad (13)$$

$$= -\frac{8\pi^2 m}{\sqrt{v_0} k_0^2 h^2} \frac{e^{ikr_2}}{r_2} B^2 F(r_0), \quad \ldots\ldots\ldots (13.1)$$

where $$F(r_0) = \frac{k_0}{B}\int_{r_0}^{\infty} V(r_1) \sin\frac{k_0 r_1}{B} r_1 dr_1. \quad \ldots\ldots\ldots (13.2)$$

Coulomb Potential.

It is evident that the potential field may be taken as Coulombian, if it is assumed that the orbital electrons of the scattering atom do not take any part in the interaction. Thus the interaction takes place between the bare nucleus and the incident particle. For such an inelastic collision the energy given out by the incident particle is evidently taken by the nucleus for its excitation.

Now supposing the incident particle to be an electron we have for the potential

$$V(r_1) = -\frac{ze^2}{r_1},$$

z being the atomic number of the scattering atom. Hence from (13.2) we have

$$F(r_0) = -ze^2 \cos\frac{k_0 r_0}{B}. \quad \ldots\ldots\ldots (13.3)$$

Therefore from (13.1)

$$\lambda_1 \chi_1(r_2, \theta_2) = \frac{1}{\sqrt{v_0}} \cdot \frac{2ze^2}{mv_0^2} B^2 \frac{e^{ikr_0}}{r_2} \cos\frac{k_0 r_0}{B}. \quad \ldots\ldots (14)$$

Hence the relative intensity of scattering is

$$I_s = \frac{2ze^2}{(mv_0)^2} B^4 \cos^2\frac{k_0 r_0}{B}. \quad \ldots\ldots\ldots (15)$$

(*a*) If now $v \triangleq v_0$, we have from (11.3)

$$B = \frac{1}{2}\left(\frac{v_0}{v}\right)^{\frac{1}{2}} \operatorname{cosec}\frac{\theta_2}{2}. \quad \ldots\ldots\ldots (15.1)$$

Hence from (15)

$$I_s = \left(\frac{ze^2}{2mvv_0}\right)^2 \operatorname{cosec}^4\frac{\theta_2}{2} \cos^2\left\{2(kk_0)^{\frac{1}{2}} \sin\frac{\theta_2}{2} r_0\right\}, \quad \ldots (15.2)$$

which, it is interesting to note, reduces to the Rutherford formula if $v=v_0$ and $r_0=0$.

(*b*) If, on the other hand, $v \ll v_0$, we have from (11.3)

$$B=\frac{1}{\left(1-\frac{2v}{v_0}\cos\theta_2\right)^{\frac{1}{2}}}, \quad \ldots\ldots \quad (15.3)$$

and so

$$I_s=\left(\frac{2ze^2}{mv_0^2}\right)^2 \frac{\cos^2\left\{k_0r_0\left(1-\frac{v}{v_0}\cos\theta_2\right)\right\}}{\left(1-\frac{2v}{v_0}\right)^2}. \quad \ldots \quad (15.4)$$

Critical approach (r_0) :—

The critical approach is determined from the boundary conditions :—

$$\left.\begin{aligned}\frac{d}{dr}(\chi_0+\lambda_1\chi_1)&=0,\\ \chi_0+\lambda_1\chi_1&=0.\end{aligned}\right|_{r=r_0} \quad \ldots\ldots \quad (16)$$

Now, because

$$\chi_0=\frac{1}{\sqrt{v_0}}e^{ik_0x}=\frac{1}{\sqrt{v_0}}\Sigma_s(2s+1)i^s\sqrt{\frac{\pi}{2k_0r}}\,J_{s+\frac{1}{2}}(k_0r)P_s(\cos\theta)$$
$$=\frac{1}{\sqrt{v_0}}\cdot\frac{e^{ik_0r}}{r}\Sigma_s\frac{2s+1}{k_0}P_s(\cos\theta) \quad \text{(asymptotic).}$$

We have from the first boundary condition, using the value of $\lambda_1\chi_1$ given in (14),

$$\frac{1}{\sqrt{v_0}}\Sigma_s\frac{2s+1}{k_0}P_s(\cos\theta)\cdot\frac{d}{dr}\left(\frac{e^{ik_0r}}{r}\right)$$
$$+\frac{1}{\sqrt{v_0}}\cdot\frac{2ze^2}{mv_0^2}B^2\cos\frac{k_0r_0}{B}\frac{d}{dr}\left(\frac{e^{ikr}}{r}\right)=0\Big|_{r=r_0},$$

which gives

$$\left.\begin{aligned}\tan k_0r_0&=k_0r_0,\\ \tan kr_0&=kr_0.\end{aligned}\right\} \quad \ldots\ldots \quad (17)$$

On solving the above graphically we find

$$\left.\begin{aligned}k_0r_0&=1{\cdot}43\pi,\ 2{\cdot}459\pi,\ 3{\cdot}471\pi,\ \ldots \text{ etc.},\\ kr_0&=1{\cdot}43\pi,\ 2{\cdot}459\pi,\ 3{\cdot}471\pi,\ \ldots \text{ etc.}\end{aligned}\right\} \quad \ldots\ldots \quad (18)$$

The different values correspond to the different orders of diffraction. If, for example, k_0r_0 is given the second order value, then kr_0 should be given the first order, since $k<k_0$. If, however, we give first order value to k_0r_0, then kr_0 may be given the same value when $k \doteq k_0$, but should be taken zero when $k \ll k_0$.

It should be noted that (17) and the following results are obtained for *any* potential field provided it is spherically symmetrical. It follows immediately if (13.1) is used instead of (14) as above. The value of r_0 obtained from (18) is independent of the angle of scattering, depending on the velocity of the bombarding particles. Thus r_0 so obtained may be taken as the average value of the critical approach for a given velocity of incidence. The small variation of r_0 with the angle of scattering is readily obtained from the second boundary condition. We have

$$\frac{1}{\sqrt{v_0}}e^{ik_0r\cos\theta}=+\frac{1}{\sqrt{v_0}}\cdot\frac{2B^2}{mv_0^2}\frac{e^{ikr}}{r}F(r_0)\Big|_{r=r_0}.$$

Proceeding as in the previous paper (*l. c.*) we obtain

$$r_0=-\frac{2B^2}{mv_0^2}e^{2ik_0r_0\left(\frac{1}{2B}-\sin^2\frac{\theta}{2}\right)}e^{i(k-k_0)r_0}G(r_0), \quad . \ . \ . \quad (19)$$

where
$$G(r_0)=\frac{k_0}{B}\int_{r_0}^{\infty}V(r_1)\sin\frac{k_0}{B}(r_1-r_0)r_1\,dr_1. \quad . \ . \ . \ . \quad (19.1)$$

For a Coulomb potential $G(r_0)=-ze^2$. Hence we have from (19), since $e^{-in\pi}=1$ when n is even,

$$r_0=\frac{2ze^2}{mv_0^2}.B^2\sin\left\{2\rho\left(\frac{1}{2B}-\sin^2\frac{\theta}{2}\right)+(k-k_0)r_0\right\}, \quad . \ . \quad (20)$$

where
$$2\rho=2k_0r_0-n\pi. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (20.1)$$

(*a*) If $v\doteq v_0$, *i. e.*, $k\doteq k_0$, B is given in (15.1). And so

$$r_0=\frac{ze^2}{2mvv_0}\operatorname{cosec}^2\frac{\theta}{2}\sin\left\{2\rho\left(\sqrt{\frac{v}{v_0}}\sin\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\right\}$$

$$=\rho\cdot\frac{ze^2}{mvv_0}\left(\sqrt{\frac{v}{v_0}}\operatorname{cosec}\frac{\theta}{2}-1\right), \text{ approx.,} \quad . \ . \ . \ . \ . \quad (20.2)$$

reducing to the corresponding formula for elastic scattering when $v=v_0$.

(*b*) If $v\ll v_0$ *i. e.*, $k\ll k_0$, B is given in (15.3). Hence from (20)

$$r_0=\frac{2ze^2}{mv_0^2\left(1-\frac{v}{v_0}\cos\theta\right)}\sin\left\{\rho\left(1-\frac{v}{v_0}\cos\theta\right)-2\rho\sin^2\frac{\theta}{2}-1{\cdot}43\,\pi\right\}$$

$$=\frac{2ze^2}{mv_0^2\left(1-\frac{v}{v_0}\cos\theta\right)}\sin\{\rho(1-\cos\theta)\}, \text{ approx.,} \quad . \ . \ . \ . \ . \quad (20.3)$$

where $\rho=\cdot43\,\pi$, from (18) and (20.1).

Wave Statistical Potential.

For normal atomic hydrogen the wave statistical potential is

$$V(r_2)=e^2\int\frac{1}{r_{12}}\chi_0\chi_{1n}\,d\tau_1, \quad \ldots\ldots\ldots \quad (21)$$

where
$$\chi_0=\left(\frac{z^3}{\pi a^3}\right)^{\frac{1}{2}}e^{-\frac{zr_1}{a}} \quad \ldots\ldots\ldots \quad (21.1)$$

$$\chi_{nlm}=\left(\frac{2z}{na}\right)^{\frac{3}{2}}\frac{\{(n-l-1)\,!\}^{\frac{1}{2}}}{(2n)^{\frac{1}{2}}\{(n+l)\,!\}^{\frac{3}{2}}}\left(\frac{2zr_1}{na}\right)^{l}L_{n+l}^{2l+1}\left(\frac{2zr_1}{na}\right)e^{-\frac{zr_1}{na}}$$
$$\times\sqrt{\frac{2l+1}{4\pi}}\cdot\frac{(l-m)\,!}{(l+m)\,!}P_l^m(\cos\theta)e^{im\phi}, \quad \ldots \quad (21.2)$$

in which $z=1$.

The wave function for the *normal* helium atom is

$$\chi_0(z,r_1)\chi_0(z,r_2)=\frac{z^3}{\pi a^3}e^{-\frac{z(r_1+r_2)}{a}}, \quad \ldots\ldots \quad (22.1)$$

in which $z=1{\cdot}69$.

The wave function for the excited helium atom according to Eckart * is

$$2^{-\frac{1}{2}}\{\chi_0(2,r_1)\chi_{nlm}(1,r_2)+\chi_0(2,r_2)\chi_{nlm}(1,r_1)\}, \quad \ldots \quad (22.2)$$

taking $z=2$ for (100)-electron and $z=1$ for (nlm)-electron, the suffixes 1 and 2 representing the first and the second electron. Hence the wave statistical potential at r_3 is

$$V(r_3)=\frac{e^2}{2^{\frac{1}{2}}}\int\left(\frac{1}{r_{13}}+\frac{1}{r_{23}}\right)\chi_0(z,r_1)\chi_0(z,r_2)$$
$$\times\{\chi_0(2,r_1)\chi_{nlm}(1,r_2)+\chi_0(2,r_2)\chi_{nlm}(1,r_1)\}d\tau_1d\tau_2,$$

and as the integrand is symmetrical with respect to the suffixes 1 and 2

$$=\sqrt{2}e^2\int\frac{1}{r_{13}}\chi_0(z,r_1)\chi_0(z,r_2)\{\chi_0(2,r_1)\chi_{nlm}(1,r_2)$$
$$+\chi_0(2,r_2)\chi_{nlm}(1,r_1)\}d\tau_1d\tau_2,$$

and on account of the orthogonal property

$$=\sqrt{2}e^2\int\frac{1}{r_{13}}\chi_0(z,r_1)\chi_0(z,r_2)\chi_0(2,r_2)\chi_{nlm}(1,r_1)d\tau_1\,d\tau_2,$$

and integrating with respect to r_2, θ_2, ϕ_2

$$=32\frac{z^{\frac{5}{2}}}{(z+2)^3}e^2\int\frac{1}{r_{13}}\chi_0(z,r_1)\chi_{nlm}(1,r_1)\,d\tau_1. \quad \ldots\ldots \quad (23)$$

The integrals in (21) and (23) are the same except the difference in the value of z in χ_{nlm}.

---

* Eckart, Phys. Rev. xxxvi. p. 878 (1930).

Also $$\frac{1}{r_{12}}=\frac{1}{[r]}\sum_{\nu}x^{\nu}\mathrm{P}_{\nu}(\cos\theta_1)\mathrm{P}_{\nu}(\cos\theta_2),$$

where $[r]$ is r_1 or r_2, whichever is greater, and x is the ratio of the r's, but is less than unity. Thus from (23) we have for the wave statistical potential at r_2

$$\mathrm{V}(r_2)=32.\frac{z^3}{(z+2)^3}\cdot\frac{e^2}{a^3}\left(\frac{8}{\pi n^3}\right)^{\frac{1}{2}}\frac{\{(n-l-1)\,!\}^{\frac{1}{2}}}{(2n)^{\frac{1}{2}}\{(n+l)\,!\}^{\frac{3}{2}}}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)\,!}{(l+m)\,!}}$$
$$\times\int\frac{1}{[r]}\sum_{\nu}x^{\nu}\mathrm{P}_{\nu}(\cos\theta_1)\mathrm{P}_{\nu}(\cos\theta_2)e^{-\frac{zr_1}{a}\left(1+\frac{1}{zn}\right)}.\left(\frac{2r_1}{na}\right)^{l}\mathrm{L}_{n+l}^{2l+1}\left(\frac{2r_1}{na}\right)$$
$$\times\mathrm{P}_l^m(\cos\theta_1)e^{im\phi_1}d\tau_1.$$

The integrations with respect to θ and ϕ vanish except for $\nu=l$ and $m=0$ respectively. So after integration the above reduces to

$$\mathrm{V}(r_2)=\frac{128}{(2l+1)^{\frac{1}{2}}}\cdot\frac{e^2}{a^3}\frac{z^3}{(z+2)^3}\cdot\frac{\{(n-l-1)\,!\}^{\frac{1}{2}}}{n^2\{(n+l)1\,!\}^{\frac{3}{2}}}\mathrm{P}_l(\cos\theta_2)\int\frac{1}{[r]}x^l\left(\frac{2r_1}{na}\right)^{l}$$
$$\times e^{-\alpha r_1}\mathrm{L}_{n+l}^{2l+1}(\beta r_1)r_1^{\,2}\,dr_1, \quad . \quad (23.1)$$

where $$\alpha=\frac{z}{a}\left(1+\frac{1}{zn}\right),\quad \beta=\frac{2}{na}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.2)$$

Or, we have $$\mathrm{V}(r_2,\theta_2)=\mathrm{P}_l(\cos\theta_2)\mathrm{V}_1(r_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $$\mathrm{V}_1(r_2)=\frac{128z^3e^2}{a^3(z+2)^3}\cdot\frac{\{(n-l-1)\,!\}^{\frac{1}{2}}}{n^2(2l+1)^{\frac{1}{2}}\{(n+l)\,!\}^{\frac{3}{2}}}\left(\frac{2}{na}\right)^{l}(\mathrm{I}_1+\mathrm{I}_2). \quad . \quad (24.1)$$

$$\mathrm{I}_1=\frac{1}{r_2^{l+1}}\int_0^{r_2}e^{-\alpha r_1}\mathrm{L}_{n+l}^{2l+1}(\beta r_1)r_1^{\,2l+2}\,dr_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24.2)$$

$$\mathrm{I}_2=r_2^{l}\int_{r_2}^{\infty}e^{-\alpha r_1}\mathrm{L}_{n+l}^{2l+1}(\beta r_1)r_1\,dr_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24.3)$$

In the same way it may be shown that in the case of the hydrogen atom, (21) gives (24) where, however,

$$\mathrm{V}_1(r_2)=\frac{8e^2}{a^3}\cdot\frac{\{(n-l-1)\,!\}}{n^2(2l+1)^{\frac{1}{2}}\{(n+l)\,!\}^{\frac{3}{2}}}\left(\frac{2}{na}\right)^{l}(\mathrm{I}_1+\mathrm{I}_2), \quad . \quad . \quad (24.4)$$

I_1, I_2 being given in (24.2) and (24.3) with $z=1$.

It is evident from (24) that the potential is spherically symmetrical only when $l=0$. But such a transition for a normal atom is optically forbidden. We, therefore, do not consider that case. According to the selection rule, $l=1$. We discuss this case only in the following :—

Case II.—Axially Symmetrical Potential.

For $l=1$, we have from (24)

$$\mathrm{V}(r_2,\theta_2)=\cos\theta_2\mathrm{V}_1(r_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We have then from (8) on integrating with respect to ϕ

$$\lambda_1\chi_1(r_2,\theta_2)=-\frac{4\pi^2mk}{\sqrt{v_0}h^2}\int\cos\theta_1 V_1(r_1)\left\{\Sigma_s(2s+1)i^s\sqrt{\frac{\pi}{2k_0r_1}}J_{s+\frac{1}{2}}(k_0r_1)P_s(\cos\theta_1)\right\}$$
$$\times\left\{\Sigma_n(2n+1)\sqrt{\frac{\pi}{2kr_1}}J_{n+\frac{1}{2}}(kr_1)\sqrt{\frac{\pi}{2kr_2}}J_{n+\frac{1}{2}}(kr_2)\right.$$
$$\left.\times P_n(\cos\theta_1)P_n(\cos\theta_2)\right\}r_1^2dr_1\sin\theta_1d\theta_1. \quad (26)$$

Because the integral

$$\left.\begin{aligned}\int\cos\theta P_s(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta&=\frac{2s}{4s^2-1}\text{ for } n=s-1,\\&=\frac{2(s+1)}{(2s+1)(2s+3)}\text{ for } n=s+1,\\&=0\text{ for } n\neq s\pm1.\end{aligned}\right\}. \quad (27)$$

We have from (26) after integrating with respect to θ

$$\lambda_1\chi_1(r_2,\theta_2)=-\frac{4\pi^2mk}{\sqrt{v_0}h^2}\Sigma_s\sqrt{\frac{\pi}{2kr_2}}J_{s+\frac{1}{2}}(kr_2)P_s(\cos\theta_2)$$
$$\times\left[\frac{2s}{4s^2-1}\int_{r_0}^{\infty}V_1(r_1)(2s-1)i^{s-1}\sqrt{\frac{\pi}{2k_0r_1}}J_{s-\frac{1}{2}}(k_0r_1)\right.$$
$$\times(2s+1)\sqrt{\frac{\pi}{2kr_1}}J_{s+\frac{1}{2}}(kr_1)r_1^2dr_1+\frac{2(s+1)}{(2s+1)+(2s+3)}\int_{r_0}^{\infty}V_1(r_1)(2s+3)$$
$$\left.\times i^{s+1}\sqrt{\frac{\pi}{2k_0r_1}}J_{s+\frac{3}{2}}(k_0r_1)(2s+1)\sqrt{\frac{\pi}{2kr_1}}J_{s+\frac{1}{2}}(kr_1)r_1^2dr_1\right]. \quad . \quad . \quad (28)$$

Or, we have

$$\lambda_1\chi_1(r_2,\theta_2)=\frac{8\pi^2m.i}{\sqrt{v_0}.h^2}\cdot\frac{\pi}{2}\sqrt{\frac{k}{k_0}}\Sigma_s\sqrt{\frac{\pi}{2kr_2}}J_{s+\frac{1}{2}}(kr_2)P_s(\cos\theta_2).i^s$$
$$\times\int_{r_0}^{\infty}V_1(r_1)J_{s+\frac{1}{2}}(kr_1)\{sJ_{s-\frac{1}{2}}(k_0r_1)-(s+1)J_{s+\frac{3}{2}}(k_0r_1)\}r_1dr_1. \quad (28.1)$$

Now we have the well-known recurrence formulæ for Bessel functions

$$\left.\begin{aligned}J_{n-1}(k_0r_1)-J_{n+1}(k_0r_1)&=2J_n'(k_0r_1),\\J_{n-1}(k_0r_1)+J_{n+1}(k_0r_1)&=\frac{2n}{k_0r_1}J_n(k_0r_1).\end{aligned}\right\}. \quad . \quad . \quad . \quad (29)$$

Thus we have

$$sJ_{s-\frac{1}{2}}(k_0r_1)-(s+1)J_{s+\frac{1}{2}}(k_0r_1)$$
$$=(s+\tfrac{1}{2})\{J_{s+\frac{1}{2}-1}(k_0r_1)-J_{s+\frac{1}{2}+1}(k_0r_1)\}$$
$$-\tfrac{1}{2}\{J_{s+\frac{1}{2}-1}(k_0r_1)+J_{s+\frac{1}{2}+1}(k_0r_1)\},$$

and from (29)

$$=(2s+1)\mathrm{J}'_{s+\frac{1}{2}}(k_0r_1)-\frac{s+\frac{1}{2}}{k_0r_1}\mathrm{J}_{s+\frac{1}{2}}(k_0r_1). \quad . \quad . \quad . \quad . \quad . \quad (29.1)$$

Hence from (28.1) we get

$$\lambda_1\chi_1(r_2,\theta_2)=\frac{1}{\sqrt{v_0}}\sum_s(2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2)i^s\mathrm{P}_s(\cos\theta_2).i\eta_s, \quad . \quad (30)$$

where

$$\eta_s=-\frac{\pi}{2}\cdot\frac{8\pi^2m}{h^2}\sqrt{\frac{k}{k_0}}\int_{r_0}^{\infty}\mathrm{V}_1(r_1)\mathrm{J}_{s+\frac{1}{2}}(kr_1)\left\{\frac{1}{2k_0r_1}\mathrm{J}_{s+\frac{1}{2}}(k_0r_1)-\mathrm{J}'_{s+\frac{1}{2}}(k_0r_1)\right\}r_1dr_1.$$
$$. \quad . \quad (30.1)$$

Because the wave statistical values of V and $\mathrm{V}-\epsilon_{12}$ are the same, the higher approximations of the scattering functions are easily obtained as in (9.5), and we have for the total scattering

$$\chi(r_2,\theta_2)=\frac{1}{\sqrt{v_0}}\sum_s(2s+1)\sqrt{\frac{\pi}{2kr_2}}\mathrm{J}_{s+\frac{1}{2}}(kr_2)i^s\mathrm{P}_s(\cos\theta_2).\frac{i\eta_s}{1-i\eta_s}. \quad (30.2)$$

On returning to the first approximations we have from (30) after substituting the asymptotic value of $\mathrm{J}_{s+\frac{1}{2}}(kr_2)$ given before

$$\lambda_1\chi_1(r_2,\theta_2)=-\frac{1}{\sqrt{v_0}}\cdot\frac{e^{i\left(kr_0-\frac{\pi}{2}\right)}}{kr_2}\sum_s(2s+1)\mathrm{P}_s(\cos\theta_2).\eta_s. \quad . \quad (30.3)$$

Now on differentiating with respect to k_0, we have from the relation (12)

$$\frac{1}{r_1}\frac{d}{dk_0}\left(\sqrt{\frac{k}{k_0}}\,\mathrm{B}\sin\frac{k_0r_1}{\mathrm{B}}\right)=\frac{\pi}{2}\sum_s(2s+1)\mathrm{J}_{s+\frac{1}{2}}(kr_1)\mathrm{J}'_{s+\frac{1}{2}}(k_0r_1)\mathrm{P}_s(\cos\theta_2). \quad (31)$$

On transforming (30.3) with the help of (12) and (31), we obtain

$$\lambda_1\chi_1(r,\theta)=\frac{8\pi^2m}{\sqrt{v_0}.h^2}\cdot\frac{e^{i\left(kr-\frac{\pi}{2}\right)}}{kr}\sqrt{\frac{k}{k_0}}\int_{r_0}^{\infty}\left\{\frac{1}{2k_0}\left(\sqrt{\frac{k}{k_0}}\mathrm{B}\sin\frac{k_0r_1}{\mathrm{B}}\right)\right.$$
$$\left.-\frac{d}{dk_0}\left(\sqrt{\frac{k}{k_0}}\mathrm{B}\sin\frac{k_0r_1}{\mathrm{B}}\right)\right\}\mathrm{V}_1(r_1)\,dr_1, \quad (31.1)$$

and after simplification and remembering that we have taken a bounded solution

$$=-\frac{8\pi^2m}{\sqrt{v_0}.h^2}\cdot\frac{e^{i\left(kr-\frac{\pi}{2}\right)}}{r}\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right)\mathrm{F}(r_0), \quad . \quad . \quad . \quad . \quad (31.2)$$

where

$$\mathrm{F}(r_0)=\int_{r_0}^{\infty}\sin\frac{k_0r_1}{\mathrm{B}}\mathrm{V}_1(r_1)dr_1. \quad . \quad . \quad . \quad . \quad . \quad (31.3)$$

Hence the relative intensity per unit solid angle is

$$\mathrm{I}_s=\left(\frac{8\pi^2m}{h^2}\right)^2\left\{\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right)\right\}^2.\{\mathrm{F}(r_0)\}^2. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In the following we evaluate the integral $F(r_0)$ for the special case $n=2$. In other words we consider the case in which the energy released from the incident particle causes the transition 1S $-$2P.

Now the Laguerre polynomial

$$L_{n+l}^{2l+1}(x)=\frac{d^{2l+1}}{dx^{2l+l}}\left\{e^x\frac{d^{n+l}}{dx^{n+i}}(x^{n+l}e^{-x})\right\}.$$

So that for $n=2$, $l=1$, we have

$$L_3^3(x)=-6.$$

Hence from (24.2)

$$I_1=-\frac{6}{r_2^2}\left\{-\frac{e^{-\alpha r_2}}{\alpha}r_2^4-4\frac{e^{-\alpha r_2}}{\alpha^2}r_2^3-12\frac{e^{-\alpha r_2}}{\alpha^3}r_2^2-24\frac{e^{-\alpha r_2}}{\alpha^4}r_2-24\frac{e^{-\alpha r_2}}{\alpha^5}+\frac{24}{\alpha^5}\right\}, \quad . \quad . \quad . \quad (33.1)$$

and from (24.3)
$$I_2=-6r_2\left(\frac{e^{-\alpha r_2}}{\alpha}r_2+\frac{e^{-\alpha r_2}}{\alpha^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (33.2)$$

So that from (24.1)

$$V_1(r_2)=\frac{16\sqrt{2}z^3e^2}{\alpha^2a^4(z+2)^3}\left\{e^{-\alpha r_2}r_2+4\frac{e^{-\alpha r_2}}{\alpha}+8\frac{e^{-\alpha r_2}}{\alpha^2}\cdot\frac{1}{r_2}+\frac{8}{\alpha^3}\cdot\frac{1}{r_2^2}(e^{-\alpha r_2}-1)\right\}. \quad (33.3)$$

On substituting the above value of V_1 in (31.3) we obtain

$$F(r_0)=\frac{16\sqrt{2}z^3e^2}{\alpha^4a^4(z+2)^3}f(r_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad (33.4)$$

where

$$f(r_0)=\frac{\sin\gamma\alpha r_0+\gamma\cos\gamma\alpha r_0}{1+\gamma^2}e^{-\alpha r_0}(4+\alpha r_0)$$
$$+\frac{(1-\gamma^2)\sin\gamma\alpha r_0+2\gamma\cos\gamma\alpha r_0}{(1+\gamma^2)^2}e^{-\alpha r_0}+\frac{8}{\alpha r_0}(e^{-\alpha r_0}-1)\sin\gamma\alpha r_0$$
$$+8\gamma J, \quad . \quad . \quad . \quad . \quad (33.5)$$

in which
$$\gamma=\frac{k_0}{B\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33.6)$$

and *
$$J=\int_{r_0}^{\infty}(e^{-\alpha r_0}-1)\cos\gamma\alpha r_0\cdot\frac{1}{r}dr. \quad . \quad . \quad . \quad . \quad (33.7)$$

$$=\log\frac{\gamma}{(1+\gamma^2)^{\frac{1}{2}}}+\alpha r_0-\tfrac{1}{4}\alpha^2r_0^2(1-2\gamma^2). \quad . \quad . \quad . \quad (33.8)$$

Therefore from (32)

$$I_s=\frac{2^{15}\pi^4m^2z^6e^4}{\alpha^8a^8(z+2)^6h^4}\left\{\frac{d}{dk_0}\left(\frac{B}{k_0}\right)\right\}^2\cdot f^2. \quad . \quad . \quad . \quad . \quad (34)$$

---

* See Bromwich, 'Theory of Infinite Series.'

(*a*) If $v \doteq v_0$, i. e., $k \doteq k_0$ and so from (15.1)

$$\frac{\mathrm{B}}{k_0} = \frac{\operatorname{cosec}\frac{\theta}{2}}{2(kk_0)^{\frac{1}{4}}},$$

$$\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right) = -\frac{\mathrm{B}}{2k_0^2}.$$

Hence from (34)

$$\mathrm{I}_s = 2^{11}\frac{\pi^2 z^6 e^4}{\alpha^{10}a^8(z+2)^6h^2v_0^2}\cdot\left(\frac{f}{\gamma}\right)^2 \quad \ldots\ldots\ldots \quad (35)$$

(*b*) If $v \ll v_0$, i. e., $k \ll k_0$ and so from (15.3)

$$\frac{\mathrm{B}}{k_0} = \frac{1}{k_0\left(1-\frac{k}{k_0}\cos\theta\right)} \text{ approx.}$$

$$\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right) = -\left(\frac{\mathrm{B}}{k_0}\right)^2.$$

Hence from (34)

$$\mathrm{I}_s = 2^{13}\frac{\pi^2 z^6 e^4 \mathrm{B}^2}{\alpha^{10}a^8(z+2)^6h^2v_0^2}\cdot\left(\frac{f}{\gamma}\right)^2 \quad \ldots\ldots\ldots \quad (36)$$

Critical Approach (r_0).

It is evident that the first boundary condition (16) gives (17) even when the scattering function is (31.2). Thus the values in (18) may, in the present case, be used to find the mean critical approach for a given velocity of the bombarding particle. The variation of r_0 with the angle of scattering may be obtained from the second boundary condition (16). We have

$$\frac{1}{\sqrt{v_0}}e^{ik_0r\cos\theta} = \frac{8\pi^2 m}{\sqrt{v_0}h^2}\cdot\frac{e^{i\left(kr-\frac{\pi}{2}\right)}}{r}\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right)\cdot\mathrm{F}\Big|_{r=r_0}.$$

Proceeding as before, we have

$$r_0 = \pm\frac{8\pi^2 m}{\sqrt{v_0}\cdot h^2}e^{2ik_0r_0\left(\frac{1}{2\mathrm{B}}\pm\sin^2\frac{\theta}{2}\right)}\cdot e^{i(k-k_0)r_0}\cdot e^{-i\frac{\pi}{2}}\frac{d}{dk_0}\left(\frac{\mathrm{B}}{k_0}\right)\mathrm{G}(r_0). \quad (37)$$

where $$\mathrm{G}(r_0) = \int_{r_0}^{\infty}\sin\gamma\alpha(r_1-r_0)\mathrm{V}_1(r_1)dr_1. \quad \ldots\ldots \quad (37.1)$$

On integrating we find

$$\mathrm{G}(r_0) = \frac{16\sqrt{2}z^3e^2}{\alpha^4a^4(z+2)^3}g(r_0), \quad \ldots\ldots\ldots \quad (37.2)$$

where $$g(r_0) = e^{-\alpha r_0}\cdot\frac{2\gamma}{(1+\gamma^2)^2} + e^{-\alpha r_0}(4+\alpha r_0)\frac{\gamma}{1+\gamma^2} + 8\gamma\mathrm{J}_1. \quad \ldots \quad (37.3)$$

and * $$J_1=\int_{r_0}^{\infty}(e^{-\alpha r_0}-1)\cos\gamma\alpha(r-r_0)\cdot\frac{1}{r}\,dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.3)$$

$$=\cos\gamma\alpha r_0\left\{\log\frac{\gamma}{(1+\gamma^2)^{\frac{1}{2}}}+\alpha r_0-\frac{1}{4}\alpha^2 r_0^2(1-2\gamma^2)\right\}$$

$$+\sin\gamma\alpha r_0\left\{\tan^{-1}\gamma-\frac{\pi}{2}+\frac{1}{2}\alpha^2 r_0{}^2\gamma\right\}. \quad . \quad (37.5)$$

(*a*) If $v\doteq v_0$, i. e., $k\doteq k_0$, we have for the critical approach corresponding to (35)

$$r_0=\pm\frac{2^4\sqrt{2}\pi z^3e^2}{\alpha^5a^4(z+2)^3hv_0}\sin\left\{2\rho\left(\frac{1}{2\mathrm{B}}\pm\sin^2\frac{\theta}{2}\right)+(k-k_0)r_0-\frac{\pi}{2}\right\}\cdot\left(\frac{g}{\gamma}\right). \quad (37.6)$$

$$=\mp\frac{2^5\sqrt{2}\pi z^3e^2}{\alpha^5a^4(z+2)^3hv_0}\cos\left\{2\rho\left(\frac{1}{2\mathrm{B}}\pm\sin^2\frac{\theta}{2}\right)\right\}\cdot\frac{g}{\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.7)$$

(*b*) If $v\ll v_0$ *i. e.*, $k\ll k_0$, we have for the critical approach corresponding to (36)

$$r_0=\mp\frac{2^6\sqrt{2}\pi z^3e^2\mathrm{B}}{\alpha^5a^4(z+2)^3hv_0}\cos\left\{2\rho\left(\frac{1}{2\mathrm{B}}\pm\sin^2\frac{\theta}{2}\right)-k_0r_0\right\}\cdot\frac{g}{\gamma}, \quad . \quad (37.8)$$

where $$k_0r_0=1{\cdot}43\,\pi,$$

and after simplification

$$=\pm\frac{2^6\sqrt{2}\pi z^3e^2\mathrm{B}}{\alpha^5a^4(z+2)^3hv_0}\cos\left\{\rho\left(\frac{v}{v_0}\cos\theta\pm2\sin^2\frac{\theta}{2}\right)\right\}\cdot\frac{g}{\gamma}. \quad . \quad . \quad (37.9)$$

It is obvious that the sign of (37.6) and (37.8) should be so selected that r_0 becomes positive.

Experimental.

Formula (35) gives the variation of the relative intensity of inelastic scattering by a helium atom with the angle of scattering when the incident and scattered velocities are nearly the same. As only 21 volts are required for the excitation 1S—2P, we may use the above formula with sufficient approximation, when the incident velocities are 100 volts or higher. The curves giving this variation for 100, 200, and 400 volts are given in fig. 1. They show that the intensity becomes infinite at zero angle for all incident velocities. But it rapidly falls off with the increase of the angle of scattering and becomes insignificant at angles lying between 5 and 10 degrees. At lesser incident velocity the intensity of scattering decreases much more slowly.

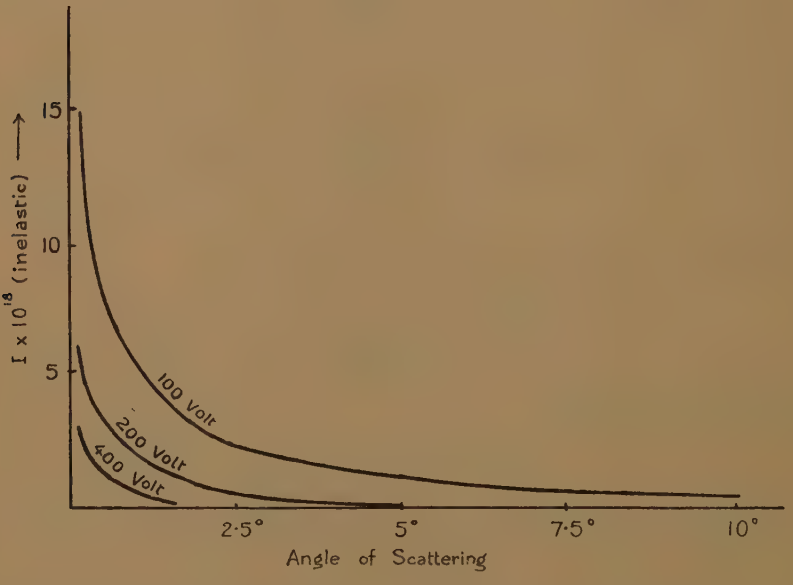
Experiments for determining the intensity of inelastic scattering by helium, resulting in excitation to the 2P level, have so far been carried

---

* See Bromwich, *l. c.*

out by Dymond and Watson *, Van Atta †, and Mohr and Nicoll ‡. All of them have expressed their data in arbitrary units, which are evidently unsuitable for quantitative verification of the present theory. Mohr and Nicoll (*l. c.*) have shown, by fitting their experimental data at one particular angle with the theoretical value, that the observed variation of intensity with the angle of scattering is well given by the Born formula at small angles. However, this agreement cannot be taken as quantitative. It simply shows that the intensity decreases uniformly with the increase of angle and tends to become infinite at zero angle of

Fig. 1.



scattering. It is evident from the curves in fig. 1 that a similar variation is given by formula (35). Thus Mohr and Nicoll's experiment indirectly supports the present theory. Their results also seem to support the theory in so far as at small angles the decrease of intensity for a given increase of the angle of scattering is less for smaller incident velocities. It should be noted that one is not justified in using the Born formula for elastic scattering in the present case.

---

* Dymond and Watson, Proc. Roy. Soc. cxxii. p. 571 (1929).

† Van Atta, Phys. Rev. xxxviii. p. 876 (1931).

‡ Mohr and Nicoll, Proc. Roy. Soc. A, cxxxviii. p. 229 (1932); *ibid.* cxlii. p. 320 (1933).

In his experiment mentioned above Van Atta has placed his electron collector against the directed beam of incident electrons. Its aperture is 1° 40′. Thus only the electrons scattered inelastically or elastically at angles between 0° and 1° 40′ are collected. He has determined the ratio of the inelastic to elastic intensities for the same scattered energy. Thus, for example, if the incident velocity for inelastic scattering is 200 volts the scattered velocity is 179 volts, which is made the incident velocity in the corresponding elastic scattering. It is obvious that the ratio thus obtained is independent of any unit, and is therefore verifiable from theory. Now the ratio considered by Van Atta may be expressed as

$$I_{ratio}(179)=\frac{\int_0^{1^\circ 40'} I_{(inel.\ 179)} \sin\theta\, d\theta}{\int_0^{1^\circ 40'} I_{(elast.\ 179)} \sin\theta\, d\theta} \quad . \quad . \quad . \quad . \quad (38)$$

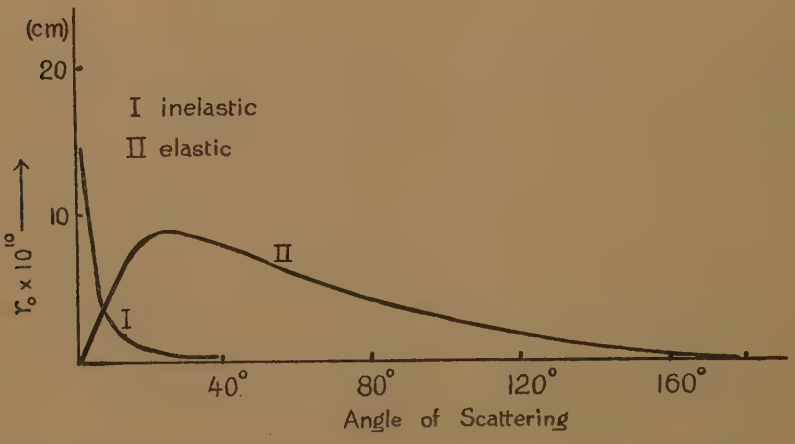
On evaluating the intensity of inelastic scattering from eq. (35) and the corresponding intensity of elastic scattering from the formula given in my previous paper * and then calculating the integrals in (38) by the graphical methods, it is found that the ratio comes out to be 2·8 per cent., 6·7 per cent., and 20 per cent. for scattered velocities 379, 179, and 79 volts respectively. The corresponding experimental values are approximately 2·8 per cent., 2·5 per cent., and 1·5 per cent. respectively for a gas pressure of 0·0106 mm. It may be noted that the theoretical value of the ratio becomes 3 per cent. for 179 volt scattering energy, when the integrals in (38) are taken over the whole range 0° to 180° ; the ratio is practically halved, obviously because the elastic scattering decreases much more slowly than the inelastic. Now at a pressure of 0·0028 mm. Van Atta finds that the ratio is about 0·5 per cent., being nearly constant over the whole range of velocity. Thus the gas pressure has an important effect on the intensity ratio of Van Atta. It is obvious that if the pressure is too high there is likely to be multiple scattering ; on the other hand, if the pressure is too low there may be cases of no scattering. These non-scattered electrons evidently enter the collector at zero angle and increase the intensity of elastic scattering. Therefore the effects of lowering the pressure would be to decrease the experimental value of the ratio down to zero for zero pressure. The theory is, however, based on the assumption that collision between the atom and the bombarding electron is a certainty. Accordingly the gas pressure should have no effect on the theoretical value of the ratio except through multiple scattering. This, it seems, explains the divergence between the theoretical

* Kar, Phil. Mag. *l. c.*

and experimental values of the ratio of intensities. It may also be pointed out that the increase of the intensity ratio with the decrease of the scattered velocity as predicted by the theory is also confirmed qualitatively by Mohr and Nicoll's observation that at smaller velocities the intensity of inelastic scattering falls off much more slowly with the scattering angle than at higher velocities. Lastly we must mention that the values for the total intensity ratio (inelastic/elastic), viz., 34 per cent. for 102 volt and 76 per cent. for 422 volt as observed by Dymond and Watson (*l. c.*), are too high.

The important feature of the wave-statistical theory of scattering is that it gives in every case the value of the critical approach (r_0) of the interacting particles. It is, therefore, advisable to compare its values

Fig. 2.



for the inelastic and elastic collisions. Curves giving the variation of r_0 with the scattering angle are given in fig. 2. The value for inelastic scattering is calculated from (37.7), whereas that for the elastic scattering is calculated from the formula *

$$r_0 = \frac{\rho\left(\frac{2e^2}{mv_0^2}\right)\left(\operatorname{cosec}\frac{\theta}{2}-1\right)}{1+\beta^2\operatorname{cosec}^2\frac{\theta}{2}} e^{-\frac{2\pi r_0}{a}} \left\{\left(1+\frac{zr_0}{a}\right)+\frac{\beta^2}{\sin^2\frac{\theta}{2}+\beta^2}\right\}.$$

by the graphical method.

Before concluding, we shall discuss from the theoretical standpoint, the recent observations of Hughes and West † on the inelastic scattering

* K. C. Kar, Phil. Mag. *l. c.*

† Hughes and West, Phys. Rev. l. p. 320 (1936).

of fast electrons by helium. For such high velocity collisions the process of scattering by the atom may be separated into two parts. First, the elastic scattering by the helium nucleus, and secondly, the inelastic scattering by the two atomic electrons. As in the latter case the bound electrons are knocked out resulting in ionization, the formula (35) is obviously not applicable. However, since the two processes are more or less independent of each other, the interaction potential may be taken as Coulombian in both the cases. The intensity of elastic scattering by the nucleus is given by the Mukherjee-Ghosh * formula

$$I=\left(\frac{ze^2}{2mv_0^2}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}\cos^2\left\{2k_0r_0\sin\frac{\theta}{2}\right\}, \quad . \quad . \quad . \quad . \quad (39)$$

where
$$r_0=\rho.\frac{ze^2}{mv_0^2}\left(\operatorname{cosec}\frac{\theta}{2}-1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39.1)$$

It may be noted that if the velocity is very high, the critical approach $r_0\sim 0$, except at very small angles ; therefore at large angles

$$\cos^2\left\{2k_0r_0\sin\frac{\theta}{2}\right\}\sim 1.$$

Thus (39) reduces to the Born-Rutherford formula which is used by Hughes and West (*l. c.*) to calculate the elastic scattering. It would, however, be more logical to use in this case the Mukherjee and Ghosh formula with $z=1{\cdot}69$. As the scattering by the atomic electron is inelastic, and, moreover, the effect of electron exchange is very important in this case, the Mukherjee and Ghosh formula is obviously not applicable here. In a recent paper on proton-proton scattering, Kar and Basu † have derived the following formula for the relative intensity of elastic scattering with exchange, assuming the potential to be Coulombian,

$$I=\left(\frac{e^2}{mv_0^2}\right)^2[\operatorname{cosec}^4\theta+\sec^4\theta-\operatorname{cosec}^2\theta\sec^2\theta\cos\{2\rho\alpha(\sin\theta-\cos\theta)\}]4\cos\theta, \quad . \quad . \quad . \quad (40)$$

where ρ has the same meaning as in (20) and $\alpha=\frac{2\pi e^2}{hv_0}$. The formula (40) is different from that derived by Mott, although both of them reduce to the approximate form

$$I=\left(\frac{e^2}{mv_0^2}\right)^2[\operatorname{cosec}^4\theta+\sec^4\theta-\operatorname{cosec}^2\theta\sec^2\theta]4\cos\theta. \quad . \quad (40.1)$$

It is this MKB-formula (40.1) which has been used by Hughes and West for calculating the intensity of scattering by the atomic electron. But, as has been pointed out before, the formula must be modified for

* See Kar, *l. c.*
† K. C. Kar and D. Basu, Phil. Mag. xxix. p. 200 (1940).

inelasticity. Now inelastic scattering under a Coulomb potential has already been considered in the present paper. Eq. (15.2) giving the relative intensity in this case, shows that the only modification due to inelasticity is that we have to take simply the geometrical mean of the incident and scattered velocities in place of the initial velocity. Accordingly (40.1) is modified, and we have for the relative intensity of inelastic scattering of the incident electron by the atomic electrons of helium, on taking into account the effect of electron exchange,

$$I=\left(\frac{e^2}{mv_0v}\right)^2[\mathrm{cosec}^4\theta+\sec^4\theta-\mathrm{cosec}^2\theta\sec^2\theta]4\cos\theta, \quad . \quad . \quad (41)$$

where v_0 and v are the incident and the scattered velocities.

Physical Laboratory,
Presidency College,
Calcutta, India.

XLVI. *The Dielectric Constants of some Metallic Sulphates containing various amounts of Water of Crystallization.*

By C. K. Cheng, Ph.D.,
Assistant Demonstrator, Department of Physics,
University of Toronto *.

[Received August 4, 1939.]

Abstract.

The dielectric constants of salts with varying amounts of water of crystallization have been measured. Samples of the powdered salt are suspended in a graded series of mixtures of two liquids which give liquid media of different dielectric constants. The mixture is found which has the same dielectric constant as the salt; this is determined by finding experimentally the particular mixture, the dielectric constant of which is not changed by the addition of the powdered salt. The capacity of the condenser cell containing the samples is measured in the circuit of a crystal controlled oscillator of about two megacycles frequency. The molecular polarization of the water in the salts is practically constant and only slightly higher than that of ice. The results indicate that the water molecules retain their integrity as individual molecules bound in much the same manner as those in ice.

I. *Introduction.*

This investigation is a determination of the dielectric constants of salts having varying amounts of water of crystallization and the consequent deduction of the molecular polarization of water in this state. It involves improvements on the technique used recently by L. G. Turnbull [1].

Attempts to measure the dielectric constant of a salt in the dry solid state meet with many practical difficulties, and consequently the method of mixtures by which the powdered salt is suspended in an inactive liquid is experimentally more promising. The dielectric constant of any sample is determined by finding the capacity of a test condenser filled with the liquid mixture.

The method here used depends on obtaining a liquid medium, the dielectric constant of which can be changed at will so as to be either

* Communicated by Professor E. F. Burton.

above or below that of the salt to be used. The value of the dielectric constant of the salt is then determined by finding that of the particular liquid which has the same value of reading for the capacity of the test condenser, whether the latter is filled with the liquid alone or with the liquid in which some of the powdered salt is suspended.

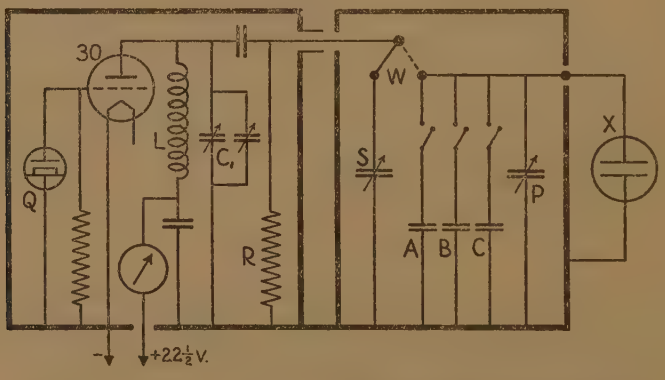
The dielectric constants of salts range around the values 5 to 6, whereas most insulating liquids, such as oils, have a constant of 2 to 3. Of course, the liquid must not dissolve or react chemically with the salt used. In order to obtain a liquid satisfactory for the purpose it is necessary to mix two liquids, one with a dielectric constant higher than 6 and one with the constant much lower. The two components finally chosen for the purpose were benzol ($\epsilon=2{\cdot}283$) and ethyl malonate ($\epsilon=8{\cdot}2$).

II. *Experimental Arrangements.*

(i.) *The Circuit.*

Many different forms of high frequency circuit have been used in the determination of capacities of condensers such as involved here [2].

Fig. 1.



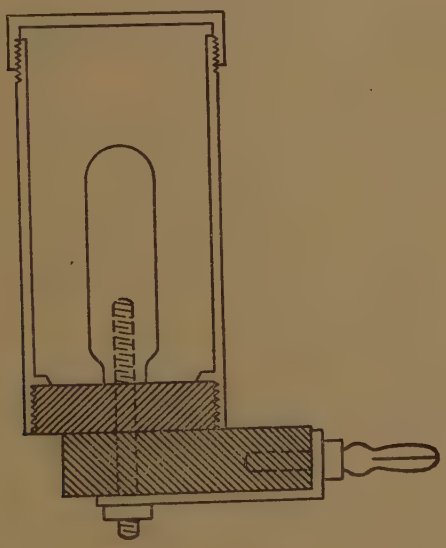
Details of circuit used to measure dielectric constant.

The circuit used in this experiment is built similarly to that described in the catalogue " Dipol 38 " of the P. J. Kipp company. The advantage in such a circuit lies in the simplicity of the manipulation without loss of the desired high sensitivity and accuracy. Details are shown diagrammatically in fig. 1. The left-hand side represents the oscillator shielded in a brass box. The crystal Q, which has a natural frequency of about 2 megacycles, is connected to the grid of a vacuum tube, triode RCA–30. The plate lead of the tube is connected to an inductance-capacity circuit LC_1, which is coupled through a large condenser to the condenser array

shown on the right-hand side of the diagram. A direct current milliammeter is placed in the circuit of the 22½ volt battery supply. The filament of the tube is heated by a two-volt storage cell.

The condenser array consists of two branches divided at the double throw switch W. S is an adjustable condenser fixed at an appropriate value. A, B, and C are three very small fixed condensers to extend the range of P, which is the standard. P is of the cylindrical type. The inner cylinder is connected to the ground, and is movable along the common axis by means of a micrometer screw. The change in capacity is therefore directly proportional to the motion of the screw. The pitch

Fig. 2.



The test condenser X.

of the screw is 1/16 inch; the change of capacity is about 0·3 cm. per turn of the screw. However, no attempt is made to determine the capacity in absolute units, because only relative values of capacity are dealt with in the results. Therefore all values are given in terms of the scale reading of the standard condenser P. The test condenser cell, marked X in the diagram, is also of cylindrical type. It is shown approximately to scale in fig. 2; the outer cylinder, which is connected to ground, serves as the container of the dielectric to be measured and also acts as an electric shield. Both cylinders are silver plated and firmly secured to the fibre bottom, which is the only solid connexion between the cylinders. The leads are connected out through the bottom to plugs

so that the cell can be detached from the measuring box for cleaning. The cell can hold about 11 c.c. of the dielectric and has a calculated capacity of about 5 cm. when empty.

(ii.) *Measurement of Capacity.*

When the capacity in the plate circuit of the oscillator is gradually increased from a small value, the direct current in the circuit as shown in the milliammeter gradually decreases until it reaches a minimum. A very slight further increase of the capacity will cause the plate current to increase very rapidly to a constant value at which the oscillator stops functioning. When the capacities are so adjusted that the milliammeter indicates the middle of this rapid increase the set is extremely sensitive to small changes of capacity in the condenser array. In fact, it is too sensitive to be steady. A resistance R of 30,000 ohms in parallel with the condensers is required to reduce the sensitivity considerably, so that each hundredth of a turn of the micrometer screw in the standard condenser produces a noticeable but not too large change in the plate current. The resistance also serves to minimize any possible disturbance due to conductivity in the dielectrics to be measured.

The condenser P (with addition of A, B, and C, when required) is adjusted to give the same reading on the meter as when the switch is swung to S, indicating that P is equal to S. When the cell X is plugged into the circuit, P must be reduced by the amount X to give the same reading again. The value for X thus determined is made of two parts, a constant part C_0 due to electric flux linkage through the insulator bottom, and a part C due to electric lines of force traversing the dielectric of constant ϵ. This latter part is just C when the condenser is empty. Since both C and C_0 are unknowns in addition to the sought ϵ, another set of readings is needed for their evaluation. The cell is therefore measured three times instead of only twice. Let the capacity of the cell when empty be X_A. Let X_B be the value when it is filled with benzene, which has a dielectric constant of 2·283 ; and let X_C be the capacity when the cell is filled with the dielectric. Then

$$X_A = C_0 + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$X_B = C_0 + 2{\cdot}283\, C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$X_C = C_0 + \epsilon C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Therefore the dielectric constant is given by

$$\epsilon = \frac{X_C - X_A}{X_B - X_A}(1{\cdot}283) + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

A calibration in this manner is carried out for each set of measurements.

(iii.) *Measurement of Dielectric Constants of Solids.*

The solid is pulverized and suspended in an inert liquid medium, consisting of a mixture of benzol and ethyl malonate. By varying the relative proportion of the two components, liquid media with dielectric constants varying from 2·3 to 8·2 can be obtained.

The experiment is carried out in the following manner :

(*a*) The dielectric constant of an appropriate liquid mixture is measured.

(*b*) A known quantity of the solid is suspended in this liquid and the change in the dielectric constant is determined.

(*c*) If the value increases, a second liquid mixture of somewhat higher dielectric constant is used and the processes (*a*) and (*b*) are repeated. And so on until a liquid medium is attained such that the dielectric constant decreases on addition of the salt by the same amount as it formerly increased. Naturally this process may be carried on in the reverse order.

(*d*) The changes in the dielectric constant on the addition of the salt are plotted against the dielectric constants of the liquid media. The point where the curve crosses the axis indicating zero change is taken to be the dielectric constant of the solid, because this is so only when the liquid and the solid are of the same dielectric constant (see figs. 3 and 4).

It is quite unnecessary to keep the solid evenly distributed in the suspension, as was done in preliminary experiments by employing rather viscous liquids. The solids can settle down to the bottom of the cell as long as the same condition is true for all measurements. It is, as a matter of fact, disadvantageous to use viscous liquids, which make it almost impossible to eliminate air bubbles trapped in the suspension.

In the choice of liquids a few conditions must be fulfilled. For the sake of simplicity not more than two pure components should be used in the mixture. They should evidently be mutually miscible in all proportions. Their mixtures should cover the range of dielectric constants of the solids to be measured. They should not dissolve nor react with any of the solids used. They should not mix easily with water lest they alter the state of hydration of the solids. In the early part of the experiment mixtures of castor oil and absolute alcohol were used which proved to be unsatisfactory on this account. Finally, ethyl malonate was employed to advantage. It has a dielectric constant of 8·2 and it mixes well with benzene to give the required liquid media. As a precaution against any possible solubility, the clear liquids measured in process (*a*) were obtained by shaking with some of the solid to be added and then filtered. The effect of the subsequent addition of the solid is therefore solely due to the solid particles and is not due to any possible mount of dissolved matter. The size and shape of the particles are

immaterial in this experiment as long as the particles are small compared to the dimensions of the cell.

(iv.) *Preparation of the Solids.*

The solids used in this work were crystals of metallic sulphates with different amounts of water of crystallization. $CuSO_4 . 5H_2O$ and $MgSO . 7H_2O$ were obtained from chemical supplies as such. They were pulverized and preserved in well stoppered bottles. Measurements were made as immediately after the pulverization as time permitted, usually within two days. The rest of the powders after the measurements were

TABLE I.

Calculated Water of Crystallization.

Sulphates.	Weight of hydrate. A grams.	Temperature.	Loss in weight (wt. of water). B grams.	No. of mols. water per mol. of salt. X.
Copper *	3·4154	ca. 360° C.	0·3512	1·016
Magnesium † ..	2·8889	ca. 360° C.	0·3645	0·963
Ditto †	4·2278	orange-red	0·6052	1·11
Zinc **	3·9542	orange-red	0·4032	1·02
Ditto **.....	6·0029	ca. 360° C.	0·6048	1·00
Zinc ***.....	3·8103	ca. 360° C.	1·6804	7·05
Ditto ***.....	5·3256	orange-red	2·3564	7·10

Note.—* Prepared by heating bluestone to 150° C.
† Prepared by heating crystals of $MgSO_4 . 7H_2O$ to 150° C.
** Taken from supply marked $ZnSO_4 . 2H_2O$.
*** Recrystallized from water under room temperature, ca. 22° C.

heated in an oven at 150° C. for about six to twelve hours until the weight no longer decreased. Small samples of the salts were then carefully weighed and heated to about 500° C. for a few hours ; the loss of weight showed that they had been reduced to $CuSO_4 . H_2O$ and $MgSO_4 . H_2O$. The zinc sulphate was obtained in supplies marked $ZnSO_4 . 2H_2O$, but heating to a bright orange colour for three hours failed to reduce the weight by any more than exactly one molecular weight of water in each molecular weight of the salt. Thus it was concluded that the original sample had only one water of crystallization. This salt was then dissolved in boiling water and the solution allowed to cool slowly to room temperature. Needle-like crystals were formed from the solution. But attempts to dry the crystals in the air caused them to lose the crystalline form,

indicating loss of water of crystallization. Hence they were washed with methylated spirit to remove the adhered water and exposed to air just long enough for the alcohol to evaporate. Heating then established the composition to be $ZnSO_4 . 7H_2O$.

Table I. gives the details of the calculations of the water of crystallization from the loss of weight due to heating at definite temperatures.

The last column in the table is calculated in the following manner: Let W be the molecular weight of water. The molecular weight of the hydrate is then M+XW. Therefore

$$\frac{A}{B} = \frac{M+XW}{XW} \text{ or } X = \frac{M}{W} \cdot \frac{B}{A-B}.$$

The temperatures at which the samples were heated until they attained a constant weight were chosen from the following information *:

$$CuSO_4.5H_2O \xrightarrow{27^\circ} \ldots . 3H_2O \xrightarrow{93^\circ} \ldots . H_2O \xrightarrow{155^\circ} CuSO_4 \xrightarrow{653^\circ} CuO, SO_3$$

$$MgSO_4.7H_2O \xrightarrow{19^\circ} \ldots . 6H_2O \xrightarrow{38^\circ} \ldots . 2H_2O \xrightarrow{112^\circ} \ldots . H_2O \xrightarrow{203^\circ} MgSO_4 \xrightarrow{890^\circ} MgO, SO_3$$

$$ZnSO_4.7H_2O \xrightarrow{25^\circ?} \ldots . 6H_2O \xrightarrow{28^\circ} \ldots . 2H_2O \xrightarrow{115^\circ} \ldots . H_2O \xrightarrow{225^\circ} ZnSO_4 \xrightarrow{702^\circ} ZnO, SO_3$$

III. *Results.*

In Table II. are given the results of the determination of the dielectric constant (ϵ_∞) of the salts; $ZnSO_4 . 7H_2O$, $ZnSO_4H_2O$, $FeSO_4 . 7H_2O$, $CuSO_45H_2O$, $CuSO_4H_2O$, $MgSO_4 . 7H_2O$, and $MgSO_4H_2O$. The symbol ϵ stands for the dielectric constants of the liquid mixtures used and $\Delta\epsilon$ for the change brought about in these samples by the addition of 4 gm. of each of the salts to the cell full.

For the sake of clearness the results in Table II. are illustrated graphically in two figures, figs. 3 and 4.

From the regularity of the curves it appears that this immersion method of measuring the dielectric constants of solids is quite satisfactory. It gives the dielectric constant to three significant figures (about 0·4 per cent. uncertainty on the average). In as far as the electrical apparatus alone is concerned very much higher precision can be attained. But the uncertainty in the purity of the samples makes it appear superfluous to refine the apparatus, as the slightest amount of moisture adhering to the solids changes the dielectric constant to a much higher value.

---

* 'Handbook of Chemistry and Physics' (Chemical Rubber Co., Cleveland), 21st ed. p. 1042.

Table II.

Dielectric Constants of Salts.

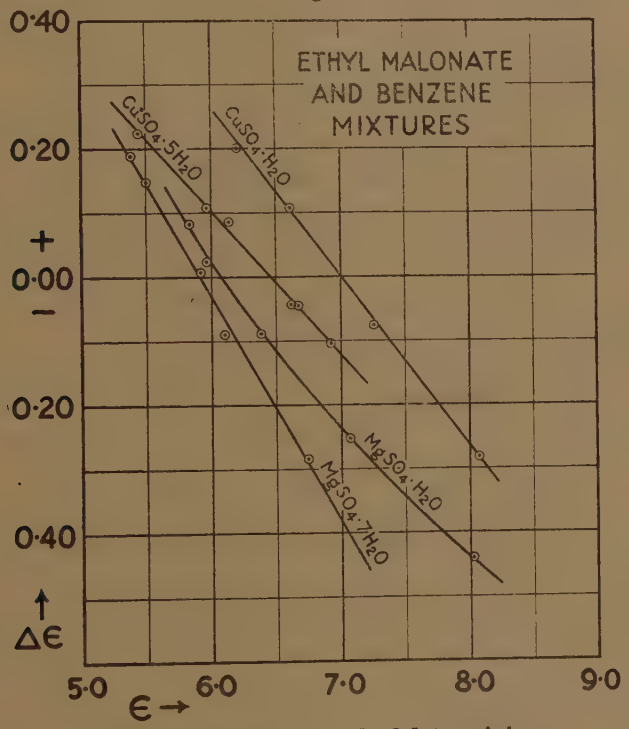
$CuSO_4 . 5H_2O$.		$CuSO_4 . H_2O$.		$MgSO_4 . 7H_2O$.		$MgSO_4 . H_2O$.		$ZnSO_4 . 7H_2O$.		$ZnSO_4 . H_2O$.		$FeSO_4 . 7H_2O$.	
ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.	ϵ.	$\Delta\epsilon$.
5·44	0·222	6·20	0·200	5·38	0·189	5·833	0·082	5·68	0·182	6·12	0·457	6·47	0·065
5·97	0·107	6·61	0·107	5·50	0·148	5·96	0·024	5·86	0·148	6·70	0·348	6·70	0·047
6·14	0·087	7·26	− 0·077	5·92	0·009	6·38	−0·089	6·25	−0·021	7·04	0·274	6·97	−0·080
6·62	−0·044	8·07	−0·282	6·10	−0·090	7·07	−0·252	6·61	−0·133	7·64	0·146	7·05	−0·060
6·67	−0·045	—	—	6·74	−0·284	8·02	−0·440	8·12	−0·498	9·00	0·030	7·52	−0·179
6·92	−0·102	—	—	—	—	—	—	—	—	—	—	—	—
ϵ_8=6·46		ϵ_8=7·00		ϵ_8=5·91		ϵ_8=6·09		ϵ_8=6·20		ϵ_8=8·25		ϵ_8=6·74	

The dielectric constants obtained from the graphs are tabulated in Table III., together with the molar polarization (P) calculated according to the equation :

$$P = \frac{\epsilon - 1}{\epsilon + 2} \cdot \frac{M}{d},$$

where M and d are the gram molecular weight and density of the salt respectively. Now if the water and the salt molecules in the hydrate crystal remain as individual molecules, and if they do not interfere with each other in their polarization behaviour, then the polarization, P,

Fig. 3.



Curves showing method of determining ϵ.

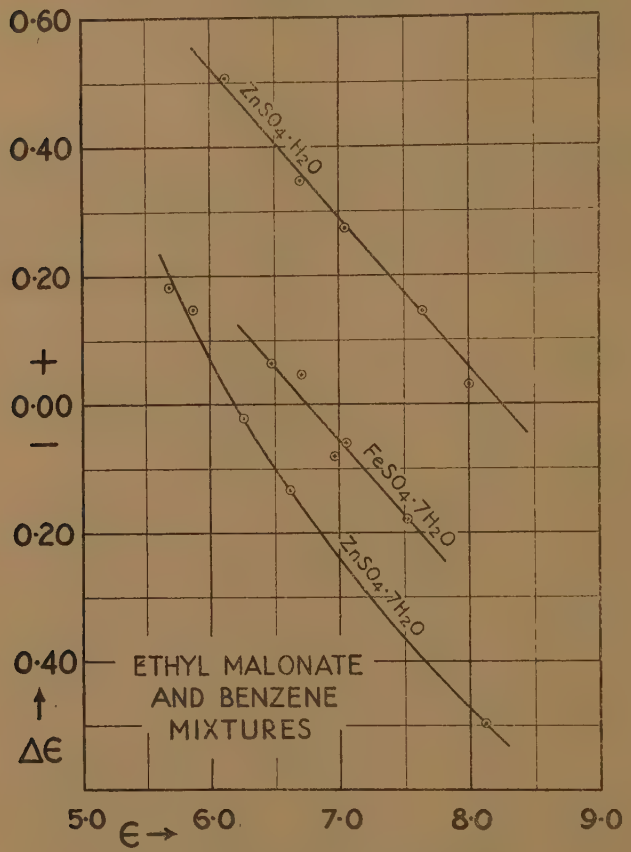
of the crystal is just the sum of the polarizations of the individual molecules multiplied by the number of them in each molecule of the hydrate.

Thus

$$P = P_S + qP_W,$$

where q is the number of water molecules in one molecule of the hydrate, P_S and P_W being the polarizations for the salt and water molecules respectively. From the results for a salt in any two different states of

Fig. 4.



Curves showing method of determining ϵ.

TABLE III.

Salt.	ϵ.	P.	Ps.	Pw.
$CuSO_4 . 5H_2O$.....	6·46	70·3	27·4	8·56
$CuSO_4 . H_2O$.....	7·00	36·0		
$MgSO_4 . 7H_2O$	5·91	92·7	26·1	9·30
$MgSO_4 . H_2O$	6·09	38·7		
$ZnSO_4 . 7H_2O$.....	6·20	91·2	29·7	9·00
$ZnSO_4 . H_2O$.....	8·25	35·4		
$FeSO_4 . 7H_2O$	6·74	96·7		
Ice	3·2			8·3

hydration P_S and P_W can be calculated. These results are assembled in Table III., and for comparison ϵ and P_w for ice are added.

The fact that the values of P_W are all close to that obtained from ice indicates definitely that the water molecules do retain their integrity in the crystals in a manner similar to those in ice. It is also observed that all these values are slightly higher than the pure ice value. This is expected, because the neighbouring molecules in such solids do affect each other.

Gratitude is hereby extended to Professor E. F. Burton for his guidance in carrying out this work. Thanks are also due to Professor A. Pitt for his suggestions in the choice and building of the electrical apparatus.

Bibliography.

(1) Burton, E. F., and Turnbull, L. G., Proc. Roy. Soc. London, A, clviii. p. 182 (1937).

(2) Körner, *Z. für Phys.* ciii. 3/4, Oct. 28, 1936, p. 170 ; Busch, G., *Helv. Phys. Acta*, x. 4, p. 261 ; Bantle and Busch, *Helv. Phys. Acta*, x. 4, p. 262 ; Evans, R. C., Phil. Mag. xxiv. 159, p. 70 (July 1937) ; Le Fèvre, R. J. W., Trans. Faraday Soc. xxxiv. pt. ix. (Sept. 1938) ; Beauvilain, M., *Ann. d. Phys.* Oct. 1936.

XLVII. *A Table of Complete Elliptic Integrals.*

By Alan Fletcher, M.A., Ph.D., University of Liverpool *.

[Received August 2, 1940.]

The following table was made in connexion with work † on Laplace coefficients in dynamical astronomy. Doubtless this is not the only case in which the modulus k is preferable to $\sin^{-1}k$ or k^2 as argument and the functions of the complementary modulus k' are not required.

In the notation of the above work, the complete elliptic integrals K and E are given by

$$K=\pi/2M, \quad E=K(1-\tfrac{1}{2}S).$$

M is the Gaussian arithmetic-geometric mean, determined from the equations

$$a_0=1, \quad b_0=\sqrt{(1-k^2)}=k',$$

$$a_n=\tfrac{1}{2}(a_{n-1}+b_{n-1}), \quad b_n=\sqrt{(a_{n-1}b_{n-1})},$$

as the limit to which a_n and b_n rapidly converge. The quantity S is given by

$$S=\tfrac{1}{2}k^2+d_1+d_2^2+2d_3^2+4d_4^2+8d_5^2+\ldots,$$

where

$$d_n=a_{n-1}-b_{n-1}.$$

Miss K. Mounfield, B.Sc., had deduced 11-decimal values of K and E from the Laplace coefficients, and differenced them. The author decided to deduce 12-decimal values directly from his original computations of M and S, difference the three tabulated functions to 12 decimals, investigate doubtful roundings to 10 decimals, and make check calculations for high values of k (·88 and over) by the Landen formulæ

$$k_1=(1-k')/(1+k'), \quad K=(1+k_1)K_1, \quad E=E_1-k'(K-E_1).$$

It is believed that all the following tabular values are correctly rounded to 10 decimals. Special provision for interpolation has not been made, since it is anticipated that the table will be chiefly useful in preparing other tables with k as argument.

---

* Communicated by the Author.

† Mon. Not. Roy. Astr. Soc. xcix. p. 259 (1939).

TABLE

k.	K.	E.	M.
·00	1·57079 63268	1·57079 63268	1·00000 00000
·01	1·57083 55989	1·57075 70562	·99997 49992
·02	1·57095 34418	1·57063 92354	·99989 99875
·03	1·57114 99350	1·57044 28380	·99977 49367
·04	1·57142 52114	1·57016 78196	·99959 97998
·05	1·57177 94575	1·56981 41184	·99937 45110
·06	1·57221 29134	1·56938 16544	·99909 89855
·07	1·57272 58741	1·56887 03298	·99877 31191
·08	1·57331 86891	1·56828 00287	·99839 67887
·09	1·57399 17642	1·56761 06169	·99796 98513
·10	1·57474 55615	1·56686 19420	·99749 21442
·11	1·57558 06010	1·56603 38329	·99696 34850
·12	1·57649 74611	1·56512 60995	·99638 36705
·13	1·57749 67807	1·56413 85331	·99575 24770
·14	1·57857 92598	1·56307 09052	·99506 96597
·15	1·57974 56613	1·56192 29679	·99433 49523
·16	1·58099 68130	1·56069 44534	·99354 80666
·17	1·58233 36090	1·55938 50735	·99270 86917
·18	1·58375 70122	1·55799 45192	·99181 64937
·19	1·58526 80561	1·55652 24607	·99087 11153
·20	1·58686 78475	1·55496 85462	·98987 21745
·21	1·58855 75687	1·55333 24023	·98881 92646
·22	1·59033 84811	1·55161 36326	·98771 19528
·23	1·59221 19274	1·54981 18178	·98654 97800
·24	1·59417 93356	1·54792 65148	·98533 22595
·25	1·59624 22221	1·54595 72561	·98405 88759
·26	1·59840 21958	1·54390 35491	·98272 90846
·27	1·60066 09623	1·54176 48753	·98134 23103
·28	1·60302 03282	1·53954 06898	·97989 79459
·29	1·60548 22061	1·53723 04199	·97839 53512
·30	1·60804 86199	1·53483 34649	·97683 38515
·31	1·61072 17105	1·53234 91945	·97521 27364
·32	1·61350 37416	1·52977 69482	·97353 12577
·33	1·61639 71065	1·52711 60339	·97178 86282
·34	1·61940 43356	1·52436 57270	·96998 40196
·35	1·62252 81034	1·52152 52689	·96811 65605
·36	1·62577 12374	1·51859 38659	·96618 53345
·37	1·62913 67272	1·51557 06874	·96418 93775
·38	1·63262 77338	1·51245 48649	·96212 76757
·39	1·63624 76008	1·50924 54899	·95999 91627

TABLE (*cont.*).

k.	K.	E.	M.
·40	1·63999 98659	1·50594 16124	·95780 27166
·41	1·64388 82730	1·50254 22386	·95553 71570
·42	1·64791 67867	1·49904 63297	·95320 12414
·43	1·65208 96064	1·49545 27988	·95079 36620
·44	1·65641 11832	1·49176 05093	·94831 30413
·45	1·66088 62376	1·48796 82719	·94575 79281
·46	1·66551 97786	1·48407 48421	·94312 67926
·47	1·67031 71256	1·48007 89174	·94041 80216
·48	1·67528 39313	1·47597 91341	·93762 99130
·49	1·68042 62078	1·47177 40636	·93476 06693
·50	1·68575 03548	1·46746 22093	·93180 83916
·51	1·69126 31908	1·46304 20023	·92877 10720
·52	1·69697 19877	1·45851 17971	·92564 65859
·53	1·70288 45089	1·45386 98670	·92243 26832
·54	1·70900 90521	1·44911 43989	·91912 69788
·55	1·71535 44956	1·44424 34883	·91572 69420
·56	1·72193 03517	1·43925 51324	·91222 98850
·57	1·72874 68247	1·43414 72243	·90863 29498
·58	1·73581 48763	1·42891 75454	·90493 30941
·59	1·74314 63000	1·42356 37576	·90112 70751
·60	1·75075 38029	1·41808 33944	·89721 14321
·61	1·75865 10996	1·41247 38519	·89318 24665
·62	1·76685 30174	1·40673 23774	·88903 62194
·63	1·77537 56165	1·40085 60582	·88476 84469
·64	1·78423 63259	1·39484 18088	·88037 45916
·65	1·79345 40997	1·38868 63557	·87584 97511
·66	1·80304 95951	1·38238 62220	·87118 86412
·67	1·81304 53779	1·37593 77089	·86638 55554
·68	1·82346 61602	1·36933 68757	·86143 43173
·69	1·83433 90753	1·36257 95167	·85632 82263
·70	1·84569 39984	1·35566 11356	·85105 99960
·71	1·85756 39226	1·34857 69158	·84562 16810
·72	1·86998 54005	1·34132 16876	·84000 45938
·73	1·88299 90676	1·33388 98894	·83419 92058
·74	1·89665 02641	1·32627 55240	·82819 50323
·75	1·91098 97808	1·31847 21080	·82198 04955
·76	1·92607 47573	1·31047 26123	·81554 27617
·77	1·94196 97735	1·30226 93937	·80886 75469
·78	1·95874 81843	1·29385 41136	·80193 88809
·79	1·97649 37678	1·28521 76419	·79473 88210

TABLE (*cont.*).

k.	K.	E.	M.
·80	1·99530 27777	1·27634 99432	·78724 71007
·81	2·01528 65256	1·26723 99388	·77944 06933
·82	2·03657 46658	1·25787 53404	·77129 32667
·83	2·05931 94243	1·24824 24459	·76277 44915
·84	2·08370 11184	1·23832 58872	·75384 91547
·85	2·10993 54676	1·22810 83135	·74447 60045
·86	2·13828 34421	1·21756 99905	·73460 62247
·87	2·16906 47830	1·20668 82829	·72418 13795
·88	2·20267 69671	1·19543 69771	·71313 05908
·89	2·23962 24748	1·18378 53745	·70136 65671
·90	2·28054 91384	1·17169 70528	·68877 98646
·91	2·32631 18943	1·15912 81248	·67523 03208
·92	2·37807 11768	1·14602 47138	·66053 37730
·93	2·43745 75568	1·13231 91467	·64444 04837
·94	2·50686 45110	1·11792 39344	·62659 80151
·95	2·59001 12309	1·10272 16483	·60648 24384
·96	2·69314 29647	1·08654 64634	·58325 76834
·97	2·82799 53489	1·06914 64692	·55544 51592
·98	3·02098 04456	1·05009 22614	·51996 24278
·99	3·35660 05234	1·02847 58090	·46797 23774
1·00	Infinity.	1·00000 00000	·00000 00000

The computations have been carried out in the Mathematical Laboratory of the University of Liverpool. Acknowledgments made in the earlier paper are gratefully re-affirmed.

XLVIII. *Notices respecting New Books.*

Molecular Spectra and Molecular Structure.—I. *Diatomic Molecules.* By Gerhard Herzberg (translated by J. W. T. Spinks). [Pp. xxviii+592.] (New York: Prentice-Hall, 1939. Price $6.50.)

Many of those who are attracted to the study of molecular spectra by the immediate appeal of an intricate physical phenomenon, or by an interest in the chemical bearings of the information obtainable about molecular structure, must have been discouraged in their approach by inchoate presentations of much of the subject-matter. The stumbling block will be largely removed for them by this book. There are many good short surveys of the main features of molecular spectra, and of the leading ideas in their theoretical interpretation. Beyond this stage, observational details and theoretical refinements crowd on each other to such an extent that a presentation aiming merely at comprehensiveness may serve little useful purpose. The marshalling of the material is all important. Herzberg's powers of exposition have been shown in his previous book, 'Atomic Spectra and Atomic Structure.' Here, in the more formidable task of writing a comprehensive treatise rather than an introductory survey, characteristics of the earlier book show to even greater advantage: the orderliness of the treatment, the avoidance of irrelevant complication, the good choice of illustrative material, and the care taken with such accessories as indexes and captions of figures and tables. Further, there is consideration for the reader who is relatively unfamiliar with the subject by the relegation of details not essential to the main arguments to smaller type sections.

After a *résumé* of the elements of atomic structure, a survey is given of the observational material on molecular spectra, and its empirical representation, with an extensive and well-chosen series of spectrograms in illustration. Rotation and vibration spectra are then described, and their interpretation is given. After a general account of electronic states and transitions, finer details are fully discussed. Building-up principles are considered, with full accounts of the determination of the term manifold from states of the separated atoms, and of the united atom, and from the electron configuration, this chapter concluding with a survey of atomic, ionic, and van der Waals binding. The phenomena of dissociation and pre-dissociation are considered in relation to continuous and diffuse spectra. An excellent final chapter summarizes some of the main results—there is a most useful table, complete to July 1939, of ground state vibrational and rotational constants for over 250 diatomic molecules—and gives an account of applications to a wide variety of fields in physics, chemistry, and astrophysics. There is a bibliography with references to 50 books and some 700 papers.

The book does not deal with experimental methods for the production of spectra, nor with some of the more difficult mathematical details on the theoretical side. Apart from this, however, it appears to cover every aspect of diatomic spectra. It is a masterly exposition, which should meet the needs, and stimulate the interest, of a very wide variety of readers. A promised second volume is to deal with polyatomic molecules. If the author can maintain a standard comparable to that in the present volume, the whole work will be an outstanding achievement. E. C. S.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

INDEX TO VOL. XXX.

ACOUSTICAL source-observer problems, some, 223.
Analogy between the theories of potential and vibrations, 161.
Aruja (E.) and Perlitz (H.), a redetermination of the crystal structure of lithium, 55.
Bickley, (W. G.), notes on Mathieu functions.—I. A class of hyperbolic Mathieu functions, 312.
Books, new :—American Mathematical Society Semicentennial Publications (in two volumes), 79 ; Internal Constitution of the Earth (Edited by Beno Gutenberg), 82 ; M. M. Macdermott, Vowel Sounds in Poetry, 84 ; G. Tintner, The Variate Difference Method, 172 ; H. Le Boiteux and R. Boussard, Elasticité et Photoélasticimétrie ; Dr. E. G. Richardson, Sound, 172 ; Profs. J. E. Mack and M. J. Martin, The Photographic Process, 259 ; H. T. Pledge, Science since 1500. A short History of Mathematics, Physics, Chemistry, Biology, 259 ; Prof. O. K. Rice, Electronic Structure and Chemical Binding, 260 ; H. S. Allen and R. S. Maxwell, A Text-book of Heat.—Part II., 260 ; D. Laugharne Thornton, Mechanics applied to Vibrations and Balancing, 344 ; J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, 345 ; W. W. Rouse Ball, Mathematical Recreations and Essays, 345 ; Prof. L. E. Dickson, Modern Elementary Theory of Numbers, 346 ; Prof. I. S. Sokolnikoff, Advanced Calculus, 347 ; Dr. Harris Hancock, Development of the Minkowski Geometry of Numbers, 347 ; Blamey Stevens, The Psychology of Physics, 347 ; Reports on Progress in Physics.—Vol. VI., 348 ; Bruce Chalmers, The Physical Examination of Metals.—Vol. I. Optical Methods, 432 ; I. H. B. and A. G. H. Spiers, The Physical Treatises of Pascal, 433 ; Stephen Timoshenko : 60th Anniversary Volume, 434 ; Otto Klemperer, Electron Optics, 434 ; Drs. G. D. Parkes and J. W. Mellor, Mellor's Modern Inorganic Chemistry, 435 ; N. Rashevsky, Advances and Applications of Mathematical Biology, 435 ; J. W. L. Glaisher, British Association Mathematical Tables.—Vol. VIII. Number-Divisor Tables, 435 ; Prof. F. Klein, Elementary Mathematics from an Advanced Standpoint, 436 ; The Raman Jubilee Volume, 436 ; Gerhard Herzberg (trans. by J. W. T. Spinks), Molecular Spectra and Molecular Structure.—I. Diatomic Molecules, 520.
Brainerd (J. G.) and Weygandt (C. N.), solutions of Mathieu's equation, I., 458.
Carslaw (H. S.), a simple application of the Laplace transformation, 414.

Castigliano's principle of minimum strain-energy, and the conditions of compatibility for strain, 252.
Chakravarti (S. P.) on the nature of negative resistances and negative resistance sections, 294.
Cheng (C. K.), the dielectric constants of some metallic sulphates containing various amounts of water of crystallization, 505.
Chubb (W. F.), the thermal and electrical conductivities of metals and alloys, 323.
Combined stress and anisotropy, research on, 350.
Contribution to the theory of elasticity of non-isotropic materials (with applications to problems of bending and torsion), 418.
Cooper (J. L. B.), the Fermi-Dirac functions, 187.
Crystal structure of lithium, a redetermination of the, 55.
Dark bands in the spectra of double diffraction gratings, 68.
Dasannacharya (B.) and Gautom (B. S.), the sine relation and minimum ellipticity in Foucault's pendulums with copper suspensions, 151.
Dielectric constants of some metallic sulphates containing various amounts of water of crystallization, 505.
—— ——, the use of the wave guide for measurement of microwave, 1.
Einstein's distant parallelism and Dirac's equation, 478.
Elasticity of non-isotropic materials (with applications to problems of bending and torsion), contributions to the theory of, 418.
End-corrections, some notes on, 173.
Erdélyi (A.), some integral representations of the associated Legendre functions, 168.
Fabric theory of protein structure, 64.
Fahmy (M.), the idea of minimum proper time, and some consequences of it, 331.
Fermi-Dirac functions, 187.
Finite complexes in crystals: a classification and review, 103.
Fischer (O. F.), Lorentz transformation and Hamilton's quaternious, 135.
Fletcher (A.), a table of complete elliptic integrals, 516.
Foucault's pendulums with copper suspensions, the sine relation and minimum ellipticity in, 151.
Gautom (B. S.) and Dasannacharya (B.), the sine relation and minimun ellipticity in Foucault's pendulums with copper suspensions, 151.
Goldstein (S.), three-dimensional vortex motion in a viscous fluid, 85.
Guest (J. J.), research on combined stress and anisotropy, 349.
Haskey (H. W.), Einstein's distant parallelism and Dirac's equation, 478.
High frequency permeability, some measurements of, 247.
Hilton (W. F.), measurements of the noise from aerofoils and streamline wires, 237.
Houstoun (R. A.), dark bands in the spectra of double diffraction gratings, 68.
Huxley (L. G. H.), the lateral diffusion of a stream of ions in a gas, 396.
Idea of minimum proper time, and some consequences of it, 331.
Ignition of inflammable gases by hot moving particles.—II., 437.
Inelastic scattering of a beam of particles by hydrogen and helium, on the, 487.
Ingram (W. H.) on the integral equations of continuous dynamical systems, 16.
Jackson (W.), some measurements of high frequency permeability, 247.
Jeffreys (H.), an analogy between the theories of potential and vibrations, 161.
Kar (K. C.) on the inelastic scattering of a beam of particles by hydrogen and helium, 487.

Kimball (W. S.), partial derivatives of derivatives, 190.
Lamont (H. R. L.), the use of the wave guide for measurement of microwave dielectric constants, 1.
Laplace transformation, a simple application of the, 414.
Lateral diffusion of a stream of ions in a gas, 396.
Lorentz transformation and Hamilton's quaternions, 135.
McCrea (W. H.), quaternion analogy of wave-tensor calculus, 261.
Mathieu functions, notes on.—I. A class of hyperbolic Mathieu functions, 312.
Mathieu's equation, solution of.—I., 458.
Matrix theory of four-terminal networks, 370.
Measurements of the noise from aerofoils and streamline wires, 237.
Mechanical appliance for the smoothing of time series, 39.
Modification of a relativity postulate, 49.
Mutual inductance of two circles: a problem in approximation, 340.
Negative resistances and negative resistance sections, on the nature of, 294.
Neville (E. H.), the mutual inductance of two circles: a problem in approximation, 340.
Note on the barrier-layer resistance of selenium photo-elements, 428.
Notes on Mathieu functions.—I. A class of hyperbolic Mathieu functions, 312.
Obituaries.—Sir Oliver Joseph Lodge, 341; Sir Joseph John Thomson, 430.
On the integral equations of continuous dynamical systems, 16.
—— —— nature of negative resistances and negative resistance sections, 294.
Partial derivatives of derivatives, 190.
Paterson (S.), the ignition of inflammable gases by hot moving particles.—II., 437.
Perlitz (H.) and Aruja (E.), a redetermination of the crystal structure of lithium, 55.
Pipes (L. A.), the matrix theory of four-terminal networks, 370.
Propagation of electromagnetic waves in an ionized medium and the calculation of the true heights of the atmosphere, 282.
Protein structure, the fabric theory of, 64.
Quaternion analogy of wave-tensor calculus, 261.
Redetermination of the crystal structure of lithium, 55.
Redfearn (S. W.), some acoustical source-observer problems, 223.
Reissner (E.), a contribution to the theory of elasticity of non-isotropic materials (with applications to problems of bending and torsion), 418.
Relativity postulate, modification of a, 49.
Research on combined stress and anisotropy, 350.
Rydbeck (O.), the propagation of electromagnetic waves in an ionized medium and the calculation of the true heights of the ionized layers of the atmosphere, 282.
Sandström (A. E.), note on the barrier-layer resistance of selenium photo-elements, 428.
Schumann (T. E. W.), a mechanical appliance for the smoothing of time series, 39.
Selenium photo-elements, note on the barrier-layer resistance of, 423.
Silberstein (L.), solution of the equation $f'(x)=f(1/x)$, 185.
Simple application of the Laplace transformation, 414.
Sine relation and minimum ellipticity in Foucault's pendulums with copper suspensions, 151.
Smoothing of time series, a mechanical appliance for the, 39.
Solution of the equation $f'(x)=f(1/x)$, 185.

Solutions of Mathieu's equation.—I., 458.
Some acoustical source-observer problems, 223.
—— integral representations of the associated Legendre functions, 168.
—— measurements of high frequency permeability, 247.
—— notes on end-corrections, 173.
Southwell (R. V.), Castigliano's principle of minimum strain-energy, and the conditions of compatibility for strain, 252.
Spectra of double diffraction gratings, dark bands in the, 68.
Sulaiman (Sir Shah), modifications of a relativity postulate, 49.
Table of complete elliptic integrals, 516.
Thermal and electrical conductivities of metals and alloys, 323.
Three-dimensional vortex motion in a viscuous fluid, 85.
Tyte (L. C.), some notes on end-corrections, 173.
Use of the wave guide for measurement of microwave dielectric constants, 1.
Wells (A. F.), finite complexes in crystals: a classification and review, 103.
Weygandt (C. N.) and Brainerd J. G.), solutions of Mathieu's equation.—I., 458.
Wrinch (D.), the fabric theory of protein structure, 64.

END OF THE THIRTIETH VOLUME

PRINTED BY TAYLOR AND FRANCIS, LTD., RED LION COURT, FLEET STREET, E.C.4.